数理解析研究所講究録
303巻 1977年 56-69

Generalized Dynamical Systems and Volterra Integral Equations

By Shin-ichi Nakagiri
(Department of Applied Mathematics, Kobe University, Nada, Kobe, Japan)

1. Introduction

The concept of local topological dynamics was first given by T.Ura [19], and he studied this theory extensively, e.g., prolongation, stability of higher order and isomorphism theory ([19] and [20]). The theory of local topological dynamics is motivated by the theory of autonomous ordinary differential equations and quite general results of topological dynamics help us to investigate many properties of their solutions such as asymptotic behavior and stability. It is well known that the solution of ordinary differential equation defines a local dynamical system (a local flow) if the equation is autonomous, but the solution defines no more a local flow in an usual manner if the equation is nonautonomous.

However, inspiring by a technique of R. K. Miller [5], G. R. Sell [16] has shown how to associate in a significant manner a local dynamical system with a nonautonomous equation. Under such constructions, he obtained many invariance properties for the limit sets of solutions of nonautonomous equations ([17]). His method to define a local flow is very natural, and therefore there appeared many further researches in this direction. See [4] and [9] for surveys, [3] and [21] for applications to stability theory and [8] and [6] for generalizations to Volterra integral equations.

In all papers cited above, it has been assumed that the solutions of the given equations satisfy some sort of uniqueness condition. On the other hand in many cases, the uniqueness condition is not needed to obtain analogous theorem about the qualitative behaviors of solutions. This suggests us the possibility of treating the given equation without uniqueness by generalized dynamical systems. Zubov [22], Seibert [15] and Roxin [13,14] gave similar types of axiomatically defined generalized dynamical systems by using a set-valued map which normally represents the solution funnel. Their axioms of generalized systems are based on differential equations without uniqueness, contingent equations and control systems governed by differential equations, but not on integral equations without uniqueness.

In the present paper we shall give the axiom of a local generalized dynamical system which can be considered a coupling of Roxin's axiom and Sell's axiom. This axiom admits us to treat Volterra integral equations without uniqueness and this treatment seems to be new. We shall show under what conditions and formulations this treatment is possible. The further theory and applications of our local generalized dynamical systems will be discussed elsewhere.

Finally we note that Sell [18] has presented an another method which allows us to define classical dynamical systems without uniqueness assumption in the case of nonautonomous differential equations. However, his method seems to be much more complicated for the case of Volterra integral equations.

2. Definitions and Notations

Let X be a metric space with metric d_X and 2^X be the set of all subsets of X. In order to avoid infinite distances between subsets of X, we will replace the given metric $d_X(a,b)$ by $\rho(a,b) = d_X(a,b)/\ 1 + d_X(a,b)$. We then define the distance $\rho(a,B)$ between a point a and a set B by

$$\rho(a,B) = \inf\{\ \rho(a,b)\ :\ b \in B\ \}$$

and the distances $\rho^*(A,B)$ and $\rho(A,B)$ between sets A and B by

$$\rho^*(A,B) = \sup\{\ \rho(a,B)\ :\ a \in A\ \} \quad \text{and}$$

$\rho(A,B) = \rho(B,A) = \max (\rho^*(A,B), \rho^*(B,A))$, and note that $\rho^*(A,B) \leq \rho(A,B)$ for any nonempty subsets A and B of X. According we define an ε-neghboring set of a given set A_0 by

$V(A_0,\varepsilon) = \{ x \in X : \rho(x,A_0) < \varepsilon \}$. Then we have easily that

$\rho(A,B) = \inf \{ \varepsilon : A \subset V(B,\varepsilon)$ and $B \subset V(A,\varepsilon) \}$.

If ψ is any function and A is a set, we write $\psi(A)$ for $\cup\{ \psi(x) : x \in A \}$, and analogously for functions of more than one variable.

The set-valued operator F defined by the variable sets $F(\alpha) \subset X$, where α belongs to some topological space $D(F)$, is said to be :

(I) continuous on $D(F)$, if for any $\alpha_0 \in D(F)$ and for every $\delta > 0$ there exists some neighborhood of α_0, say V, such that for all $\alpha \in V$: $\rho(F(\alpha),F(\alpha_0)) < \delta$;

(II) upper semi-continuous on $D(F)$, if in the condition (I) the distance ρ is replaced by the distance ρ^*.

