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# On the cohomology of the classifying space of the exceptional Lie groups.

Mamoru MIMURA (Kyoto Univesity)
Yuriko SAMBE (Tsuda College)

## 1. Introduction.

The following five cases of the cohomology $H^*(BG ; Zp)$ of the classifying space $BG$ of the 1-connected exceptional Lie groups $G$ are undetermined :

$$(G ; p) = (E_8 ; 2),\ (E_8 ; 3),\ (E_8 ; 5),\ (E_7 ; 3) \text{ and } (E_6 ; 3).$$

The first may soon be determined by M. Mori and M. Mimura and we expect to determine the second soon. In this paper we determined the remaining three cases.

We make use of the Eilenberg-Moore spectral sequence $\{E_r;d_r\}$ which has the following properties :

$$E_2 = \mathrm{Cotor}^A(Z_p, Z_p) \quad \text{where} \quad A = H^*(G ; Z_p)$$

and

$$E_\infty = \mathcal{G}r\, H^*(BG ; Z_p).$$

The $E_2$-term is determined by constructing the twisted tensor product and then we show that the spectral sequence collapses.

Here we shall explain the case $G = E_7$ and only state the results for the other two cases.

## 2. An injective resolution of $Z_3$ over $H^*(E_7 ; Z_3)$.

First we recall the Hopf algebra structure of $H^*(E_7 ; Z_3)$ over $\mathcal{A}_3$.

$$H^*(E_7 ; Z_3) = Z_3[x_8]/(x_8^3) \otimes \Lambda(x_3, x_7, x_{11}, x_{15}, x_{19}, x_{27}, x_{35}),$$

where $\deg x_i = i$ :

$$\bar{\phi}(x_i) = 0 \qquad \text{for } i = 3, 7, 8, 19,$$
$$\bar{\phi}(x_j) = x_8 \otimes x_{j-8} \qquad \text{for } j = 11, 15, 27,$$
$$\bar{\phi}(x_{35}) = x_8 \otimes x_{27} - x_8^2 \otimes x_{19},$$

where $\bar{\phi}$ is the reduced diagonal map induced from the multiplication on $E_7$ ;

$$\mathcal{P}^1 x_i = x_{i+4} \qquad \text{for } i = 3, 11,$$
$$\mathcal{P}^3 x_i = x_{j+12} \qquad \text{for } j = 7, 15,$$
$$\beta x_7 = x_8 \qquad \text{and} \qquad \beta x_{15} = x_8^2 .$$

Notation $A = H^*(E_7 ; Z_3)$ and $\bar{A} = \tilde{H}^*(E_7 ; Z_3)$.

Let $L$ be the $Z_3$-submodule generated by

$$\{x_3, x_7, x_8, x_{19}, x_{11}, x_{15}, x_{27}, x_8^2, x_{35}\}.$$

Let $\theta : A \to L$ be the projection and $\iota : L \to A$ the inclusion. We name the corresponding element under the suspension $s$ as

$$sL = \{a_4, a_8, a_9, a_{20}, b_{12}, b_{16}, b_{28}, c_{17}, e_{36}\}$$

respectively. Define $\bar{\theta} : A \to sL$ by $\bar{\theta} = s \circ \theta$ and $\bar{\iota} : sL \to A$ by $\bar{\iota} = \iota s^{-1}$. Let $T(sL)$ be the tensor algebra with the (natural) product $\psi$ . Consider the two sided ideal $I$ of $T(sL)$ generated by $\mathrm{Im}(\psi\circ(\bar\theta \times \bar\theta)\circ\phi)(\mathrm{Ker}\bar\theta)$ , where $\phi$ is the diagonal map of $A$ . Put $\bar X = T(sL)/I$ , that is,

$$\bar X = Z_3\{a_i, b_j, c_{17}, e_{36}\} \quad (i = 4, 8, 9, 20;\ j = 12, 16, 28)$$

and $I$ is generated by

$[\alpha, \beta]$ for all pairs $(\alpha, \beta)$ of generators $\alpha, \beta$ of $\bar X$ except $(a_9, b_j)$ $(j = 12, 16, 28)$ and $(a_9, e_{36})$ , $(a_9, c_{17})$ ,

$[a_9, b_j] + c_{17}a_{j-8}$ for $j = 12, 16, 28$,

$[a_9, e_{36}] + c_{17}b_{28}$,

where $[\alpha, \beta] = \alpha\beta - (-1)^*\beta\alpha$, with $* = \deg\alpha\cdot\deg\beta$.

