数理解析研究所講究録
314巻 1977年 88-104

An extention of AKTH-theory to locally compact groups

by Nobuhiko TATSUUMA

1. Let $\{\mathfrak{A}, G, \alpha\}$ be a C*-system. That is, $\mathfrak{A}$ is a C*-algebra, G is a locally compact group, and $G \ni g \mapsto \alpha_g \in \mathrm{Aut}(\mathfrak{A})$ is a continuous homomorphism. Consider an α-invariant state ω on $\mathfrak{A}$, and the unitary representation $\{\pi, U_g, \mathcal{H}, \Omega\}$ of G deduced by GNS-construction.

For any $A, B \in \mathfrak{A}$, put $f_{AB}(g) = \omega(B\,\alpha_g(A)) - \omega(A)\,\omega(B) = \langle U_g \pi(A)\Omega, \pi(B^*)\Omega\rangle - \langle \pi(A)\Omega, \Omega\rangle \langle \pi(B)\Omega, \Omega\rangle$ and $g_{AB}(g) = \omega(\alpha_g(A)B) - \omega(A)\omega(B)$. Then evidently,

(1) $$g_{AB}(g) = f_{A^*B^*}(g) = f_{BA}(g^{-1}).$$

Now we assume the existence of a norm dense α-invariant *-subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$, for which the followings are valid.

Put

$\mathcal{F}_0$ = (function algebra on G algebraically generated by $\{f_{AB}\}_{A,B \in \mathfrak{A}_0}$),

$\mathcal{F}$ = (the uniform closure of $\mathcal{F}_0$) ,

and construct $\mathcal{G}_0$ and $\mathcal{G}$ as same way from $\{g_{AB}\ ;\ A, B \in \mathfrak{A}_0\}$.

[Assumption 1] $\mathcal{F}$ is closed with respect to complex conjugation.

[Assumption 2] For any $n \geq 1$ and $A_j, B_j \in \mathfrak{A}_0$ (j=1,2, ...),

(2) $$\int_G \left(\prod_j^n f_{A_jB_j}(g) - \prod_j^n g_{A_jB_j}(g)\right) dg = 0.$$

[Assumption 3] There exist $1 \leq p, q < +\infty$ and a non-zero

element $f_0 \equiv \sum_k^n \prod_j f_{A_{j,k} B_{j,k}}$ $(A_{j,k}, B_{j,k} \in \mathfrak{A}_0)$ in $\mathfrak{F}_0$

such that (i) $f_0 \in L^p(G)$,

(ii) $g_0 \equiv \sum_k^n \prod_j g_{A_{j,k} B_{j,k}} \in L^q(G)$.

(We use a right Haar measure dg on G).

The purpose of this paper is to show the KMS-property for C*-systems which satisfy the above assumptions.

From (1) and [Assumption 1] the following lemma is direct.

<u>Lemma 1</u> $\mathfrak{F} = \mathfrak{G}$,and $\mathfrak{F}$ is closed under the operation

$$f(g) \longmapsto f(g^{-1}) \quad .$$

2. We shall give the formulation of our KMS-property on C*-systems based on Araki-Kastler-Takesaki-Haag's theory.

When G is the additive group $\mathbb{R}$ of real numbers, the ordinary KMS-property is stated as follows.

[KMS] The function $\psi_{AB}(t) = \omega(B\,\alpha_t(A))$ can be extended analytically on some strip domain $\{t\ ;\ 0 \leq \mathrm{Im}(t) \leq \beta\}$ and $\psi_{AB}(t + i\beta) = \omega(\alpha_t(A)B)$ for any t in $\mathbb{R}$ and any A, B in $\mathfrak{A}$.

In the other hand for any one-parameter subgroup $g(t)$ of G, using the Stone's theorem, we can determine its infinitesimal generator iH , as H is a self-adjoint operator on $\mathcal{H}$ and

$$e^{iHt} = U_{g(t)} \ .$$

Now in our case, denote by K the kernel in G of the homomorphism $g \longrightarrow \alpha_g$, then our main result is given as follows.

MAIN THEOREM. Under the assumptions $1 \sim 3$, there exists an one-parameter subgroup $g(t)$ of G/K, such that

$$< U_g e^{H/2} \pi(A)\Omega, e^{H/2} \pi(B^*)\Omega > = < \pi(B)\Omega, U_g \pi(A^*)\Omega >,$$

for any A, B in $\mathfrak{A}_0$.