Definition 2.1. Let X be a metric space. For each $p \in X$, let there be given a half open interval $I_p = [0,\alpha_p)$, where $\alpha_p > 0$. Let D be the set $D = \{ (t,p) : p \in X$ and $t \in I_p \}$. A set-valued function $\pi : D \longrightarrow 2^X$ is a local generalized semi-flow (or a local generalized semi-dynamical system) on X if the followings hold :

(1) The set $\pi(t,p)$ is nonempty and closed for all $(t,p) \in D$; in particular $\pi(0,p) = \{ p \}$ for each $p \in X$.

(2) If $t \in I_p$ and $s \in \cap \{ I_q : q \in \pi(t,p) \}$, then $t + s \in I_p$ and $\pi(t + s,p) = \pi(t,\pi(s,p)) = \{ \pi(t,q) : q \in \pi(s,p) \}$.

(3) Each I_p is maximal in the sense that if $I_p = [0,\alpha_p)$ then either $\alpha_p = \infty$ or the closure of the set $\cup\{\pi(t,p) : t \in [0,\alpha_p) \}$ is not compact in X.

(4) D is open in $R^+ \times X$.

(5) $\pi : D \longrightarrow 2^X$ is upper semi-continuous.

(6) for each fixed $p \in X$, $\pi(\cdot,p) : I_p \longrightarrow 2^X$ is continuous.

We note that ρ defines a metric on the class of all nonempty compact subsets of X and our flow defined by an integral equation is a (compact-connected set)-valued mapping.

3. Integral Equations and Function Spaces

We consider the Volterra integral equation,

$$P(f,g) \qquad x(t) = f(t) + \int_0^t g(t,s,x(s))ds ,$$

where t belongs to the interval $R^+ = [0,\infty)$ and x , f and g have values in R^n . Let $I \subset R^+$ be an interval containing 0 . For each I and each natural number N , we define a space $C(I,N)$ by the set of all continuous functions with domain I and range in $\{ x \in R^n : |x| \leq N \}$ with the compact open topology. We denote $\cup_{N=1}^{\infty} C(I,N)$ by $C(I)$ and $C[0,\infty)$ by C . Then $C[0,\infty)$ is a Frèchet space with the semi-norms of uniform convergence on compact subintervals of $[0,\infty)$. Especially the topology on C is generated by a metric $d_C(f,h) = \sum_{n=1}^{\infty} 2^{-n}(\max (1 , \max \{ | f(t) - h(t)| : t \in [0,n] \}) .$

For the equation $P(f,g)$ we assume the following hypotheses (A) , (B-1) - (B-3) which are used in [1].

(A) $f : R^+ \longrightarrow R^n$ is continuous.

(B-1) $g : R^+ \times R^+ \times R^n \longrightarrow R^n$ is a function such that

- (1) for each $(t,x) \in R^+ \times R^n$, $g(t,s,x)$ is L-measurable in s
- (2) for each $(t,s) \in R^+ \times R^+$, $g(t,s,x)$ is continuous in x , and
- (3) $g(t,s,x) = 0$ for all $s \geq t$.

(B-2) For each real number $l > 0$ and each natural number N , there exists a L-measurable function $m(t,s)$ in $s \in [0,t]$ for each $t \in [0,l]$ such that $|g(t,s,x)| \leq m(t,s)$ $(0 \leq s \leq t \leq l , |x| \leq N)$

and $\int_0^t m(t,s)ds < \infty$.

(B-3) For each compact interval $J \subset R^+$, each natural number N and each τ in R^+ ,

$$\lim_{t\to\tau} \sup \{ \left| \int_J [g(t,s,\phi(s)) - g(\tau,s,\phi(s))]ds \right| : \phi \in C(J,N)\} = 0 .$$

Under the Hypotheses (A) , (B-1) - (B-3) , we have from Miller [6,7], Kelly [2], Artstein [1] and Nakagiri and Murakami [10,11] the following fundamental properties of the equation $P(f,g)$.

(i) Local existence, i.e., there exist a $\beta > 0$ and a continuous function $x(t)$ on $[0,\beta]$ such that $x(t)$ satisfies $P(f,g)$ for $0 \le t \le \beta$.

(ii) Continuability, i.e., every solution $x(t)$ of $P(f,g)$ can be extended to right maximal interval $[0,\alpha(f,g,x))$, where $\alpha(f,g,x)$ is either ∞ or a finite number such that $\limsup_{t\uparrow\alpha(f,g,x)} |x(t)| = \infty$.

