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<tr>
<td><strong>Author(s)</strong></td>
<td>Kato, Hisao</td>
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<tr>
<td><strong>Citation</strong></td>
<td>数理解析研究所講究録 (1984), 509: 18-22</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1984-01</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/2433/103789">http://hdl.handle.net/2433/103789</a></td>
</tr>
<tr>
<td><strong>Type</strong></td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td><strong>Textversion</strong></td>
<td>publisher</td>
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HYPERSPACES AND WHITNEY MAPS

By Hisao Kato

Throughout this note, the word compactum means a compact metric space. A connected compactum is a continuum. A Peano continuum is a locally connected continuum. If \( x \) and \( y \) are points of a metric space, \( d(x, y) \) denotes the distance from \( x \) to \( y \). For any subsets \( A \) and \( B \) of a metric space, let \( d(A, B) = \inf \{ d(a, b) \mid a \in A, \ b \in B \} \). Also, let \( d_H(A, B) = \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \} \). \( d_H \) is called the Hausdorff metric. The hyperspaces of a continuum are the spaces \( 2^X = \{ A \subseteq X \mid A \) is compact and nonempty\} and \( C(X) = \{ A \in 2^X \mid A \) is connected\} which are metrized with the Hausdorff metric \( d_H \). Let \( F_1(X) = \{ x \in X \mid x \in X \} \). A Whitney map for a hyperspace \( H \) of a continuum \( X \) is a continuous function \( w: H \rightarrow [0, w(X)] \) such that \( w(\{ x \}) = 0 \) for each \( \{ x \} \in F_1(X) \), and if \( A, B \in H \) and \( A \subseteq B \), then \( w(A) < w(B) \) (see [9]). The notion of Whitney map is an important and convenient tool for hyperspace theory. If \( w \) is a Whitney map for \( H \) and \( t \in [0, w(X)] \), then \( w^{-1}(t) \) is called a Whitney level. Whitney levels are coverings of \( X \) which, as \( t \) gets close to zero, converge to \( w^{-1}(0) = F_1(X) \subseteq X \). It is of interest to obtain information about the structure of Whitney levels and determine those properties which are preserved by the convergence of positive Whitney levels to zero level. In [1] and [8], Curtis, Schori and West proved that for any Peano continuum (locally connected continuum) \( X \), \( 2^X \) is a Hilbert cube \( Q = \prod_{n=1}^{\infty} [-1, 1] \) and if \( X \) contains no free arc, \( C(X) \) is a Hilbert cube \( Q \). Recently, Goodykoontz and Nadler introduced the notion "admissible Whitney map" and they proved the following
Theorem (Goodykoontz and Nadler). Let $X$ be a Peano continuum and let $w$ be an admissible Whitney map for $H=2^X$ or $C(X)$. If $H=C(X)$, assume that $X$ contains no free arc. Then for any $t \in (0, w(X))$, $w^{-1}(t)$ is a Hilbert cube and $w$ is an open map.

Let $X$ be a continuum. A Whitney map $w$ for $H=2^X$ or $C(X)$ is an admissible Whitney map for $H$ if there is a homotopy $h$: $H \times [0,1] \rightarrow H$ satisfying the following conditions:

1. $h(A,1)=A$, $h(A,0) \in F_A(X)$ for each $A \in H$, and

2. if $w(h(A,t)) > 0$ for some $A \in H$ and $t \in (0,1]$, then $w(h(A,s)) < w(h(A,t))$ for each $0 \leq s < t \leq 1$.

Moreover, in [4] we proved the following

Theorem [4]. Under the same hypotheses as in above theorem, the restriction $w | w^{-1}((0,w(X)) : w^{-1}((0,w(X))) \rightarrow (0,w(X))$ of $w$ to $w^{-1}((0,w(X)))$ is a trivial bundle map with Hilbert cube fibers. If $X$ is the Hilbert cube $Q$, there is a Whitney map $w$ for $H$ such that $w | w^{-1}([0,w(X))]$ is a trivial bundle map with Hilbert cube fibers. Also, if $X$ is the $n$-sphere ($n \geq 1$), then there is a Whitney map $w$ for $H=2^{S^n}$ ($n \geq 1$) or $C(S^n)$ ($n \geq 2$) such that for some $t \in (0,w(X))$, $w | w^{-1}((0,t))$ is a trivial bundle map with $S^n \times Q$ fibers.

