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On a Minimal Flow

By

Ippei ISHII
(Keio University)

1. Preliminaries

Let (Y, ρ_t) or simply ρ_t be a flow on a compact metric space Y ; i.e. ρ_t is a homeomorphism for each real number t and $\rho_{t+s} = \rho_t \circ \rho_s$ for any two real numbers t and s . If $A \subset Y$ and $J \subset R$, we wrire $A \cdot J$ for $\{\rho_t(y) \mid t \in J,\ y \in A\}$. A subset $N \subset Y$ is said to be a <u>minimal set</u> if $\overline{y \cdot R} = N$ for any $y \in N$, especially if Y is the minimal set, then we call (Y, ρ_t) a <u>minimal flow</u> .

DEFINITION 1. A subset $\Sigma \subset Y$ is said to be a <u>local section</u> of the flow ρ_t if it satisfies :

(i) $h : \overline{\Sigma} \times (-\mu, \mu) \to \overline{\Sigma} \cdot (-\mu, \mu)$ defined by $h(y, t) = \rho_t(y)$ is a homeomorphism for some $\mu > 0$.

(ii) $\Sigma \cdot J$ is open for any open $J \subset R$.

Moreover if Σ is compact, then we call it a <u>global section</u>.

LEMMA 1. (see [1]) Let (Y, ρ_t) be a minimal flow and $S = y_0 \cdot Z$. If $\overline{S} \neq Y$, then $\overline{S}$ is a global section of (Y, ρ_t) .

LEMMA 2. (see [2]) Let (Y, ρ_t) be a minimal flow and Σ be a local section. Then for each y Y there exists a sequence $\{t_j\}$ of reals such that $\delta_1 < t_{j+1} - t_j < \delta_2$ for some positive numbers δ_1, δ_2 and $\rho_t(y) \in \Sigma$ iff $t = t_j$ for some j.

2. A Flow Associated with a Local Section

Throughout this and the next sections (M, ξ_t) will be a minimal flow on a compact metric space M and Σ will be a local section. Let B be the set of all continuous functions on the real line with the compact-open topology, and η_t be a flow on B defined by

$$\eta_t(g)(s) = g(t + s) \quad (g \in B\ ,\ t, s \in R)\ .$$

Now take a point x_o M, and let $\{t_j\}$ be the sequence for x_o as in LEMMA 2. Then we can construct a uniformly continuous function f which satisfies that $f(t) > \varepsilon > 0$ for all t and that

$$\int_{t_j}^{t_{j+1}} f(t)dt = 1 \quad (\ j = 0, \pm 1, \pm 2, \ldots\ldots\ )\ .$$

Define a flow on $M \times B$ by $\zeta_t(x, g) = (\xi_t(x), \eta_t(g))$ $(x \in M,\ g \in B)$. Since the orbit closure of f is compact, there is a compact minimal set $\tilde{M}$ of the flow ζ_t in $\overline{\{\zeta_t(x_o, f) \mid -\infty < t < \infty\}}$, so $(\tilde{M}, \zeta_t)$ is a minimal flow. By p we denote the natural projection $\tilde{M} \to M$. It is easy to see that $p \circ \zeta_t = \xi_t \circ p$.

Using LEMMA 1, we obtain

LEMMA 3. $\overline{p^{-1}(\Sigma)}$ is a global section of $(\tilde{M}, \zeta_t)$.

And more careful investigation shows that

LEMMA 4. There exists a minimal flow $(\tilde{M}, \zeta_t)$ with the following properties :

(i) $\tilde{M}$ is a compact metric space,

(ii) There is a homomorphism $p : (\tilde{M}, \zeta_t) \to (M, \xi_t)$,

(iii) $\overline{p^{-1}(\Sigma)}$ is a global section of $(\tilde{M}, \zeta_t)$,

(iv) $\overline{p^{-1}(\Sigma)}$ is totally disconnected, i.e. $\dim(\overline{p^{-1}(\Sigma)}) = 0$.

3. Cohomology Theory

Let Y be any topological space and Γ be a presheaf of R-module on Y . Then we denote by $\bar{H}^*(Y)$ the Alexander cohomology of Y with the real coefficients and by $\check{H}^*(Y; \Gamma)$ the Čech cohomology of Y with coefficients Γ .

In the following we shall investigate the first cohomology of $X = M \setminus \Sigma \cdot (0, \mu)$. In this section p denotes the restriction of $p : \tilde{M} \to M$ onto $\tilde{X} = \tilde{M} \setminus \overline{p^{-1}(\Sigma)} \cdot (0, \mu)$ where $(\tilde{M}, \zeta_t)$ is that in LEMMA 4 .

Let Γ_1 and Γ_2 be presheaves on X defined by $\Gamma_1(U) = \bar{H}^0(U)$ and $\Gamma_2(U) = \bar{H}^0(p^{-1}(U))$ respectively, where U is an open subset of X . Then p induces a homomorphism $p^* : \Gamma_1 \to \Gamma_2$. Since p^* is a monomorphism, $0 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 0$ ($\Gamma_3 = \mathrm{Coker}(p^*)$) is an exact sequence. Hence we have

LEMMA 5. There is an exact sequence

$$0 \to \check{H}^0(X; \Gamma_1) \to \check{H}^0(X; \Gamma_2) \to \check{H}^0(X; \Gamma_3) \to \check{H}^1(X; \Gamma_1) \to \check{H}^1(X; \Gamma_2) \to \dots$$

LEMMA 6. $\check{H}^q(X; \Gamma_1) \simeq \bar{H}^q(X)$ and $\check{H}^q(X; \Gamma_2) \simeq \bar{H}^q(\tilde{X})$ for any q .