(iii) Kneser's property, i.e., the solution space $S(f,g)$ $= \{ x(\cdot) \in C[0,\alpha(f,g)) : x(t)$ is a solution of $P(f,g)$ on $[0,\alpha(f,g)) \}$ is compact and connected in the Frèchet space $C[0,\alpha(f,g))$, where $\alpha(f,g) = \inf \{ \alpha(f,g,x) : x$ is a solution of $P(f,g) \}$.

To obtain the semi-group property (2.1-(2)) of a local generalized semi-flow, we must consider the translated equations of $P(f,g)$. Let $x(t)$ be a solution of $P(f,g)$ on $[0,\alpha(f,g,x))$. Then for any $\tau \in [0,\alpha(f,g,x))$, $y(t) = x(t + \tau)$ satisfies the τ-translated equation

$$y(t) = \{ f(t + \tau) + \int_0^\tau g(\tau + t,s,x(s))ds \} + \int_0^t g(t + \tau,s + \tau,y(s))ds$$

on $[0,\alpha(f,g,x) - \tau)$. Then we define the translated function f_τ and g_τ by $f_\tau(t) = f(t + \tau)$ for all $t \in R^+$ and $g_\tau(t,s,x) = g(t + \tau,s + \tau,x)$ for all $(t,s,x) \in R^+ \times R^+ \times R^n$, and define the operator G by

$$G(\tau,\phi,g)(t) = \int_0^\tau g(t + \tau,s,\phi(s))ds \quad \text{for all} \quad t \in R^+ .$$

Now define G_0 by the set of all functions g which satisfy (B-1) and the followings (B-2*) and (B-3*) which are weaker than those used in [6].

(B-2*) Besides the conditions in (B-2), we add the following condition:

$\sup \{ \int_0^r m(t + r,s)ds : r \in J \} < \infty$ for any compact interval $J \subset [0,l]$ and t such as $t + r \in [0,l]$ for all $r \in J$.

(B-3*) Under the same condition in (B-3), the stronger relation

$$\lim_{t \to \tau} \sup\{ | \int_0^l [g(t,s,\phi(s)) - g(\tau,s,\phi(s))]ds | : l \in J \text{ and } \phi \in C(J,N) \} = 0$$ holds.

We remark that if $\sup \{ \int_0^t m(t,s)ds : t \in [0,l] \} < \infty$ in (B-2), then the condition (B-2*) is satisfied. our stronger Hypotheses (B-2*) and (B-3*) are needed to establish the continuity of G . In this section, we shall characterize a subspace $G \subset G_0$ such that the equation $P(f,g)$ where $(f,g) \in C \times G$ defines a local generalized semi-flow.

Definition 3.1. A metric space (G,d) is said to be a base space if the following conditions hold :

(1) $G \subset G_0$

(2) If $\{ g_n, g \} \subset G$ and $d(g_n,g) \to 0$ as $n \to \infty$, then for any two compact intervals J_1 and J_2 and each natural number N ,

$$\sup \{ | \int_0^l [g_n(t,s,r(s)) - g(t,s,r(s))]ds | : l \in J_1,\ t \in J_2 \text{ and } r \in C(J_1,N)\} \longrightarrow 0 \text{ as } n \to \infty .$$

(3) G is translation invariant : the mapping $(\tau,g) \to g_\tau$ is a continuous mapping of $R^+ \times G$ into G .

Then we have the following fundamental lemma which shows the continuity of G.

Lemma 3.1. *The operator* $G : R^+ \times C \times G \longrightarrow C$ *defined by* $G(\tau,\phi,g)(t)$

$$= \int_0^\tau g(\tau + t, s, \phi(s))ds \quad \textit{is continuous.}$$

Proof. For any $(\tau,\phi,g) \in R^+ \times C \times G$, by (B-1) and (B-2) the integral $G(\tau,\phi,g)(t)$ is possible and by (B-3) $G(\tau,\phi,g)(t)$ is continuous on $[0,\infty)$, that is, $G(\tau,\phi,g) \in C$. To show the continuity, assume that $(\tau_n,\phi_n,g_n) \longrightarrow (\tau,\phi,g)$ as $n \to \infty$. We have

$$|G(\tau_n,\phi_n,g_n)(t) - G(\tau,\phi,g)(t)|$$
$$\leq |G(\tau_n,\phi_n,g_n)(t) - G(\tau_n,\phi_n,g)(t)| + |G(\tau_n,\phi_n,g)(t) - G(\tau,\phi_n,g)(t)|$$
$$+ |G(\tau,\phi_n,g)(t) - G(\tau,\phi,g)(t)| = I_1 + I_2 + I_3 .$$

