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On a characterization of finite groups of p-rank 1.

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Let $G$ be a finite group. Let $p$ be a prime number. Define the p-rank $r_p(G)$ of $G$ by the maximal integer $k$ such that $G$ contains the elementry abelian p-group $(\mathbb{Z}_p)^k$ of rank $k$.

It is obvious that $G$ is of p-rank 0 if and only if the p-Sylow subgroup $G_p = e$. According to Cartan-Eilenberg [2], we see that $G$ is of p-rank 1 if and only if $G_p$ is either a cyclic group $\mathbb{Z}_{p^r}$ or a generalized quaternionic group if $p = 2$.

It is also shown [2] that a finite group $G$ with p-rank 0 or 1 for any $p$ is characterized by having the periodic cohomology.

Such a group is called an Artin-Tate group.

Now the purpose of the present note is to give a characterization
of finite groups of p-rank 1 in terms of stable homotopy groups.

Let $|G|$ be the order of $G$ and let $\Sigma_n$ denote the symmetric group on $n$ letters. We denote by $\rho = \rho_G : G \to \Sigma_{|G|}$ the regular permutation representation, and $B\rho : BG \to B\Sigma_{|G|}$ denotes the induced map on classifying spaces. Let

$$\omega : \bigwedge_n B\Sigma_n \to \Omega \bigwedge_n B\Sigma_n \simeq Q(S^0)$$

be the Barratt–Priddy–Quillen map [1], where $Q(S^0) = \lim_k \Omega^k S^k$.

Then as the adjoint of the composition

$$BG_+ \xrightarrow{B\rho} B\Sigma_{|G|} \simeq B\Sigma_n \xrightarrow{\omega} Q(S^0)$$

we obtain a stable map of spectra

$$f : S(BG_+) \xrightarrow{} S$$

where $BG_+ = BG \cup$ disjoint base point. Then we obtain a homomorphism
\[ \phi = \phi_G : \pi_n^{S(BG)} \rightarrow \pi_n^S(S^0) \]

of stable homotopy groups. Note that \( \pi_n^{S(BG)} \cong \pi_n^S(BG) \oplus \pi_n^S(S^0) \),

direct sum. The restriction \( \phi|_{\pi_n^S(BG)} \) is also denoted by \( \phi \).

Now let \( J : \pi_n^S(O) \rightarrow \pi_n^S(S^0) \) denote the J-homomorphism, where

\[ O = \lim \pi(n). \]

Restricting \( J : \pi_n^S(O) \rightarrow \pi_n^S(S^0) \) on \( \pi_n(U) \) or \( \pi_n(S_p) \),

we obtain the complex J-homomorphism \( J_C \) or the quaternionic J-

homomorphism \( J_H \).

For a finite abelian group \( A \), we denote by \( A(p) \) the p-component

of \( A \). Then we can state our theorems.

**Theorem 1.1.** Let \( G \) be a finite group of p-rank 1. If \( p \)

is odd, then

\[ \text{Im}[\phi : \pi_*^{S(BG)} \rightarrow \pi_*^S(S^0)] \supset (\text{Im } J)_p = (\text{Im } J_C)_p. \]

If \( p = 2 \), then

\[ \text{Im}[\phi : \pi_*^{S(BG)} \rightarrow \pi_*^S(S^0)] \supset (\text{Im } J_H)_2. \]
Theorem 1.2. Let $G$ be a finite group. Then the $p$-rank of $G$ is equal to 1 if and only if \( \phi : \pi_{2p-3}^S(BG)(p) \rightarrow \pi_{2p-3}^S(S^0)(p) \)

\( \phi : \pi_3^S(BG)(2) \rightarrow \pi_3^S(S^0)(2) \) if $p = 2$) is an epimorphism.

Concerning with the 2-component, it may be worth showing the following

Proposition 1.3. \( \phi : \pi_1^S(BG) \rightarrow \pi_1^S(S^0) \) is an epimorphism if and only if the 2-Sylow subgroup $G_{(2)}$ is a non trivial cyclic group.

From this proposition it follows immediately that if $G_{(2)}$ is non trivial cyclic, then $G$ is not perfect, hence not simple unless $G = Z_2$ (Burnside's theorem).

If one uses the Feit - Thompson theorem [3], one can show the following.

Corollary 1.4. Let $G$ be an Artin - Tate group. Suppose that

$H_i(G ; Z) = 0, 1 \leq i \leq 3$, then $G$ is trivial.
Proof. By the assumption, \( \pi_3^{S}(BG) = 0 \). Hence by Theorem 1.2, we see that \( G(2) = e \), i.e., \( G \) is of odd order. Then by the Feit-Thompson theorem, \( G \) is solvable. Then \( H_1(G : Z) = 0 \) implies \( G = e \). q.e.d.

Now for a finite group \( G \) of p-rank 1, Theorem 1.1 shows the non-triviality of \( \pi_{2p-3}^{S}(BG) \) if \( p = 2 \). We remark that such a non-triviality of \( \pi_i^{S}(BG) \) for \( i < 2p-3 \) does not hold as the following examples show. If \( p \) is odd, then \( \sum_{p} \) is of p-rank 1. It is known [5] that \( H_i(B_p^+ : Z_p) = 0 \) for \( i < 2p-3 \). Then by Serre's class theory, \( \pi_i^{S}(B_p^+)(p) = 0 \) if \( i < 2p-3 \). For \( p = 2 \), consider the binary icosahedral group \( I^* \). This is a subgroup of order 120 of \( Sp(1) = S^3 \). Hence \( I^* \) is an Artin-Tate group and \( I^*_p(2) \) is the quaternionic group. It is well-known [7] that \( H_1(BI^*) = H_2(BI^*) = 0 \). Hence \( \pi_i^{S}(BI^*) = 0 \) for \( i < 2 \).

The non-triviality of \( \pi_{2p-3}^{S}(BG) \) clearly fails.
for general finite groups as the following Quillen's example shows.

Let $F_q$ be the finite field with $q = p^d$ elements. Then Quillen has shown [4] that $H^i(BGL(n, F_q) : Z_p) = 0$ for $0 < i < d(p-1)$.

Thus $\pi^S_1(BGL(n, F_q))(p) = 0$ for $i < d(p-1)$.

For a cyclic group $Z_p$ of prime order, Theorem 1.1 is a direct consequence of the Kahn – Priddy theorem [4], that is

$\phi : \pi^S_*(BZ_p) \to \pi^S_*(S^0)(p)$ is an epimorphism ($* > 0$). We shall show that the Kahn – Priddy theorem fails for cyclic group of order $2^r$, $r \geq 2$.

Theorem 1.5. Let $r$ be an integer $\geq 2$. Let $f : SBZ_{2^r} \to S$ be an arbitrary stable map. Then $f_* : \pi^S_7(BZ_{2^r}) \to \pi^S_7(S^0)(2)$ is not epimorphism.

For an odd prime, the problem seems to be more difficult. For example, a direct computation shows that the element

$\beta_1 \in \pi^S_{2p(p-1)-2}(S^0)(p)$ is in the image of $\phi : \pi^S_*(BZ_{p^r}) \to \pi^S_*(S^0)$ for any $r$. 

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