If the Main Theorem is proved, the function

$$\psi(t) = \omega(B\alpha_{g(t)}(A)) = < U_{g(t)} \pi(A)\Omega, \pi(B^*)\Omega >$$

has the analytical extension

$$\psi(t+is) = < U_{g(t)} e^{sH/2} \pi(A)\Omega, e^{sH/2} \pi(B^*)\Omega >$$

and $\psi(t+i) = < \pi(B^*)\Omega, U_{g(t)} \pi(A)\Omega > = \omega(\alpha_{g(t)}(A)B)$.

This shows that the subsystem $\{\mathfrak{A}, \mathbb{R}, \alpha_{g(t)}\}$ is just a KMS-C*-system as originally defined.

3. At first we discuss under slightly more general situation and prove a useful Proposition 1.

Let F_0 be a set of bounded uniformly continuous functions on G, and F be the uniform closure of F_0. For any $f \in F$, put $G_f = \{g \in G \ ; \ f(gg_1) = f(g_1), \forall g_1 \in G\}$ and $G_{F_0} = \bigcap_{f \in F_0} G_f$, $G_F = \bigcap_{f \in F} G_f$.

Lemma 2. $G_F = G_{F_0}$, and G_F is a closed subgroup of G.

Proof. Because f is continuous, G_f is closed. Hence G_F, G_{F_0} are closed.

For any $k_1, k_2 \in G_f$, $g \in G$, $f(k_1^{-1}k_2g) = f(k_1(k_1^{-1}k_2g)) = f(k_2g) = f(g)$. Thus $k_1^{-1}k_2 \in G_f$, therefore G_f and G_F, G_{F_0} are subgroups.

Obviously $G_F \subset G_{F_0}$, If $G_F \neq G_{F_0}$ there exists $g_1 \in G_{F_0}$ and $\notin G_F$. That is, $\exists\, g_2$, $\exists f \in F$ and $f(g_1g_2) \neq f(g_2)$ and for $\forall \varphi \in F_0$, $\varphi(g_1g) =$

$\varphi(g)$. On the other hand, $\forall \varepsilon > 0, \exists \varphi_1 \in F_0$ such that $\|f - \varphi_1\|_\infty < \varepsilon/2$, Put $\varepsilon = |f(g_1g_2) - f(g_2)|$, then $|f(g_1g_2) - f(g_2)| \leq |f(g_1g_2) - \varphi(g_1g_2)| + |\varphi(g_1g_2) - \varphi(g_2)| + |\varphi(g_2) - f(g_2)| < \varepsilon/2 - \varepsilon/2 = |f(g_1g_2) - f(g_1)|$. That is contradiction.

<u>Lemma 3.</u> If there is a non-trivial function f_0 of zero at ∞ in F_0, then the subgroup G_{f_0} and $G_{F_0} = G_F$ are compact.

<u>Proof.</u> If G_{f_0} is not compact, there exists a sequence $\{k_j\} \subset G_{f_0}$ such that $k_j \to \infty$. Therfore for some $g_0 \in G$, and for all j $0 \neq f_0(g_0) = f_0(k_jg_0)$. This contradicts to the assumption for f_0.

<u>Corollary 1.</u> In such a case, $L^p(G_F \backslash G)$ is imbedded into $L^p(G)$ as a space of functions which are constant on G_F-left cosets.

Hereafter we write $H = G_F$.

<u>Lemma 4.</u> If a uniformly continuous function f on G belongs to $L^p(G)$ for some $p < +\infty$, f is zero at ∞.

<u>Proof.</u> If f is not zero at ∞, there exists a sequence $\{k_j\} \subset G$ and $a > 0$ such that $k_j \to \infty$, $|f(k_j)| > a$ for any j. Uniform continuity of f asserts the existence of a compact neighborhood V of e, such that $|f(g_1) - f(g_2)| < a/2$ for any $\forall g_1, g_2$ such that $g_1g_2^{-1} \in V$. Since $k_j \to \infty$, if it is necessary, taking a subsequence, we can assume $Vk_j \cap Vk_\ell = \emptyset$ ($j \neq \ell$). Thus,

$$\int_G |f(g)|^p dg \geq \sum_j \int_{Vk_j} |f(g)|^p dg \geq \sum_j \int_{Vk_j} [|f(k_j)| - (a/2)]^p dg \geq$$

$$\geq (a/2)^p \sum_j \mu(V) = \infty.$$ That is contradiction.