Since $\tau_n \longrightarrow \tau$ and $\phi_n \longrightarrow \phi$ as $n \to \infty$, for any compact interval J there exist two compact intervals J_1 and J_2 such that $\tau_n \in J_1$ and $t + \tau_n \in J_2$ for all n and $t \in J$, and $|\phi_n(t)| \leq N$ for some N and for all n and $t \in J_1$. Then $I_1 \leq \sup \{ \ |\int_0^l [g_n(t,s,\phi(s)) - g(t,s,\phi(s))]ds| : l \in J_1,\ t \in J_2 \text{ and } \phi \in C(J_1,N)\}$ and since $g_n \longrightarrow g$, we have by (3.1-(2)) that $I_1 \longrightarrow 0$ as $n \to \infty$ uniformly in $t \in J$.

$$I_2 \leq \sup \{ \ |\int_0^l [g(t + \tau_n,s,\phi(s)) - g(t + \tau,s,\phi(s))]ds| : l \in J_1 \text{ and } \phi \in C(J_1,N)\}$$
$$+ \int_{\tau_n}^{\tau} m(t + \tau, s)ds = I_{2,1} + I_{2,2} ,$$
where $m(t + \tau, \cdot)$ is the measurable function defined in (B-2) corresponding to $t + \tau \in J_2$ and N. By (B-3*) and (3.1-(1)), $I_{2,1} \longrightarrow 0$ and by (B-2), $I_{2,2} \longrightarrow 0$, and hence $I_2 \longrightarrow 0$ as $n \to \infty$. That $I_3 \longrightarrow 0$ is a consequence of Lebesgue theorem. To show that this convergence is uniform in $t \in J$, it is sufficient to assure the compactness of $\{ G(\tau_n,\phi_n,g)(\cdot) \}$ in $C(J)$. The equi-continuity follows from (B-3*) and the boundedness of the set $\{ G(\tau_n,\phi_n,g)(t) \}$ for each $t \in J$ follows from (B-2*). Then by Ascoli-Arzela's Theorem, $\{ G(\tau_n,\phi_n,g)(\cdot)\}$ is compact in $C(J)$. Therefore the convergence $G(\tau_n,\phi_n,g_n) \longrightarrow G(\tau,\phi,g)$ in C as $n \to \infty$ is proved.

As examples of the base space, we can give :

Example 3.1. A metric space (G_0, d_{G_0}) is a base space. Here the metric
$d_{G_0}(g,h) = \sum_{i,j=1}^{\infty} 2^{-(i+j)} \min(1, d_{ij}(g,h))$, where
$d_{ij}(g,h) = \sup\{\int_0^t |g(t,s,x) - h(t,s,x)|ds : |x| \le i \text{ and } t \in [0,j]\}$.
This metric topology is introduced by Miller [6].

Example 3.2. Let G be the set of all functions g such that $g(t,s,x)$ is continuous in $(t,s,x) \in R^+ \times R^+ \times R^n$ and $g(t,s,x) = 0$ for $s \ge t$. We define the metric d_G by

$$d_G(g,h) = \sum_{n=1}^{\infty} 2^{-n} \min(1, \max\{|g(t,s,x) - h(t,s,x)| : |t|, |s|, |x| \le n\}).$$

Then the space (G, d_G) is a base space.

In the following we assume that the space G is a base space. We see that

(3.2) for each compact interval $J \subset R^+$, each natural number N and each $\tau \in R^+$,

$$\lim_{t\to\tau} \sup\{ |\int_J [g_k(t,s,\phi(s)) - g_k(\tau,s,\phi(s))]ds| : \phi \in C(J,N) \text{ and } k=1,2,\cdots \}$$
$$= 0 \text{ if } g_k \longrightarrow g \text{ in } G .$$

This is a consequence from (B-2*) and (3.1-(2)) by a standard argument of uniformity on compact sets. Moreover the metric topology generated by d is a jointly continuous topology in the sense of Artstein [1]. That is, for any fixed $t > 0$ the mapping $(g,\phi) \in G \times C[0,t] \longrightarrow \int_0^t g(t,s,\phi(s))ds \in R^n$ is continuous in the product topology of G and $C[0,t]$. This is an easy consequence from Lemma 1 (put $t = 0$). Then we have by [1] the following continuous dependence result on $C \times G$.