We define a map

$$d = \psi\circ(\bar\theta \otimes \bar\theta)\circ\phi\circ\bar\iota : sL \to T(sL)$$

and extend it naturally over $T(sL)$ as a derivation. Since $d(I) \subset I$ holds, $d$ induces a map $\bar X \to \bar X$, which is denoted by $d$. It is easy to check that $d\circ d = 0$ and so $\bar X$ is a differential algebra over $Z_3$. By the relation

$$d\circ\bar\theta + \psi\circ(\bar\theta \times \bar\theta)\circ\phi = 0$$

we can construct the twisted tensor product $W = A \otimes \bar X$ with respect to $\bar\theta$. Namely, $W$ is an A-comodule with the differential operator

$$\bar d = 1 \otimes d + (1 \otimes \psi)\circ(1 \otimes \bar\theta \otimes 1)\circ(\phi \otimes 1).$$

Explicitly, the differential operators $\bar d$ and $d$ are given by

$$\bar{d}(x_i \otimes 1) = a_{i+1} \otimes 1 \qquad \text{for } i = 3, 7, 8, 19,$$
$$\bar{d}(x_8^2 \otimes 1) = 1 \otimes c_{17} - x_8 \otimes a_9,$$
$$\bar{d}(x_j \otimes 1) = 1 \otimes b_{j+1} + x_8 \otimes a_{j-7} \quad \text{for } j = 11, 15, 27,$$
$$\bar{d}(x_{35} \otimes 1) = 1 \otimes e_{36} + x_8 \otimes a_{28} - x_8^2 \otimes a_{20},$$
$$da_i = 0 \qquad \text{for } i = 4, 8, 9, 20,$$
$$dc_{17} = a_9^2,$$
$$db_j = -a_9 a_{j-8} \qquad \text{for } j = 12, 16, 28,$$
$$de_{36} = -a_9 b_{28} + c_{17} a_{20}.$$

Now we define weight in $W = A \otimes \bar{X}$ as follows :

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $A$ : | $x_3$, | $x_7$, | $x_{19}$, | $x_8$, | $x_8^2$, | $x_{11}$, | $x_{15}$, | $x_{27}$, | $x_{35}$ |
| $\bar{X}$ : | $a_4$, | $a_8$, | $a_{20}$, | $a_9$, | $c_{17}$, | $b_{12}$, | $b_{16}$, | $b_{28}$, | $e_{36}$ |
| weight : | 0 | 0 | 0 | 1 | 2 | 2 | 2 | 2 | 6 |

The weight of a monomial is a sum of the weights of every element. Define a filtration

$$F_r = \{x \mid \text{weight } x \leq r\}.$$

Put $E_0W = F_i/F_{i-1}$. Then it is easy to see

$$E_0W = \Lambda(x_3, x_7, x_{19}, x_{11}, x_{15}, x_{27}, x_{35})$$
$$\otimes Z_3[a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}] \otimes C(Q(x_8)),$$

where $C(Q(x_8))$ is the cobar construction of $Z_3[x_8]/(x_8^3)$. The differential formulas imply that $E_0W$ is acyclic, and hence $W$ is acyclic.

<u>Theorem 1.1.</u> $W$ is an injective resolution of $Z_3$ over $A = H^*(E_7 ; Z_3)$.