<u>Corollary 2.</u> Any $f \in \mathcal{B}(G) \cap L^p(G)$ ($p < +\infty$) is zero at ∞.

Here $\mathcal{B}(G)$ is the ring of functions generated by $\{\langle U_g^{\omega} v, u \rangle\}$ (ω runs unitary representations of G, and v,u run vectors of spaces of representation ω).

Proposition 1. Assume that the above F_0 satisfies the follwings.

(i) F_0 is a function algebra, that is , closed under the operations $+$, $\times$ and scalar multiplication.

(ii) F_0 is invariant under right translations, that is , for any f in F_0 and any g_1, the function $(R_{g_1} f)(g) = f(gg_1)$ of g is in F_0.

(iii) The uniform closure F of F_0 is closed with respect to complex conjugation.

(iv) There exist an f_0 ($\neq 0$) in F_0 and $p<+\infty$, such that $f_0 \in L^p(G)$.

Then there exists a natural number n and the set

$$F_1 = \left\{ \sum_j^N \varphi_j \cdot R_{g_j} (f_0)^n \ ; \ N=1,2,\dots, \ g_j \in G, \ \varphi_j \in F_0 \oplus \mathbb{C}\, 1 \right\}$$

in $F_0 \cap L^1(H\backslash G)$, is dense in $L^q(H\backslash G)$ for $1 \leq {}^\forall q < +\infty$,and is dense in $L_c^\infty(H\backslash G) \equiv \{$ continuous function of zero at ∞ on $H\backslash G\}$ with uniform norm.

Proof. If we put $n = [p] + 1$, $(f_0)^n \in L^1(G) \cap L^\infty(G)$, therefore $F_1 \subset F_0 \cap L^1(H\backslash G) \cap L^\infty(H\backslash G)$. Thus replacing f_0^n to f_0 ,we can consider $f_0 \in L^1(H\backslash G)$ and $F_1 \subset L^1(H\backslash G) \cap L^\infty(H\backslash G) \subset L^q(H\backslash G)$ for $1 \leq {}^\forall q < +\infty$. And by Lemma 4 , $F_1 \subset L_c^\infty(H\backslash G)$. Moreover we consider

$$F_2 = \left\{ \sum_j^N \varphi_j \cdot R_{g_j} |f_0|^2 \ ; \ N=1,2,\dots, \ g_j \in G, \varphi_j \in F \oplus \mathbb{C} 1 \right\} .$$

In general $F_2 \not\subset F_0$, but by the assumption (iii) $F_2 \subset F$, since $\overline{R_{g_j} f_0}$ and therefore $R_{g_j} |f_0|^2 = (\overline{R_{g_j} f_0})(R_{g_j} f_0)$ are in F.

Lemma 5. For $\forall\varphi \in F_2, \forall\varepsilon > 0, 1 \leq \forall p \leq +\infty$, there exists $f \in F_1$ such that $\| \varphi - f \|_p < \varepsilon$.

Proof. Let $\varphi = \sum_j^N (\varphi_j \cdot \overline{R_{g_j} f_0}) R_{g_j} \cdot f_0 \in F_2$. Here $\varphi_j \overline{R_{g_j} f_0} \in F$, so there exist f_j's $\in F_0$ such that $\| \varphi_j \overline{R_{g_j} f_0} - f_j \|_\infty < (\varepsilon / N \| f_0 \|_p)$. Thus $\| \varphi - \sum f_j R_{g_j} f_0 \|_p < \sum \| \varphi_j \overline{R_{g_j} f_0} - f_j \|_\infty \| R_{g_j} f_0 \|_p =$

$$= \sum \| \varphi_j \overline{R_{g_j} f_0} - f_j \|_\infty \| f_0 \|_p < \varepsilon .$$

By the reason of Lemma 5, it is enough to show that F_2 is dense in $L^q(H \backslash G)$ and $L_c^\infty(H \backslash G)$.

Lemma 6. F_2 is (i) a subring of F, (ii) closed with respect to complex conjugation, (iii) invariant to right translations, (iv) $F_2 \subset L^1(G) \cap L^\infty(G)$, so its elements are zero at ∞, (v) separates any two points $\tilde{g}_1 \neq \tilde{g}_2$ in $H \backslash G$.

