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On a characterization of finite groups of p-rank 1.

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Let G be a finite group. Let p be a prime number. Define the p-rank $r_p(G)$ of G by the maximal integer k such that G contains the elementry abelian p-group $(Z_p)^k$ of rank k.

It is obvious that G is of p-rank 0 if and only if the p-Sylow subgroup $G_{(p)} = e$. According to Cartan - Eilenberg [2], we see that G is of p-rank 1 if and only if $G_{(p)}$ is either a cyclic group Z_{pr} or a generalized quoterniounic group if p = 2, It is also shown [2] that a finite group G with p-rank 0 or 1 for any p is characterized by having the periodic cohomology. Such a group is called an Artin - Tate group.

Now the purpose of the present note is to give a characterization of finite groups of p-rank 1 in terms of stable homotopy groups.

Let $|G|$ be the order of G and let $\sum_n$ denote the symmetric group on n letters. We denote by $\rho = \rho_G : G \to \sum_{|G|}$ the regular permutation representation, and $B\rho : BG \to B\sum_{|G|}$ denotes the induced map on classifying spaces. Let

$$\omega : \coprod_n B\textstyle\sum_n \to \Omega B(\coprod_n B\sum_n) \cong Q(S^0)$$

be the Barratt - Priddy - Quillen map [1], where $Q(S^0) = \lim_k \Omega^k S^k$. Then as the adjoint of the composition

$$BG_+ \xrightarrow{B\rho_+} B\textstyle\sum_{|G|+} \subset B\sum_n \xrightarrow{\omega} Q(S^0)$$

we obtain a stable map of spectra

$$f : S(BG_+) \to S$$

where $BG_+ = BG \cup$ disjoint base point. Then we obtain a homomorphism

$$\phi = \phi_G : \pi_n^S(BG_+) \to \pi_n^S(S^0)$$

of stable homotopy groups. Note that $\pi_n^S(BG_+) \cong \pi_n^S(BG) \oplus \pi_n^S(S^0)$, direct sum. The restriction $\phi|_{\pi_n^S(BG)}$ is also denoted by ϕ.

Now let $J : \pi_n(O) \to \pi_n^S(S^0)$ denote the J-homomorphism, where $O = \lim O(n)$. Restricting $J : \pi_n(O) \to \pi_n^S(S^0)$ on $\pi_n(U)$ or $\pi_n(S_p)$, we obtain the complex J-homomorphism J_C or the quoternionic J-homomorphism J_H.

For a finite abelian group A, we denote by $A_{(p)}$ the p-component of A. Then we can state our theorems.

<u>Theorem 1.1</u>. Let G be a finite group of p-rank 1. If p is odd, then

$$\mathrm{Im}[\phi : \pi_*^S(BG) \to \pi_*^S(S^0)] \supset (\mathrm{Im}\, J)_{(p)} = (\mathrm{Im}\, J_C)_{(p)}.$$

If $p = 2$, then

$$\mathrm{Im}[\phi : \pi_*^S(BG) \to \pi_*^S(S^0)] \supset (\mathrm{Im}\, J_H)_{(2)}.$$

Theorem 1.2. Let G be a finite group. Then the p-rank of G is equal to 1 if and only if $\phi : \pi^S_{2p-3}(BG)_{(p)} \to \pi^S_{2p-3}(S^0)_{(p)}$ ($\phi : \pi^S_3(BG)_{(2)} \to \pi^S_3(S^0)_{(2)}$ if $p = 2$) is an epimorphism.

Concerning with the 2-component, it may be worth showing the following

Proposition 1.3. $\phi : \pi^S_1(BG) \to \pi^S_1(S^0)$ is an epimorphism if and only if the 2-Sylow subgroup $G_{(2)}$ is a non trivial cyclic group.

From this proposition it follows immediately that if $G_{(2)}$ is non trivial cyclic, then G is not perfect, hence not simple unless $G = Z_2$ (Burnside's theorem).

If one uses the Feit - Thompson theorem [3], one can show the following

Corollary 1.4. Let G be an Artin - Tate group. Suppose that $H_i(G : Z) = 0$, $1 \leq i \leq 3$, then G is trivial.

<u>Proof</u>. By the assumption, $\pi_3^S(BG) = 0$. Hence by Theorem 1.2, we see that $G_{(2)} = e$, i.e., G is of odd order. Then by the Feit - Thompson theorem, G is solvable. Then $H_1(G : Z) = 0$ implies $G = e$. q. e. d.

Now for a finite group G of p-rank 1, Theorem 1.1 shows the non-triviality of $\pi_{2p-3}^S(BG)_{(p)}$ ($\pi_3^S(BG)_{(2)}$ if $p = 2$). We remark that such a non-triviality of $\pi_i^S(BG)_{(p)}$ for $i < 2p-3$ does not hold as the following examples show. If p is odd, then Σ_p is of p-rank 1. It is known [5] that $H_i(B\Sigma_p : Z_p) = 0$ for $i < 2p-3$. Then by Serre's class theory, $\pi_i^S(B\Sigma_p)_{(p)} = 0$ if $i < 2p-3$. For $p = 2$, consider the binary icosahedral group I^*. This is a subgroup of order 120 of $Sp(1) = S^3$. Hence I^* is an Artin - Tate group and $I^*_{(2)}$ is the quoternionic group. It is well-known [7] that $H_1(BI^*) = H_2(BI^*) = 0$. Hence $\pi_i^S(BI^*) = 0$ for $i \leq 2$.

The non-triviality of $\pi_{2p-3}^S(BG)_{(p)}$ ($\pi_3^S(BG)_{(2)}$) clearly fails for general finite groups as the following Quillen's example shows. Let F_q be the finite field with $q = p^d$ elements. Then Quillen has shown [6] that $H^i(BGL(n, F_q) : Z_p) = 0$ for $0 < i < d(p-1)$. Thus $\pi_i^S(BGL(n, F_q))_{(p)} = 0$ for $i < d(p-1)$.

For a cyclic group Z_p of prime order, Theorem 1.1 is a direct consequence of the Kahn - Priddy theorem [4], that is $\phi : \pi_*^S(BZ_p) \to \pi_*^S(S^0)_{(p)}$ is an epimorphism $(* > 0)$. We shall show that the Kahn - Priddy theorem fails for cyclic group of order 2^r, $r \geq 2$.

Theorem 1.5. Let r be an integer ≥ 2. Let $f : SBZ_{2^r} \to S$ be an arbitrary stable map. Then $f_* : \pi_7^S(BZ_{2^r}) \to \pi_7^S(S^0)_{(2)}$ is not epimorphism.

For an odd prime, the problem seems to be more difficult. For example, a direct computation shows that the element $\beta_1 \in \pi_{2p(p-1)-2}^S(S^0)_{(p)}$ is in the image of $\phi : \pi_*^S(BZ_{p^r}) \to \pi_*^S(S^0)$ for any r.

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