Lemma 3.2. *Let G be a base space. Suppose that the sequence (f_k, g_k) converges to (f,g) in the product topology on $C \times G$. Then the followings hold . Let $x_k(t)$ be a maximally defined solution of $P(f_k, g_k)$. Then there exist a maximally defined solution $x(t)$ of $P(f,g)$ with domain $[0, \alpha(f,g,x))$,* *and a subsequence* $\{ x_m(t) \} \subset \{ x_k(t) \}$ *such that* $x_m(t)$ *converges to* $x(t)$ *uniformly on each compact subinterval of* $[0,\alpha(f,g,x))$. *In particular, if* $[0, \alpha(f_m,g_m,x_m))$ *is the domain of* $x_m(t)$, *and if* $0 < d < \alpha(f,g,x)$ *then for* m *large enough we have* $d \le \alpha(f_m,g_m,x_m)$.

Here we note that it is sufficient to obtain Lemma 3.2 that the weaker condition (3.2) than that of Artstein's (G-3) is satisfied. We gave in [12] by Lemma 3.2 , the following Lemma.

Lemma 3.3. *For any* $(f,g) \in C \times G$, *we can find a solution* x *of* $P(f,g)$ *such that* $\alpha(f,g) = \alpha(f,g,x)$. *Moreover the function* $\alpha : C \times G \longrightarrow R^+ \cup \{\infty\}$ *is a lower semi-continuous function, and hence the set*

$D = \{ (t,f,g) : f \in C , g \in G$ and $t \in [0,\alpha(f,g)) \}$ *is open in* $R^+ \times C \times G$

4. Flows defined by Integral Equations

To construct a local generalized semi-flow, we must define the operator $T(f,g) : [0,\alpha(f,g)) \times C[0,\alpha(f,g)) \longrightarrow C$ for each $(f,g) \in C \times G$ by

$$(T(f,g)(t,\phi))(\theta) = f(t + \theta) + \int_0^t g(t + \theta,s,\phi(s))ds \text{ , where } \phi \in C[0,\alpha(f,g)).$$

Let $S(f,g)$ be the solution family of $P(f,g)$ in $C[0,\alpha(f,g))$, and for simplicity we denote the image $T(f,g)(t,S(f,g))$ by $T(t,f,g)$.

Our purpose in this section is to show that the set-valued mapping $\pi(t,f,g) = (T(t,f,g),g_t) \in 2^C \times G$ which is defined for all $(t,f,g) \in D$ is a Local generalized flow. Let $X = C \times G$ and a metric d_X is a sum of the metrics of each spaces C and G. Clearly $\pi(D) \subset 2^C \times G \subset 2^X$.

Lemma 4.1. *For any compact interval* $J \subset [0,\alpha(f,g))$ *and* $(f,g) \in C \times G$, *the set* $T(J,f,g) = \cup \{ T(t,f,g) : t \in J \}$ *is compact and connected in* C . *Especially* $T(t,f,g)$ *is also compact and connected in* C *for every* $(t,f,g) \in D$.

Proof. By Lemma 3.1, the operator $T(f,g)$ is continuous. Since the product set $J \times S(f,g)$ is compact and connected in $[0,\alpha(f,g)) \times C[0,\alpha(f,g))$ (See Section 3. (iii)), then the continuous image $T(J,f,g)$ is also compact and connected in C.

By considering the sets $T(t,f,g)$ as a function of t and a function of (t,f,g), we can define two operators

$T(\cdot,f,g) : [0,\alpha(f,g)) \longrightarrow 2^C$ and $T : D \longrightarrow 2^C$.

Lemma 4.2. *The operator* $T(\cdot,f,g) : [0,\alpha(f,g)) \longrightarrow 2^C$ *is continuous for each* $(f,g) \in C \times G$.