Proof. F_2 is the ideal of F generated by $\Lambda \equiv \{ R_g |f_0|^2 ; g \in G \}$, thus (i) is evident. The fact that $R_g |f_0|^2$ are real-valued, and the assumption (iii) in Proposition 1, give (ii). (iii) is direct result of right invariant properties of F_0, F and Λ. $R_g |f_0|^2$ are in $L^1(G)$ and F is in $L^\infty(G)$, hence (iv) is true. At last, if $f_0(g_1) \neq f_0(g_2)$ then (v) is true for such g_1, g_2. And if $0 \neq f_0(g_1 g_0) = f_0(g_2 g_0)$ for some g_0 in G, by the definition of $H = G_F$, there exists a $\varphi \in F$ such that $\varphi(g_1) \neq \varphi(g_2)$, thus $\varphi \cdot R_{g_0} |f_0|^2$ separates these $\tilde{g}_1$, $\tilde{g}_2$.

Corollary 3. For $\forall \varphi \in L_c^\infty(H \backslash G), \forall \varepsilon > 0$, there exists $f \in F_2$ such that $\| \varphi - f \|_\infty < \varepsilon$, that is, F_2 is dense in $L_c^\infty(H \backslash G)$.

Proof. Consider the one point compactification space X of $H \backslash G$. We apply the Stone-Weierstrass's theorem to $F_2 \oplus \mathbb{C}1$ on $C(X)$. Thus we get $f_1 = f + a1 \in F_2 \oplus \mathbb{C}1$ and $\| \varphi - f_1 \|_\infty < \varepsilon/2$. But φ is zero at ∞ and $f \in F_2$ is too. Hence $|a| < (\varepsilon/2)$, and $\|\varphi - f\|_\infty \leq \|\varphi - f_1\|_\infty + (\varepsilon/2) < \varepsilon$.

Since $C_0(H\backslash G) = \{$continuous functions on $H\backslash G$ with compact supports$\}$ is dense in $L^p(H\backslash G)$ $(p < \infty)$, the following Lemma 7 gives directly a proof of Proposition 1.

<u>Lemma 7.</u> For $\forall \varphi \in C_0(H\backslash G), \forall \varepsilon > 0, \forall p < +\infty$, there exists $f \in F_2$ such that $\|\varphi - f\|_p < \varepsilon$.

<u>Proof.</u> Put $C = [\varphi]$ (support of φ), $a = \mu(C)$ (measure of C) and $M = \|\varphi\|_\infty$. Using the regularity of Haar measure, there exists a relative compact open set U containing C such that $\mu(U) < 2a$. Moreover we can take a $\psi \in C_0(H\backslash G)$ such that $\psi(g) = 1$ for $g \in C$, and $= 0$ for $g \notin U$, $0 \leq \psi(g) \leq 1$ $\forall g \in G$.

By Corollary 3, take $f_1 \in F_2$ such that

$$\|\varphi - f_1\|_\infty < \rho < \mathrm{Min}(\, 1,\ \varepsilon(2^{p+1}a + 1)^{-1/p}).$$

Put $m = \int_{G-U} |f_1(g)|^p \, d g$, and $0 < \delta < \mathrm{Min}(\, 1,\ \rho/(M+\rho), \rho m^{-1/p})$. Again by Corollary 3, take $f_2 \in F_2$ such that $\|\psi - f_2\|_\infty < \delta$ and put $f = f_1 \cdot f_2$. Then $|\varphi(g) - f(g)| = |\varphi(g) - f_1(g) f_2(g)|$ is less than $|\varphi(g) - f_1(g)| + |1 - f_2(g)|\,|f_1(g)| < \rho + \delta(M + \rho) < 2\rho$ for $g \in C$,

$|f_1(g)|\,|f_2(g)| < \rho(\,|\psi(g)| + \delta\,) < \rho(\,1 + \delta\,) < 2\rho$ for $g \in U - C$,

$|f_1(g)|\,|f_2(g)| < |f_1(g)|\delta < \rho\, m^{-1/p} |f_1(g)|$ for $g \notin U$.