Also, in [5] we showed the following

Theorem [5]. Let $P_i$ be a 1 or 2 dimensional connected polytope for each $i=1,2,\ldots,n$. Then there is a Whitney map $w$ for $H=\bigotimes_{i} P_i$ or $C(\bigotimes_{i} P_i)$ ($n \geq 2$) such that for some $t \in (0,w(\bigotimes_{i} P_i))$,
$w|w^{-1}((0,t))$ is a trivial bundle map with $\prod_{w_{i}}^{n} P_{1} \times Q$ fibers.

In relation to above theorems, we have the following

**Proposition [5].** Let $X$ be a compact ANR but not AR. Let $H=2^{X}$ or $C(X)$. If $H=C(X)$, assume that $X$ contains no free arc. If $w$ is any Whitney map for $H$, there is a point $t_{0} \in (0,w(X))$ such that $w|w^{-1}((0,t_{0}))$ is not a trivial bundle map.

**Example [5].** Let $X=S$ be the unit circle. Let $A \in H=2^{X}$ or $C(X)$. For each $n \geq 2$, let $F_{n}(A)=\{K \subset A | K \neq \emptyset \}$ and the cardinality of $K$ is $\leq n \}$. Define $\lambda_{n} : F_{n}(A) \rightarrow [0,\omega)$ by letting $\lambda_{n}(\{a_{1},a_{2},...,a_{n}\}) = \min\{d(a_{i},a_{j}) | i \neq j \}$ for each $\{a_{i}\} \in F_{n}(A)$, where $d$ is the arc length metric for $S$. Also, let $w_{n}(A)=\sup_{n} \lambda_{n}(F_{n}(A))$ and let $w(A)= \sum_{n=2}^{\infty} w_{n}(A)/2^{n-1}$ for each $A \in H$. Then $w$ is a Whitney map for $H$. Then $w|w^{-1}((0,\pi/2)) : w^{-1}((0,\pi/2)) \rightarrow (0,\pi/2)$ is a trivial bundle map with $S \times Q$ fibers, but $w|w^{-1}((0,\pi/2))$ is not a trivial bundle map. In fact, $w|w^{-1}((0,\pi/2))$ is not an open map.

**Example [5].** There is a Whitney map $w$ for $H=2^{[0,1]}$ such that for every $t \in (0,w([0,1]))$, $w|w^{-1}((0,t))$ is not a trivial bundle map.

**Question [5].** Is it true that if $P$ is a $n$-dimensional ($n \geq 3$) polytope, there is a Whitney map $w$ for $H=2^{P}$ or $C(P)$ such that for some $t \in (0,w(P))$, $w|w^{-1}((0,t))$ is a trivial bundle map with $P \times Q$ fibers? (If $H=C(P)$, assume that $P$ contains no free arc.)
As an application of hyperspace theory, we obtain the following

Theorem [6]. If a compactum $X$ has a scalene metric, then $X$ is an absolute retract. Moreover, if a locally compact space $X$ has a locally scalene metric, then $X$ is an absolute neighborhood retract.

A metric $d$ defined in a space $X$ is a scalene metric [6] if $x_1$ and $x_2$ are different two points of $X$, then there is a point $x_0$ of $X$ such that for each point $x$ of $X$ either $d(x,x_1) > d(x,x_0)$ or $d(x,x_2) > d(x,x_0)$ holds. This notion is a generalization of norm of a linear space. A metric $d$ defined in a space $X$ is a locally scalene metric [6] if for each point $x \in X$ there is a neighborhood $U$ of $x$ in $X$ such that $d|_{U \times U}$ is a scalene metric.

Remark [6]. Every 1-dimensional AR has a scalene metric but there is a 2-dimensional AR which does not admit a scalene metric.

Question [6]. Is it true that every locally compact polytope has a locally scalene metric?

Question [6]. Is it true that every compact strongly convex metric space admits a scalene metric? Is it true that every compact scalene metric space admits a strongly convex metric?

Proposition [6]. If $d$ is a scalene metric and convex, then $d$ is a strongly convex metric.
References