Proof. Let $t\in[0,\alpha(f,g))$ be fixed. First we shall show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $t + \delta< \alpha(f,g)$ and $T(t + s,f,g) \subset V(T(t,f,g),\varepsilon)$ for every $|s| < \delta$, where V is the neighboring set defined in Section 2 corresponding to the metric d_C . Assume to the contrary that there exists a sequence $\{ x_n(\cdot)\}$ in C and a sequence $\{ t_n \}$ tending to t as $n \to \infty$ such that $x_n(\cdot) \in T(t_n,f,g)$ and $x_n(\cdot) \notin V(T(t,f,g),\varepsilon_0)$ for some $\varepsilon_0 > 0$, where $x_n(\cdot)$ is given by $x_n(\cdot) = f(t_n+ \cdot) + \int_0^{t_n} g(T_n + \cdot,s,\phi_n(s))ds$ for some $\phi_n(\cdot) \in S(f,g)$. Since $[t - \delta,t + \delta]$ is a compact interval , Lemma 4.1 implies that $T([t-\delta,t+\delta],f,g)$ is compact in C , and hence $\{ x_n(\cdot)\}$ has a subsequence $\{ x_{nk}(\cdot)\}$ converging to some $x(\cdot) \notin V(T(t,f,g),\varepsilon_0)$. Since the solution space $S(f,g)$ is compact in $C[0,\alpha(f,g))$, then the solution subsequence $\{ \phi_{nkj}(\cdot)\}$ (for brevity we write $\{ \phi_{nj}(\cdot) \}$) such that $\phi_{nj}(\cdot)$ converges to some solution $\phi(\cdot)$ of $P(f,g)$ in $[0,\alpha(f,g))$. Hence by the continuity of $T(f,g)$, we can verify that

$$x(\cdot) = f(t + \cdot) + \int_0^t g(t + \cdot,s,\phi(s))ds \quad \text{for} \quad \phi(\cdot) \in S(f,g) \text{ , i.e.,}$$

$x(\cdot) \in T(t,f,g)$, a contradiction.

In order to verify the continuity of $T(t,f,g)$ at $t \in [0,\alpha(f,g))$, by the definition of $\rho(T(t,f,g),T(t + s,f,g))$ we must show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $t + \delta < \alpha(f,g)$ and $T(t,f,g) \subset V(T(t + s,f,g),\varepsilon)$

for every $|s| < \delta$. Also in this case, assume the contrary. Then we can suppose that there exist a sequence $\{ x_n(\cdot) \}$ in C and a sequence $\{ t_n \}$ tending to t as $n \to \infty$ such that $x_n(\cdot) \in T(t_n,f,g)$ and $x_n(\cdot) \notin V(T(t_n,f,g), \varepsilon_0)$ for some $\varepsilon_0 > 0$. Here we can assume without loss of generality that $x_n(\cdot)$ is converging to a limit $x(\cdot) \in T(t,f,g)$ in C . For any element $y(\cdot) = y(\cdot;t,f,g) \in T(t,f,g)$, we define $y_n(\cdot)$ by $y_n(\cdot) = y(\cdot;t_n,f,g) \in T(t_n,f,g)$, then by Lemma 3.1 we have $\lim_{n\to\infty} d_C(y(\cdot),y_n(\cdot)) = 0$. Therefore

$$
\begin{aligned}
d_C(y(\cdot),x(\cdot)) &\leq \lim_{n\to\infty} |d_C(y(\cdot),y_n(\cdot)) - d_C(y_n(\cdot),x_n(\cdot))| \\
&\leq \lim_{n\to\infty} d_C(y_n(\cdot),x_n(\cdot)) \geq \varepsilon_0 .
\end{aligned}
$$

This implies that $x(\cdot) \notin V(T(t,f,g),\varepsilon_0)$, a contradiction. This completes the Lemmma.

Lemma 4.3. *The mapping* $T : D \longrightarrow 2^C$ *is upper semi-continuous.*

Proof. In order to show T is continuous, let $(t,f,g) \in D$ and $W((t,f,g),\delta) = \{(t',f',g') : |t - t'| < \delta,\ d_C(f,f') < \delta$ and $d_G(g,g') < \delta \}$. since D is open in $R^+ \times C \times G$ (Lemma 3.3), there exists a $\delta_0 > 0$ such that $W((t,f,g),\delta) \subset D$ for all $\delta \in (0, \delta_0)$. We shall show that for any $\varepsilon > 0$ we can take $\delta > 0$ such that $T(W(t,f,g),\delta)) \subset V(T(t,f,g),\varepsilon)$. If this inclusion were false, then there would exist an $\varepsilon_0 > 0$, a sequence $\{(t_n,f_n,g_n)\} \subset W((t,f,g),\delta_0)$ tending to (t,f,g) as $n \to \infty$ and a sequence $\{ x_n(\cdot) \}$ in C such that $x_n(\cdot) \in T(t_n,f_n, g_n)$ and $x_n(\cdot) \notin V(T(t,f,g),\varepsilon_0)$ for every n . Since the closure of the set $\{ (t_n,f_n,g_n) \}$ is compact, by Lemma 3.1 $T(\cup\{ (t_n,f_n,g_n) : n = 1,2,\cdots \})$ has a compact closure in C . Then we can assume without loss of generality that $x_n(\cdot) \longrightarrow x(\cdot) \in V(T(t,f,g),\varepsilon_0)$ in C as $n \to \infty$ for some $x(\cdot) \in C$. Here $x_n(\cdot)$ has the form

$$
x_n(\cdot) = f_n(t_n + \cdot) + \int_0^{t_n} g_n(t_n + \cdot, s, \phi_n(s))ds , \quad \text{where } \phi_n(\cdot) \in S(f_n,g_n) .
$$