Thus $\|\varphi - f\|_p^{\,p} = \int_G |\varphi(g) - f(g)|^p \, d g = \int_C{}'' + \int_{U-C}{}'' + \int_{G-U}{}'' <$
$< 2^p \rho^p \mu(C) + 2^p \rho^p \mu(U-C) + \rho^p m^{-1} \int_{G-U} |f_1(g)|^p \, d g <$
$< (2^{p+1}a + 1)\rho^p \leq \varepsilon^p$.

4. Now we return to our problem concerning to the C*-system $\{\mathfrak{A}, G, \alpha\}$. We apply Proposition 1 twice, at first to the case

$F_0 = \mathcal{F}_0$ and second to the case $F_0 = \mathcal{G}_0$.

Lemma 8. In both cases, G_{F_0} (= K) are same one and compact normal subgroup of G.

Proof. If $F_0 = \mathcal{F}_0$, $K = G_{\mathcal{F}_0} = G_{\mathcal{F}}$, and if $F_0 = \mathcal{G}_0$, $K = G_{\mathcal{G}_0} = G_{\mathcal{G}}$, But by Lemma 1, $\mathcal{G} = \mathcal{F}$, thus $G_{\mathcal{F}} = G_{\mathcal{G}}$.

For $\forall k \in K$, $f_{AB}(kg) = f_{AB}(g)$, that is, for $\forall g \in G$ and $\forall A, B \in \mathfrak{A}_0$,

$$\langle U_k U_g \, \pi(A)\Omega, \, \pi(B^*)\Omega \rangle = \langle U_g \, \pi(A)\Omega, \, \pi(B^*)\Omega \rangle$$

Thus $U_k v = v$, for $\forall v \in \mathcal{H}$, This shows $U_k = I$, therefore K is the kernel of this representation, hence normal. [Assumption 3] and Lemma 3, Corollary 2 assure the compactness of K.

Based on Lemma 8, replacing the factor group $K \backslash G$ to G, hereafter we can assume $K = \{e\}$. Moreover we take $p_0 = [\max(p,q)]+1$ and replace $f_0^{p_0}$, $g_0^{p_0}$ to f_0, g_0 in Assumption 3. Thus we can assume that $f_0, g_0 \in L^1(G) \cap L^\infty(G)$.

Lemma 9. $\mathcal{G}_0 = \{\overline{f_1(g)} \; ; \; f_1 \in \mathcal{F}_0\} = \{f_1(g^{-1}) \; ; \; f_1 \in \mathcal{F}_0\}$.

Proof. Since $\mathfrak{A}_0$ is *-invariant, by (1) we obtain the result.

Proposition 1 leads us to the following lemma.

Lemma 10. The following spaces are dense in $L^p(G)$ $(1 \leq \forall p < \infty)$ and in $L_c^\infty(G)$.

$$\mathcal{F}_1 = \left\{ \sum_j^N f_j(R_{g'_j} f_0) \; ; \; N=1,2,\ldots, \; g'_j \in G, \; f_j \in \mathcal{F}_0 \oplus \mathbb{C}1 \right\},$$

$$\mathcal{G}_1 = \left\{ \sum_j^N g_j(R_{g'_j} g_0) \; ; \; N=1,2,\ldots, \; g'_j \in G, \; g_j \in \mathcal{G}_0 \oplus \mathbb{C}1 \right\}.$$

Now define a map S from $\mathcal{F}_1$ onto $\mathcal{G}_1$ by

$$S : \mathcal{F}_1 \ni \sum_k^N \prod_j^n f_{A_{j,k}B_{j,k}} \longmapsto \sum_k^N \prod_j^n g_{A_{j,k}B_{j,k}} \in \mathcal{G}_1.$$

Lemma 11. (i) The map S is welldefined. That is, for $\forall f_1 \in \mathcal{F}_1$, Sf_1 does not depend on the form $f_1 = \sum \prod f_{A_{j,k}B_{j,k}}$.

(ii) As a map defined on dense space in $L^p(G)$ (resp. $L_c^\infty(G)$, S is closable.

Proof. Summing up the relations (2) in [Assumption 2], we obtain for any f_1, f_2 in $\mathcal{F}_1$,

$$(3) \qquad \int_G f_1(g)f_2(g)\,dg = \int_G (Sf_1)(g)(Sf_2)(g)\,dg.$$

If f_2 runs over $\mathcal{F}_1$, Sf_2 runs over $\mathcal{G}_1$. Thus if $f_1 \equiv 0$, $\int_G (Sf_1)(g)k(g)\,dg = 0$ for $\forall k \in \mathcal{G}_1$. Because $\mathcal{G}_1$ is dense in $L^1(G)$, $Sf_1 \equiv 0$. This shows, S is welldefined.