This lemma can be proved by the next lemma (see [3]).

LEMMA 7. Let $h : Y' \to Y$ be a closed continuous map between paracompaxt Hausdorff spaces. Suppose $\bar{H}^q(h^{-1}(y)) = 0$ for all y Y and $0 < q < n$. Let Γ be the presheaf on Y defined by $\Gamma(U) = \bar{H}^0(h^{-1}(U))$. Then there are isomorphisms $\check{H}^q(Y;\ \Gamma) \approx \bar{H}^q(Y')$ for $q < n$.

Since $\overline{p^{-1}(\Sigma)}$ is a deformation retract of $\tilde{X}$ and totally disconnected, $\bar{H}^1(\tilde{X})$ is trivial. Therefore, combining LEMMA 5 and 6, we get

LEMMA 8. There is an exact sequence

$$\check{H}^0(X;\ \Gamma_2) \to \check{H}^0(X;\ \Gamma_3) \to \bar{H}^1(X) \to 0$$

THEOREM 1. $\bar{H}^1(X) \approx \check{H}^0(X;\ \Gamma_3)/\check{H}^0(X;\ \Gamma_2)$.

4. The Case of 3-Manifolds

In this section let M be a differentiable 3-dimensional manifold and ξ_t be a minimal flow on M generated by a C^1-vector field. Let Σ be a local section homeomorphic to a 2-disk.

NOTATIONS

(a) Let F be a real valued function defined on a subset D of . Then by F we denote a map $D \to M$ defined by $F(x) = \xi_{F(x)}(x)$.

(b)

$T : \overline{\Sigma} \to R$ defined by $T(x) = \inf\{t > 0 \mid \xi_t(x) \in \overline{\Sigma}\}$

$A_0 \subset \partial\Sigma : A_0 = \{x \in \partial\Sigma \mid \hat{T}(x) \in \partial\Sigma\}$

$A_j \subset \partial\Sigma : A_j = \{x \in \partial\Sigma \mid \hat{T}(x) \in A_{j-1}\}$ $(j = 1, 2, \ldots\ldots)$

$A \subset \Sigma : A = \{x \in \Sigma \mid \hat{T}(x) \in A_0\}$

$C \subset \Sigma : C = \{x \in \Sigma \mid \hat{T}(x) \in \partial\Sigma\}$

DEFINITION 2. A local section Σ is said to be <u>regular</u> if A is a finite set and $A_j = \phi$ for $j \geqq 1$.

Using the transversality theorem, we can show the following lemma.

LEMMA 9. There is a regular local section.

In the following we assume that Σ is a regular local section and $A = \{a_1, a_2, \ldots\ldots, a_N\}$. Let Σ' be a local section such that Σ' $\overline{\Sigma}$. Then we can choose a neighborhood U_k Σ of a_k with the following properties :

(1) There are continuous functions $\sigma_{k,j} : U_k \to R$ $(j = 1, 2, 3)$ such that $\hat{\sigma}_{k,j}(U_k) \subset \Sigma'$ $(j = 1, 2)$, $\hat{\sigma}_{k,3}(U_k) \subset \Sigma$ and $\hat{\sigma}_{k,j}(a_k) = \hat{T}^j(a_k)$ $(j = 1, 2, 3)$.

(2) $U_k \cap (C \setminus A)$ has exactly three connected components $\gamma_{k,j}$ $(j = 1, 2, 3)$ such that $\hat{\sigma}_{k,2}(\gamma_{k,1}) \subset \Sigma$, $\hat{\sigma}_{k,2}(\gamma_{k,2}) \cap \overline{\Sigma} = \phi$ and $\hat{\sigma}_{k,2}(\gamma_{k,3}) \subset \partial\Sigma$.

It can be easily seen that $C \setminus A$ has $2N$ connected components, by $C_1, C_2, \ldots\ldots, C_{2N}$ we denote these components. For $1 \leqq k \leqq N$, let $k(j)$ $(j = 1, 2, 3, 4)$ be integers such that $C_{k(j)} \cap \gamma_{k,j} \neq \phi$ $(j = 1, 2, 3)$ and $\hat{T}(a_k) \in \overline{C_{k(4)}}$. Now let $u = (u_1, u_2, \ldots\ldots, u_{2N})$ be the $2N$-vector and define a linear equation $u\Lambda = 0$ (Λ is a $2N \times 2N$ matrix) by

$$u_{k(1)} - u_{k(2)} = 0 , \quad u_{k(2)} - u_{k(3)} + u_{k(4)} = 0 \quad (k = 1, 2, \ldots, N).$$

Then we can prove the following theorem.

THEOREM 2. If $\dim(\ker \Lambda) = m$, then $\bar{H}^1(X) \simeq R^m$.

REFERENCES

[1] W.H.Gottchalk and G.A.Hedlund, " Topological Dynamics," A.M.S. Colloquim Publications, 1955.

[2] V.V.Nemytskii and V.V.Stepanov, " Qualitative Theory of Differential Equations," Princeton, 1960.

[3] E.H.Spanier, " Algebraic Topology," McGraw-Hill, 1966.