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# NONWANDERING SET AND MINIMAL SETS

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## 1. INTRODUCTION.

Let there be given a dynamical system $(X, R, \pi)$ defined by usual axioms where the phase space X is supposed to be compact and metric. Let W be the set of all the wandering points of this dynamical system. Then $N = X - W$, the set of all the nonwandering points, is called the nonwandering set. As is obvious from the definition, W is open and N is closed and they are both invariant sets.

Since a wandering point can never be a limit point of any orbit, every orbit in W has its (both positive and negative) limit sets in N. So, if we enclose N by a neighbourhood U arbitrarily small, then, as $t \to \infty$, every orbit in W eventually comes into U and stays in U thereafter. The same situation takes place as $t \to -\infty$. So, in a sense, the study of the behaviour of orbits in W can be regarded as a local problem around N.

Suppose that this problem has been solved. Then we have to investigate the inner structure of N. For that purpose, we consider the restriction of our dynamical system onto N which we denote by $(N, R, \pi)$. Then we split N into the wandering part $W_1$ and the nonwandering part $N_1$ as before. Evidently the same reasoning works in N and the study of the orbits in $W_1$ is reduced to the local problem around $N_1$.

Repeating this procedure, we finally come to the set $N_\alpha$ such that $(N_\alpha, R, \pi)$ no longer has any wandering points. Here $\alpha$ is a certain ordinal which is, in general, transfinite and $N_\alpha$ is called the <u>set of central motions</u>. Our splitting process terminates here and we cannot separate the local-theoretic part any more. At this step, there comes up the essentially global part of the study.

This scheme of studying the dynamical systems was established by G. D. Birkhoff. To carry out our study along this scheme, we have to solve the following three problems.

The first problem should of course be

1) the splitting of the phase space into W and N.

Next we have to investigate the behaviour of orbits around N. This amounts to be

2) the investigation of the behaviour of orbits in the vicinity of a closed invariant set.

The final step is the study of the orbits in the set of central motions, namely

3) the global study of dynamical systems with no wandering points.

The second problem has been discussed by many mathematicians and we have a rich stock of results. The third problem is a more difficult one and we are still far from its solution. However we already have several good results - for example, many remarkable theorems concerning the minimal flow, the almost periodic minimal flow or the measure preserving flow. Compared with those, the results concerning the first problem are rather poor. This problem seems to be the most neglected one among these three. It is this first problem that we are now going to discuss in this note and we shall show that there exists a certain very intimate connection between N and the minimal sets, especially saddle minimal sets.

## 2. NOTATION AND PRELIMINARIES.

For any $x$ $X$, we denote by:

$C(x)$, the orbit through $x$;

$C^+(x)$, the positive semiorbit from $x$;

$C^-(x)$, the negative semiorbit from $x$;

$L^+(x)$, the positive limit set of $x$;

$L^-(x)$, the negative lomit set of $x$;

$J^+(x)$, the positive prolongational limit set of $x$;

$J^-(x)$, the negative prolongational limit set of $x$.

The concept of a saddle set, which was first introduced by Ura [1], plays an important role in the following arguments. Since the concept might not be very familiar, we shall give the definition.

DEFINITION. A closed invariant set $M$ is called a <u>saddle set</u> if $M$ has a neighbourhood $U$ such that every neighbourhood of $M$ contains a point $x$ with

$$C^+(x) \not\subset U, \quad C^-(x) \not\subset U.$$

In other words, there exists a sequence $\{x_n\}$ tending to a point of $M$ such that

$$C^+(x_n) \not\subset U, \quad C^-(x_n) \not\subset U.$$

Otherwise it is called a <u>nonsaddle set</u>.

A nonsaddle compact invariant set has a following very remarkable property.

THEOREM A. Let $M$ be a nonsaddle compact invariant set. Then $x \notin M$ and $L^+(x) \cap M \neq \emptyset$ imply $M \supset J^+(x)$ $(\supset L^+(x))$, and $x \notin M$ and $L^-(x) \cap M \neq \emptyset$ imply $M \supset J^-(x)$ $(\supset L^-(x))$. [2]

## 3. CHARACTERIZATION OF NONWANDERING SET VIA PROLONGATIONAL LIMIT SET.

Let $S_1$, $S_2$, ... be the saddle minimal sets and $F_1$, $F_2$, ... be the nonsaddle minimal sets of $(X, R, \pi)$. We first prove the following

PROPOSITION 1. $N \subset (\bigcup F_k) \cup (\bigcup (J^+(S_k) \cap J^-(S_k)))$.