Next if $f_1 \to 0$ and $Sf_1 \to f_3$ in $L^p(G)$ (resp. $L_c^\infty(G)$), since $\mathcal{F}_1 \subset L^q(G)$ $((1/p)+(1/q)=1)$ (resp. $L^1(G)$), the left hand side of (3) tends to zero, and the right hand side tends to $\int_G f_3(g)Sf_2(g)\,dg$ for any f_2 in $\mathcal{F}_1$. Again by the denseness of $\mathcal{G}_1 = \{Sf_2\ ;\ f_2 \in \mathcal{F}_1\}$ in $L^q(G)$ (resp. in $L^1(G)$), f_3 must be zero.

Corollary 4. For any f_1, $f_2 \in \mathcal{F}_1$,

$$(4) \qquad \langle Sf_1, \overline{Sf_2}\rangle = \langle f_1, \overline{f}_2\rangle .$$

Proof. A direct result of (3).

Let T_2 (resp. T_∞) be the closure of S as an orerator on $L^2(G)$ (resp. $L_c^\infty(G)$), and $D_2 \equiv D(T_2)$ (resp. $D_\infty = D(T_\infty)$) be the domains of T_2 (resp. T_∞).

Lemma 12. For $\forall \varphi \in D_2$, $\forall \psi \in D_\infty$, $\psi\cdot\varphi \in D_2$ and

$$T_2(\psi\cdot\varphi) = T_\infty(\psi)\cdot T_2(\varphi).$$

Proof. Let $\mathcal{F}_1 \ni f_j \to \varphi$, $Sf_j \to T_2(\varphi)$ in $L^2(G)$, and $\mathcal{F}_1 \ni k_j \to \psi$, $Sk_j \to T_\infty(\psi)$ in $L_c^\infty(G)$, then $\mathcal{F}_1 \ni (k_jf_j) \to \psi\cdot\varphi$, $(Sk_j)(Sf_j) \to T_\infty(\psi)T_2(\varphi)$ in $L^2(G)$. By the definition of S, $(Sk_j)(Sf_j) = S(k_jf_j)$ for $\forall k_j, f_j \in \mathcal{F}_1$. Thus we get the result.

Lemma 13. S commutes with right and left translations R_g, L_g.

(We use the notatins, $R_g f(g_1) = f(g_1 g)$ and $L_g f(g_1) = f(g^{-1} g_1)$.)

Proof. It is enough to show that S commutes with R_g, L_g on generators $\{f_{AB}\}$ of $\mathcal{F}_1$. And

$$(L_{g_1} R_{g_2} f_{AB})(g) = \omega(B\,\alpha_{g_1^{-1} g g_2}(A)) - \omega(A)\omega(B)$$
$$= \omega(\alpha_{g_1}(B)\,\alpha_g(\alpha_{g_2}(A))) - \omega(\alpha_{g_2}(A))\omega(\alpha_{g_1}(B))$$
$$= f_{\alpha_{g_2}(A),\,\alpha_{g_1}(B)}(g),$$

in just same way

$(L_{g_1} R_{g_2} g_{AB})(g) = g_{\alpha_{g_2}(A)\,\alpha_{g_1}(B)}(g)$. Therefore

$$S(L_{g_1} R_{g_2} f_{AB})(g) = S(f_{\alpha_{g_2}(A)\,\alpha_{g_1}(B)})(g) = g_{\alpha_{g_2}(A)\alpha_{g_1}(B)}(g)$$
$$= (L_{g_1} R_{g_2} g_{AB})(g) = (L_{g_1} R_{g_2} S f_{AB})(g).$$

Lemma 14. For $\forall \varphi \in D_2$, $\forall \psi \in L^1(G) \wedge L^\infty_c(G)$, the function $\langle R_g \varphi, \psi \rangle$ is in D_∞ and

(5) $$T_\infty(\langle R_g \varphi, \psi \rangle) = \langle R_g T_2 \varphi, \psi \rangle \ .$$