By the definition of Cotor we have

<u>Corollary 1.2.</u> $H(\bar{X} : d) = \mathrm{Ker}\, d/\mathrm{Im}\, d = \mathrm{Cotor}^{A}(Z_3, Z_3)$.

There are reduced power operations in $\bar{X}$ which are induced from those in $H^*(E_7 ; Z_3)$. In particular,

$$\mathcal{P}^1 a_i = a_{i+4} \qquad \text{for } i = 4, 8,$$
$$\mathcal{P}^3 b_j = b_{j+12} \qquad \text{for } j = 8, 16.$$

There are reduced power peorations in the Eilenberg-Moore spectral sequence :

$$\mathcal{P}^i : E_r^{s,t} \longrightarrow E_r^{s,t+2i(p-1)} \qquad 2i \leq t$$
$$\beta : E_r^{s,t} \longrightarrow E_r^{s,t+1} \quad .$$

In the $E_2$-term they coincide, since they are both induced through the cobar construction. (See M. Mori's paper.)

## 3. Calculation of $H(\bar{X} : d)$.

We define an operator $\partial$ by

$$\partial(a_i) = 0 \qquad \text{for } i = 4, 8, 20.$$
$$\partial(b_j) = -a_{j-8} \qquad \text{for } j = 12, 16, 28,$$
$$\partial(e_{36}) = -b_{28},$$

and extend it over $Z_3\{a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}\}$ by

$$\partial(\alpha\beta) = \partial(\alpha)\cdot\beta + \alpha\cdot\partial(\beta)$$

Then for a polynomial $P$ in $Z_3\{a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}\}$, we have

(3.1) $\partial^3 P = 0,$

$[a_9, P] = c_{17}\partial P,$

$dP = a_9\partial P + c_{17}\partial^2 P,$

and hence

$$dP = 0 \text{ if and only if } \partial P = 0.$$

We now state our results without proof :

(3.2) d-cocycles in $Z_3\{a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}\}$ are as follows :

$$
\begin{aligned}
a_4 &= \partial(-b_{12}),\\
a_8 &= \partial(-b_{16}),\\
a_{20} &= \partial^2 e_{36},\\
x_{36} &= b_{12}^3,\\
x_{48} &= b_{16}^3,\\
x_{84} &= b_{28}^3 = \partial^2(-b_{28}e_{36}^2),\\
y_{20} &= a_4b_{16} - a_8b_{12},\\
y_{32} &= a_8b_{28} - a_{20}b_{16} = \partial^2(b_{12}e_{36}),\\
y_{36} &= a_8b_{28} - a_{20}b_{16} = \partial^2(-b_{16}e_{28}),\\
y_{68} &= \partial^2(b_{12}^2b_{16}^2b_{28}),\\
y_{80} &= \partial^2(b_{12}^2b_{16}b_{28}^2),\\
y_{84} &= \partial^2(b_{12}b_{16}^2b_{28}^2),\\
y_{96} &= \partial^2(b_{12}^2b_{16}^2b_{28}^2).
\end{aligned}
$$

We have d-cocycles with $e_{36}$ and without $a_9$, $c_{17}$ in the following degrees, all of which except $x_{108} = e_{36}^3$ are in $\partial^2$-image.

(3.3) 108, 56, 44, 48, 52, 68, 72, 60, 64, 76, 104, 116, 120, 132, 80, 84, 88, 96, 100, 112, 96, 100, 108, 112, 116, 124, 128, 112, 124, 128, 136, 140, 144, 140, 152, 156, 168.

(3.4) Those with $a_9, c_{17}$ are

$a_9$, $y_{26}$, $y_{21}$ and $y_{25}$.

Our result is

Theorem 3.5. $\mathrm{Cotor}^A(Z_3, Z_3) = H(\bar{X} : d)$ is generated as an algebra by the elements listed in (3.2), (3.3), (3.4).