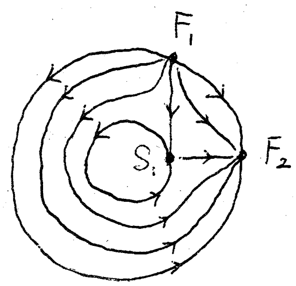
PROOF. Suppose that $x \in N - \bigcup F_k$. Since $x$ is a nonwandering point, $x \in J^+(x) \cap J^-(x)$. Since $J^+(x) \cap J^-(x)$ is a closed, hence in our case compact invariant set, $L^+(x)$ is nonempty and compact. Let $M$ be a minimal set in $L^+(x)$. Then $J^+(x) \supset M$ and $J^-(x) \supset M$ imply $x \in J^-(M)$ and $x \in J^+(M)$ respectively. So we have only to prove that $M$ is a saddle set.

Suppose that $M$ is a nonsaddle set. Then $x \notin M$ by assumption. Then $L^+(x) \cap M \neq \emptyset$ implies $M \supset J^+(x) \ni x$ by Theorem A, which is a contradiction.

In general the set

$$(\bigcup F_k) \cup (\bigcup (J^+(S_k) \cap J^-(S_k)))$$

is larger than $N$. This can be shown by an example illustrated below.




Here the phase space is a ring-shaped closed domain, $F_1$, $F_2$ are the nonsaddle minimal sets and $S_1$ is the saddle minimal set. All the points outside minimal sets, namely all the noncritical points are wandering. But $J^+(S_1) \cap J^-(S_1)$ is the inner boundary of the phase space.

To obtain more precise relation between $N$ and the minimal sets, we introduce the following concept.

DEFINITION. For any $x \in X$, the set $E(x)$ is defined as follows: $y \in E(x)$ if and only if there exist a sequence $\{x_n\}$ in $X$ and two sequences of real numbers $\{t_n\}$ and $\{s_n\}$ such that

1) $x_n \to x$,

2) $t_n \to \infty$, $s_n \to -\infty$,

3) $\pi(x_n, t_n) \to y$, $\pi(x_n, s_n) \to y$.

It is obvious that $E(x) \subset J^{+}(x) \cap J^{-}(x)$, and, in general, $E(x)$ is actually smaller than $J^{+}(x) \cap J^{-}(x)$. In the example given above, $E(S_1) = S_1$ and it is smaller than $J^{+}(S_1) \cap J^{-}(S_1)$.

Then we have

PROPOSITION 2. $y \in X$ is nonwandering if and only if $y \in E(x)$ for some $x \in X$.

PROOF. 1) Suppose that $y \in E(x)$ for some $x \in X$. Then, by definition, there exist $\{x_n\} \subset X$, $\{t_n\}$, $\{s_n\} \subset R$ such that

$$x_n \to x, \quad t_n \to \infty, \quad s_n \to -\infty,$$
$$\pi(x_n, t_n) \to y, \quad \pi(x_n, s_n) \to y.$$

Put $y_n = \pi(x_n, s_n)$. Then $\pi(x_n, t_n) = \pi(y_n, t_n - s_n) \to y$. Since $y_n \to y$ and $t_n - s_n \to \infty$, this shows that $y \in J^{+}(y)$. Hence $y$ is a nonwandering point.

2) If $y$ is a nonwandering point, then $y \in J^{+}(y)$. Hence there exist $\{y_n\} \subset X$ and $\{t_n\} \subset R$ such that

$$y_n \to y, \quad t_n \to \infty \quad \text{and} \quad \pi(y_n, t_n) \to y.$$

Put $x_n = \pi(y_n, t_n/2)$. Then, without loss of generality, we may assume that $x_n \to x \in X$. Since

$$\pi(x_n, t_n/2) = \pi(y_n, t_n) \to y, \quad t_n/2 \to \infty,$$
$$\pi(x_n, -t_n/2) = y_n \to y, \quad -t_n/2 \to -\infty,$$

we have $y \in E(x)$.

4. FUNDAMENTAL PROPERTIES OF $E(x)$.

PROPOSITION 3. $E(x)$ is a closed invariant set.

Invariance is almost obvious from the definition. Closedness can be proved by a routine argument based on the diagonal process. So we shall omit the proof.

PROPOSITION 4. $E(x) = E(\pi(x, t))$.

The proof of this proposition is also omitted since it is an easy consequence of the definition.

PROPOSITION 5. 1) If $M$ is a minimal set, then $E(M) \supset M$.

2) If $M$ is a nonsaddle minimal set, then $E(M) = M$.

PROOF. 1) Let $x$ and $y$ be two points of $M$. Since $M$ is minimal, $y \in L^+(x) = L^-(x) = M$. Therefore there exist two sequences $\{t_n\}$ and $\{s_n\}$ in $R$ such that

$$t_n \to \infty, \quad s_n \to -\infty, \quad \pi(x, t_n) \to y, \quad \pi(x, s_n) \to y.$$

If we notice that $\{x\}$ can be regarded as a sequence tending to $x$, the above relation shows that $y \in E(x)$. This being valid for any $x$ and $y$ in $M$, we have $M \subset E(M)$.