Proof. For $\forall f \in \mathcal{F}_1$,

$$\langle R_g f, \psi \rangle = \int_G f(g_1 g)\overline{\psi(g_1)}\, d\, g_1 = \lim \sum_j^N f(g_j g)\overline{\psi(g_j)}\, |\Delta_j|$$
$$= \lim \sum_j^N (L_{g_j^{-1}} f)(g)\overline{\psi(g_j)}\, |\Delta_j| .$$

Because of uniform continuity of f, ψ and integrability in our case, this integral converges uniformly in $g \in G$. Moreover

$$S(\sum (L_{g_j^{-1}} f)(g)\overline{\psi(g_j)}\, |\Delta_j|) = \sum (L_{g_j^{-1}}(Sf))(g)\overline{\psi(g_j)}\, |\Delta_j| \ .$$

Thus $\sum f(g_j g)\overline{\psi(g_j)}\, |\Delta_j|$ and $S(\sum f(g_j g)\overline{\psi(g_j)}\, |\Delta_j|)$ converge to $\langle R_g f, \psi \rangle$ and $\langle R_g Sf, \psi \rangle$ in $L^\infty_c(G)$ respectively. This shows the results for such a f.

Next for $\forall \varphi \in D_2$, let $\mathcal{F}_1 \ni f_j \to \varphi$, $Sf_j \to T_2 \varphi$ in $L^2(G)$, then $\langle R_g f_j, \psi \rangle$ and $\langle R_g Sf_j, \psi \rangle$ converge to $\langle R_g \varphi, \psi \rangle$ and

$\langle R_g T_2 \varphi, \psi \rangle$ in $L_c^\infty(G)$ respectively. That is, the proof is obtained.

Corollary 5. For $\forall \varphi \in \mathfrak{F}_1$, $\forall \psi \in L^1(G) \cap L_c^\infty(G)$,

$\langle R_{g^{-1}} \psi, \overline{\varphi} \rangle \in D_\infty$, and $T_\infty(\langle R_{g^{-1}} \psi, \overline{\varphi} \rangle) = \langle R_g S \varphi, \overline{\psi} \rangle$

(Here $\overline{\varphi}$, $\overline{\psi}$ show the complex conjugations of φ, ψ respectively.)

Proof. Indeed, $\langle R_{g^{-1}} \psi, \overline{\varphi} \rangle = \overline{\langle R_g \overline{\varphi}, \psi \rangle} = \langle R_g \varphi, \overline{\psi} \rangle$.

From assumptions, $\varphi \in \mathfrak{F}_1$ and $\overline{\psi} \in L^1(g) \cap L_c^\infty(G)$, so lemma 14 leads us to the results.

5. Now we have to discuss the Katz-Takesaki operator on G, and the relation to the above operator T_2. We define a unitary operator on $L^2(G) \otimes L^2(G)$ (called the Katz-Takesaki operator) by

(4) $W(f_1 \otimes f_2)(g_1, g_2) = f_1(g_1 g_2) f_2(g_2)$.

This operator is closely related with duality theorem as follows.

Proposition 2. The operators $U \equiv R_g$ of the right regular representation of G, satisfy

(5) $W(U \otimes U) = (I \otimes U)W$.

And conversely, for any non-zero bounded operator U satisfying (5), there exists unique element g in G such that $R_g = U$.

For the proof of Proposition 2 , we refer [ ].

However for our discussion, we don't need this proposition directly, but the following which is deduced from it.

Proposition 3. For any closed operator T on $L^2(G)$ such that

(6) $W(T \otimes T) = (I \otimes T)W$,

there exist an element g_0 in G and an one parameter subgroup $g(t)$ of G with infinitesimal generator iH , such that

(7) $$T = g_0 \, e^{H} ,$$

(Here we denote the closure of algebraic tensor product of two closed operators A and B on $L^2(G)$ by $A \otimes B$.)