We have from Lemma 3.2 that there exist a subsequence $\{ nk \} \subset \{n\}$ and a solution $\phi(\cdot)$ of $P(f,g)$ such that $\lim_{k\to\infty} t_{nk} = t \in [0,\alpha(f,g))$ and $\lim_{k\to\infty} \phi_{nk}(\cdot) = \phi(\cdot)$ in $C[0,t]$. It is obvious that $f_n(t_n + \cdot) \longrightarrow f(t + \cdot)$ in C as $(t_n,f_n) \longrightarrow (t,f)$. Hence by Lemma 3.1 , we have for some $\phi(\cdot) \in S(f,g)$, $x(\cdot) = f(t + \cdot) + \int_0^t g(t +\cdot,s,\phi(s))ds$, i.e., $x(\cdot) \in T(t,f,g)$. This contradiction proves this Lemma.

Main Theorem. *Let* G *be a base space,* $X = C \times G$ *and* $D = \{ (t,f,g) : f \in C, g \in G$ and $t \in I_{(f,g)} \}$ *, where* $I_{(f,g)} = [0,\alpha(f,g))$ *. Then the mapping* π *defined by* $\pi(t,f,g) = (T(t,f,g),g_t) \subset 2^X$ *determines a local generalized semi-flow on* X .

Proof. Since G is translation invariant, by Lemma 4.3 the mapping π maps D into 2^X . For each $(t,f,g) \in D$ by Lemmma 4.1 $\pi(t,f,g) = (T(t,f,g), g_t)$ is nonempty and compact-connected. Moreover $\pi(0,f,g) = \{ (f,g)\}$. This proves (1) of Definition 2.1. For any fixed $t \in I_{(f,g)}$ and any $s \in \cap \{ I_{(f',g')} : (f',g') \in \pi(t,f,g) \}$, we have for any $x(\cdot) \in T(t + s,f,g)$ that

$$
\begin{aligned}
x(\cdot) &= f(t + s + \cdot) + \int_0^{t+s} g(t + s + \cdot,\xi,\phi(\xi))d\xi , \\
&= f_{t+s}(\cdot) + \int_0^s g(t + s + \cdot,\xi,\phi(\xi))d\xi + \int_s^{t+s} g(t + s + \cdot, \xi ,\phi(\xi))d\xi \\
&= (f_s(\cdot) + \int_0^s g(s + \cdot,\xi,\phi(\xi))d\xi)_t + \int_0^t g_s(t + \cdot,\xi,\phi_s(\xi))d\xi,
\end{aligned}
$$

for some $\phi(\cdot) \in S(f,g)$. Here by the relation

$$
\begin{aligned}
\phi_s(\tau) &= \phi(s + \tau) = f(s + \tau) + \int_0^{s+\tau} g(s + \tau,\tau,\phi(\tau))d\tau \\
&= \{ f_s(\tau) + \int_0^s g(s +\tau,\tau,\phi(\tau))d\tau \} + \int_0^\tau g_s(\tau,\xi,\phi_s(\xi))d\xi
\end{aligned}
$$

for $\tau \in [0,\alpha(f,g) - t)$, we have $\phi_s(\cdot) \in S(T(s,f,g),g_s)$. Then $x(\cdot) \in T(t,T(s,f,g),g_s)$, i.e., $T(t + s,f,g) \subset T(t,T(s,f,g),g_s)$. The proof of the inclusion $T(t,T(s,f,g),g_s) \supset T(t + s,f,g)$ is similar. This proves (2) .

