On a characterization of finite groups of p-rank 1.

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Let $G$ be a finite group. Let $p$ be a prime number. Define the $p$-rank $r_p(G)$ of $G$ by the maximal integer $k$ such that $G$ contains the elementary abelian $p$-group $(\mathbb{Z}_p)^k$ of rank $k$.

It is obvious that $G$ is of $p$-rank 0 if and only if the $p$-Sylow subgroup $G_{(p)} = e$. According to Cartan - Eilenberg [2], we see that $G$ is of $p$-rank 1 if and only if $G_{(p)}$ is either a cyclic group $\mathbb{Z}_{p^r}$ or a generalized quaternionic group if $p = 2$.

It is also shown [2] that a finite group $G$ with $p$-rank 0 or 1 for any $p$ is characterized by having the periodic cohomology.

Such a group is called an Artin - Tate group.

Now the purpose of the present note is to give a characterization
of finite groups of \( p \)-rank 1 in terms of stable homotopy groups.

Let \(|G|\) be the order of \( G \) and let \( \Sigma_n \) denote the symmetric group on \( n \) letters. We denote by \( \rho = \rho_G : G \to \Sigma_{|G|} \) the regular permutation representation, and \( B\rho : BG \to B\Sigma_{|G|} \) denotes the induced map on classifying spaces. Let

\[
\omega : \prod_{n} B\Sigma_n \to \Omega B(\prod_{n} B\Sigma_n) \simeq Q(S^0)
\]

be the Barratt–Priddy–Quillen map [1], where \( Q(S^0) = \text{lim}_k \Omega^k S^k \).

Then as the adjoint of the composition

\[
BG_+ \xrightarrow{\mathbf{B}\rho} B\Sigma_{|G|} \xrightarrow{\omega} B\Sigma_n \to Q(S^0)
\]

we obtain a stable map of spectra

\[
f : S(BG_+) \to S
\]

where \( BG_+ = BG \cup \text{disjoint base point} \). Then we obtain a homomorphism
\[ \phi = \phi_G : \pi_n^{S(BG)} \rightarrow \pi_n^{S(S^0)} \]

of stable homotopy groups. Note that \( \pi_n^{S(BG)} \cong \pi_n^{S(BG)} \oplus \pi_n^{S(S^0)} \),
direct sum. The restriction \( \phi|_{\pi_n^{S(BG)}} \) is also denoted by \( \phi \).

Now let \( J : \pi_n(O) \rightarrow \pi_n^{S(S^0)} \) denote the J-homomorphism, where
\[
O = \lim \pi_n(O(n)).
\]
Restricting \( J : \pi_n(O) \rightarrow \pi_n^{S(S^0)} \) on \( \pi_n(U) \) or \( \pi_n(S_p) \),
we obtain the complex J-homomorphism \( J_C \) or the quaternionic J-
homomorphism \( J_H \).

For a finite abelian group \( A \), we denote by \( A(p) \) the p-component
of \( A \). Then we can state our theorems.

**Theorem 1.1.** Let \( G \) be a finite group of p-rank 1. If \( p \)
is odd, then
\[
\text{Im}[\phi : \pi_n^{S(BG)} \rightarrow \pi_n^{S(S^0)}] \supset (\text{Im } J)_p = \text{Im } J_C(p).
\]

If \( p = 2 \), then
\[
\text{Im}[\phi : \pi_n^{S(BG)} \rightarrow \pi_n^{S(S^0)}] \supset (\text{Im } J_H)(2).
\]
Theorem 1.2. Let $G$ be a finite group. Then the $p$-rank of $G$ is equal to 1 if and only if $\phi : \pi_{2p-3}^S(BG)(p) \to \pi_{2p-3}^S(S^0)(p)$

$\phi : \pi_3^S(BG)(2) \to \pi_3^S(S^0)(2)$ if $p = 2$) is an epimorphism.

Concerning with the 2-component, it may be worth showing the following

Proposition 1.3. $\phi : \pi_1^S(BG) \to \pi_1^S(S^0)$ is an epimorphism if and only if the 2-Sylow subgroup $G_{(2)}$ is a non trivial cyclic group.

From this proposition it follows immediately that if $G_{(2)}$ is non trivial cyclic, then $G$ is not perfect, hence not simple unless $G = Z_2$ (Burnside's theorem).

If one uses the Feit - Thompson theorem [3], one can show the following

Corollary 1.4. Let $G$ be an Artin - Tate group. Suppose that

$H_i(G : Z) = 0, 1 \leq i \leq 3$, then $G$ is trivial.
Proof. By the assumption, $\pi_3^{S}(BG) = 0$. Hence by Theorem 1.2, we see that $G_{(2)} = e$, i.e., $G$ is of odd order. Then by the Feit-Thompson theorem, $G$ is solvable. Then $H_1(G; Z) = 0$ implies $G = e$. q. e. d.

Now for a finite group $G$ of p-rank 1, Theorem 1.1 shows the non-triviality of $\pi_{2p-3}^{S}(BG)(p) \pi_3^{S}(BG)(2)$ if $p = 2$). We remark that such a non-triviality of $\pi_i^{S}(BG)(p)$ for $i < 2p-3$ does not hold as the following examples show. If $p$ is odd, then $\sum_{p}$ is of p-rank 1. It is known [5] that $H_i(B_{\bar{p}} : Z_p) = 0$ for $i < 2p-3$. Then by Serre's class theory, $\pi_i^{S}(B_{\bar{p}})(p) = 0$ if $i < 2p-3$. For $p = 2$, consider the binary icosahedral group $I^*$. This is a subgroup of order 120 of $Sp(1) = S^3$. Hence $I^*$ is an Artin-Tate group and $I^*_<(2)$ is the quaternionic group. It is well-known [η] that $H_1(BI^*) = H_2(BI^*) = 0$. Hence $\pi_1^{S}(BI^*) = 0$ for $i < 2$.

The non-triviality of $\pi_{2p-3}^{S}(BG)(p) \pi_3^{S}(BG)(2)$ clearly fails
for general finite groups as the following Quillen's example shows.

Let $F_q$ be the finite field with $q = p^d$ elements. Then Quillen has shown \cite{4} that $H^i_{\text{BLG}}(n, F_q) : Z_p = 0$ for $0 < i < d(p-1)$.

Thus $\pi^S_{i}(\text{BLG}(n, F_q)) (p) = 0$ for $i < d(p-1)$.

For a cyclic group $\mathbb{Z}_p$ of prime order, Theorem 1.1 is a direct consequence of the Kahn - Priddy theorem \cite{4}, that is

$\phi : \pi^S_*(B\mathbb{Z}_p) \to \pi^S_*(S^0) (p)$ is an epimorphism ($*>0$). We shall show that the Kahn - Priddy theorem fails for cyclic group of order

$2^r$, $r \geq 2$.

Theorem 1.5. Let $r$ be an integer $\geq 2$. Let $f : S B\mathbb{Z}_{2^r} \to S$ be an arbitrary stable map. Then $f_* : \pi^S_7(B\mathbb{Z}_{2^r}) \to \pi^S_7(S^0) (2)$ is not epimorphism.

For an odd prime, the problem seems to be more difficult. For example, a direct computation shows that the element

$\beta_1 \in \pi^S_{2p(p-1)-2}(S^0) (p)$ is in the image of $\phi : \pi^S_*(B\mathbb{Z}_{p^r}) \to \pi^S_*(S^0)$

for any $r$.  

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REFERENCES