Relations. i) $a_4^2$, $a_8^2$, $y_{20}^2$, $a_4a_8$, $a_4y_{20}$, $a_8y_{20} \in \partial^2$-image.

ii) $a_9^2 = d(c_{17})$, $[a_9, y_{26}] = d(c_{17}^2)$

$a_9a_4 = d(-b_{12})$, $a_9a_8 = d(-b_{16})$.

$a_9\partial^2P = d(\partial P)$,

$y_{26}\partial^2P = d(-a_9P - c_{17}\partial P)$

iii) $y_{21}\partial^2P = d(a_4P + b_{12}\partial P)$, $y_{25}\partial^2P = d(a_8P + b_{16}\partial P)$,

$y_{21}a_4 = d(b_{12}^2)$, $y_{25}a_4 = d(b_{12}b_{16}) - a_9y_{20}$,

$y_{21}a_8 = d(b_{12}b_{16}) + a_9y_{20}$, $y_{25}a_8 = d(b_{16}^2)$,

$y_{21}y_{20} = d(b_{12}^2b_{16})$, $y_{25}y_{20} = d(b_{12}b_{16}^2)$

$a_9y_{21} = d(c_{17}b_{12}) - y_{26}a_4$, $a_9y_{25} = d(c_{17}b_{16}) - y_{26}a_8$

$y_{21}^2 = d(c_{17}b_{12}^2)$, $y_{25}^2 = d(c_{17}b_{16}^2)$

$y_{21}y_{25} = d(c_{17}b_{12}b_{16}) + y_{26}y_{20}$.

iv) d-cocycles in $Z_3\{a_4, a_8, a_{20}, b_{12}, b_{16}, b_{28}, e_{36}\}$ are 0 if and only if they are 0 as polynomials.

## 4. Collapsing of the Eilenberg-Moore spectral sequence.

Of the 54 elements that generate the $E_2$-term, only 19 are independent over $\mathcal{A}_3$. We have only to show that these 19 elements survive to $E_\infty$, since $\mathcal{P}^i$ and $\beta$ commute with the differentials $d_r : E_r^{s,t} \longrightarrow E_r^{s+t,\ t-r+1}$ $(r \geq 2)$. It is easy to list all the possible cases of $d_r P = Q$, which we deny one by one in the following manner.

Some survive for dimensional reasons, e.g. $y_{21}$ survives. For some P and Q, $y_{21}P = 0$ while $y_{21}Q \neq 0$ in $E_r$, thus $d_r P = Q$ does not occur. For some others, we have $\mathcal{P}^i$ for which $d_r(\mathcal{P}^i P)$ is known to be 0, while $\mathcal{P}^i Q \neq 0$ in $E_r$.

We have

Theorem 4.1. The Eilenberg-Moore spectral sequence for $E_7$ with $Z_3$-coeficient collapses and hence

$$H^*(BE_7 ; Z_3) \cong \mathrm{Cotor}^A(Z_3, Z_3) \qquad \text{with } A = H^*(E_7 ; Z_3)$$

as a module.

Remark 4.2. In our determination we did not use the properties of $E_7$ as a Lie group. Thus our results are valid for any a compact associative H-space that has the same cohomology as $E_7$.

## 5. $(G ; p) = (E_6 ; 3)$ and $(E_8 ; 5)$.

The Hopf algebra structure of $H^*(E_6 ; Z_3)$ and $H^*(E_7 ; Z_3)$ are alike and $\mathrm{Cotor}^{H^*(E_6 ; Z_3)}(Z_3, Z_3)$ can be obtained as a collorary of Theorem 3.5. It is generated by 18 elements and we have the following

<u>Theorem.</u> The Eilenberg-Moore spectral sequence for $E_6$ with $Z_3$-coefficient collapses.

We have also determined $\mathrm{Cotor}^B(Z_5, Z_5)$ with $B = H^*(E_8 ; Z_5)$. It is generated as an algebra by about 940 elements, but it is not hard to show

<u>Theorem.</u> The Eilenberg-Moore spectral sequence for $E_8$ with $Z_5$-coefficient collapses.