2) To prove the second part, assume that $E(M) \neq M$ to derive the contradiction. Let $y$ be a point in $E(M) - M$. Since $y \in E(M)$, $y \in E(x)$ for some $x$ in $M$. Therefore there exist $\{x_n\} \subset X$ and $\{s_n\}$, $\{t_n\} \subset R$ such that

$$x_n \to x, \ t_n \to \infty, \ s_n \to -\infty,$$
$$\pi(x_n, t_n) \to y, \ \pi(x_n, s_n) \to y.$$

Since $y \notin M$, there exists a neighbourhood $U$ of $M$ which does not contain $y$. Then, for sufficiently large $n$, we have

$$\pi(x_n, t_n) \notin U, \quad \pi(x_n, s_n) \notin U.$$

Hence $C^+(x_n) \not\subset U$ and $C^-(x_n) \not\subset U$. As $x_n \to x \in M$, this means that $M$ is a saddle set contrary to the assumption.

PROPOSITION 6. If $z \in L^+(x)$ or $z \in L^-(x)$, then $E(x) \subset E(z)$.

PROOF. If $z \in L^+(x)$, there exists $\{c_n\} \subset R$ such that

$$c_n \to \infty, \quad \pi(x, c_n) \to z.$$

Let $y$ be a point of $E(x)$. Then, by definition, there exist $\{x_n\} \subset X$ and $\{t_n\}$, $\{s_n\} \subset R$ such that

$$x_n \to x, \quad t_n \to \infty, \quad s_n \to -\infty,$$
$$\pi(x_n, t_n) \to y, \quad \pi(x_n, s_n) \to y.$$

Without loss of generality, we may assume that

$$t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots \quad \text{and} \quad t_n - c_n \to \infty .$$

For each fixed n, $\pi(x_k, c_n) \to \pi(x, c_n)$ as $k \to \infty$ . So there exists a number $k(n)$ such that

$$d(\pi(x_k, c_n), \pi(x, c_n)) < 1/n \quad \text{for } k \geq k(n)$$

where $d( \ , \ )$ denotes a metric in X. We may also assume that

$$k(n) \geq n, \quad k(1) < k(2) < \cdots < k(n) < k(n+1) < \cdots .$$

Then

$$d(\pi(x_{k(n)}, c_n), z) \leqq d(\pi(x_{k(n)}, c_n), \pi(x, c_n)) + d(\pi(x, c_n), z)$$
$$\leqq 1/n + d(\pi(x, c_n), z).$$

Thus if we put

$$\pi(x_{k(n)}, c_n) = z_n,$$

we have $z_n \to z$ and

$$\pi(z_n, t_{k(n)} - c_n) = \pi(x_{k(n)}, t_{k(n)}) \to y.$$

Since $k(n) \geq n$ and $t_1 < t_2 < \cdots$ ,

$$t_{k(n)} - c_n \geqq t_n - c_n.$$

Then since $t_n - c_n \to \infty$ as was assumed above, we have

$$t_{k(n)} - c_n \to \infty .$$

On the other hand,

$$\pi(z_n, s_{k(n)} - c_n) = \pi(x_{k(n)}, s_{k(n)}) \to y$$

and $s_{k(n)} - c_n \to -\infty$ . Hence $y \in E(z)$ and we have

$$E(x) \subset E(z).$$

From this proposition, we immediately have

PROPOSITION 7. If M is a minimal set, then $E(M) = E(x)$ for every x in M .

PROOF. Since $E(x) \subset E(M)$ is obvious, we have only to show that $E(x) \supset E(M)$.

Let y be an arbitrary point of M. As M is minimal, $x \in L^+(y)$. Hence $E(x) \supset E(y)$ by Proposition 6. y being arbitrary, we have $E(x) \supset E(M)$.

4. MAIN THEOREM.

Let $y \in N$, i.e. $y$ be a nonwandering point. Then, by Proposition 2, $y$ belongs to $E(x)$ for some $x \in X$. Let $M$ be a minimal set in $L^+(x)$. Then $E(x) \subset E(M)$ by Proposition 6. Hence $y \in E(M)$ for some minimal set $M$.

Conversely, if $y \in E(M)$ for some minimal set $M$, then $y$ is a nonwandering point by Proposition 2. Consequently

$$N = (\bigcup E(F_k)) \bigcup (\bigcup E(S_k)).$$

However $E(F_k) = F_k$ by Proposition 5. So we finally get the following theorem.

THEOREM. $N = (\bigcup F_k) \bigcup (\bigcup E(S_k))$.

REFERENCES

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