To verify (3) suppose $\alpha(f,g) < \infty$, then we can choose a solution $\phi(\cdot) \in S(f,g)$ by Lemma 3.3 such that $\alpha(f,g) = \alpha(f,g,x)$ and hence $\limsup_{t\uparrow\alpha(f,g)} |\phi(t)| = \infty$. Then there exists a sequence $\{ t_n \}$ tending to t such that $|\phi(t_n) - \phi(t_m)| \geq 1$ if $n \neq m$. Let $\{ x_n(\cdot) \} \subset \cup \{ \pi(t,p) : t \in [0, \alpha_p) \}$ be the sequence defined by

$$x_n(\cdot) = f(t_n + \cdot) + \int_0^{t_n} g(t_n + \cdot, s, \phi(s))ds , n = 1,2,\cdots\cdots .$$

Since $x_n(0) = \phi(t_n)$, we have $d_C(x_n(\cdot),x_m(\cdot)) = \sum_{k=1}^{\infty} 2^{-k}(1, \max \{ |x_n(t)-x_m(t)| : t \in [0,k] \}) = \sum_{k=1}^{\infty} 2^{-k} = 1$ for all n, m = 1,2,···· ($n \neq m$). This shows (3). The condition (4) follows from Lemma 3.3. Since the projection of the mapping π to G is the translation mapping and this mapping is continuous by the definition of the base space G, it is sufficient to verify the conditions (5) and (6) that T (the projection of π to C) satisfies (5) and (6). And these are already proved in Lemma 4.2 and Lemma 4.3. This completes the Theorem.

Finally we remark that our proof of this theorem is direct and simpler than that of [6] with uniqueness condition.

References

[1] Z. Artstein: Continuous dependence of solutions of Volterra integral equations, SIAM J. Math. Anal. 6 (1975), 446-456.

[2] W. G. Kelly: A Kneser theorem for Volterra integral equations, Proc. Amer. Math. Soc., 40 (1973), 183-190.

[3] J. P. LaSalle: Invariance principles and stability theory for nonautonomous systems, Proc. Greek Math. Soc. Carathēodory Symposium, Athenes, Sept. (1973), 397-408.

[4] J. P. LaSalle: Stability theory and invariance principles, Dynamical Systems, An International Symposium, Vol.I, Academic press,(1976) 211-222.

[5] R. K. Miller: Almost periodic differential equations as dynamical systems

with applications to the existence of a. p. solutions, J. Diff. Eqns. 1 (1965),337-345.

[6] R. K. Miller: Non-linear Volterra integral equations, Mathematical lecture note series, Benjamin, (1971).

[7] R. K. Miller: Some fundamental theory of Volterra integral equations, International Conference on D. E., Univ. of Southern California, Sept. (1974).

[8] R. K. Miller and G. R. Sell: Volterra integral equations and toplogical dynamics, Mem. Amer. Math. Soc. 102 (1970).

[9] R. K. Miller and G. R. Sell: Topological dynamics and its relation to integral equations and nonautonomous systems, Dynamical Systems, An International Symposium, Vol.I, Academic press, (1976) 223-249.

[10] S. Nakagiri and H. Murakami: Approximate solutions for some non-linear Volterra integral equation, Proc. Jap. Acad. 50 (1974) 212-217.

[11] S. Nakagiri and H. Murakami: Kneser's property of solution families of non-linear Volterra integral equations, Proc. Jap. Acad. 50(1974) 296-300.

[12] S. Nakagiri and H. Murakami: Continuities of solutions of non-linear Volterra integral equations, Mathematica Japonicae 20 (1975) 145-160.

[13] E. Roxin: Stability in general control systems, J. Diff. Eqns. 1 (1965), 115-150.

[14] E. Roxin: On generalized dynamical systems defined by contingent equations, J. Diff. Eqns. 1 (1965), 188-205.

[15] P. Seibert: Stability under perturbations in generalized dynamical systems, International Symposium on Nonlinear differential equations and nonlinear mechanics, Academic press (1963) 463-473.

[16] G. R. Sell: Nonautonomous differential equations and topological dynamics I and II, Trans. Amer. Math. Soc. 127 (1967), 241-283.

[17] G. R. Sell: Lectures on topological dynamics and differential equations, Van Nostrand Reinhold, London, (1971).

[18] G. R. Sell: Differential equations without uniqueness and classical topological dynamics, J. Diff. Eqns. 14 (1973) 388-396.

[19] T. Ura: Sur le Courant extérier a une région invariante, prolongements d'une caractéristique et parde de stabilité, Funkcial. Ekvac. 2 (1959), 143-200.

[20] T. Ura: Isomorphisms and local characterization of local dynamical systems, Funkcial. Ekvac. 12 (1969), 99-122.