<u>Proof.</u> Put $T^*T = A$, then A is a self-adjoint positive definite operator satisfying

(8) $$W(A \otimes A) = (I \otimes A)W .$$

Consider the projection P onto the space $\mathcal{H} = (A^{-1}(0)) = \overline{\mathrm{Range}(A)}$, then by (8) $P \neq 0$, and

(9) $$W(P \otimes P) = (I \otimes P)W .$$

Proposition 2 assures that P is unitary , therefore $P = I$. That is $\mathcal{H} = L^2(G)$, and we can define The self-adjoint **operator** $H = (1/2)\log A$ satisfying

(10) $$W(H \otimes I + I \otimes H) = (I \otimes H)W .$$

Direct calculations show that for $\forall t \in \mathbf{R}$, $U(t) = e^{iHt}$ is a bounded operator in Proposition 2. Hence we obtain an one-parameter subgroup $g(t)$ in G and

(11) $$U(t) = R_{g(t)} \qquad \text{for } \forall t \in \mathbb{R} .$$

On the other hand, the **bounded** operator $Te^{-H} = U$ satisfies (**5**) too. Again Proposition 2 gives an element g_0 in G such that $R_{g_0} = U$. This completes the proof.

We shall call that these operators given in Proposition 3 admissible. In after propositions, we show that the our operator T_2 is admissible.

At first we must remark the following.

<u>Lemma 15.</u> Using any fixed complete orthonormal system $\{\varphi_\alpha\}$ in $L^2(G)$, the Katz-Takesaki operator is expanded as follows.

(12) $\quad W(f_1\otimes f_2)(g_1,g_2) = \sum_\alpha \varphi_\alpha(g_1) < R_{g_2}f_1, \varphi_\alpha > f_2(g_2)$.

Proof. By only calculation of the expansion.

Lemma 16. $W(\mathcal{F}_1\otimes\mathcal{F}_1)$ is in the domain of $I\otimes T_2$ and

(13) $\quad (I\otimes T_2)W(f_1\otimes f_2) = W(Sf_1\otimes Sf_2)$ for $\forall f_1,f_2\in\mathcal{F}_1$.

Proof. By Schmidt's orthogonalization, we can take all φ_α's in (12) from $L^1(G)\cap L_c^\infty(G)$. Then by Lemmata 12 and 14, the function $\varphi_\alpha(g_1)<R_{g_2}f_1, \varphi_\alpha>f_2(g_2)$ are in $D(I)\otimes D(T_2)\subset D(I\otimes T_2)$ (The domain of $I\otimes T_2$), and

(14) $\quad (I\otimes T_2)(\varphi_\alpha(g_1)<R_{g_2}f_1, \varphi_\alpha>f_2(g_2)) =$

$\quad = \varphi_\alpha(g_1)T_\infty(<R_{g_2}f_1, \varphi_\alpha>)(T_2f_2)(g_2)$

$\quad = \varphi_\alpha(g_1)<R_{g_2}T_2f_1, \varphi_\alpha>(T_2f_2)(g_2)$.

Moreover, $\sum^N \varphi_\alpha(g_1)<R_{g_2}f_1, \varphi_\alpha>f_2(g_2)$ and $(I\otimes T_2)\sum^N(\varphi_\alpha(g_1)<R_{g_2}f_1, \varphi_\alpha>f_2(g_2)) = \sum^N\varphi_\alpha(g_1)<R_{g_2}T_2f_1, \varphi_\alpha>\times (T_2f_2)(g_2)$ converge to $W(f_1\otimes f_2)(g_1,g_2)$ and $W(Sf_1\otimes Sf_2)$ in $L^2(G)\otimes L^2(G)$ respectively. This gives the results.

Lemma 17. $W^{-1}(\mathcal{F}_1\otimes\mathcal{F}_1)$ is in the domain of $T_2\otimes T_2$ and

(15) $\quad (T_2\otimes T_2)W^{-1}(f_1\otimes f_2) = W^{-1}(I\otimes S)(f_1\otimes f_2)$ for $\forall f_1,f_2\in\mathcal{F}_1$.

Proof. W^{-1} is given by $W^{-1}(f_1\otimes f_2)(g_1,g_2) = f_1(g_1g_2^{-1})f_2(g_2)$. Thus for $\forall f_1,f_2\in\mathcal{F}_1$, $(S\otimes I)W^{-1}(f_1\otimes f_2)(g_1,g_2) = (S\otimes I)(R_{g_2^{-1}}f_1(g_1)\times f_2(g_2)) = (SR_{g_2^{-1}}f_1)(g_1)f_2(g_2) = (R_{g_2^{-1}}Sf_1)(g_1)f_2(g_2) =$

$\quad = (Sf_1)(g_1g_2^{-1})f_2(g_2) = \sum\varphi_\alpha(g_1)<R_{g_2^{-1}}(Sf_1), \varphi_\alpha>f_2(g_2)$.

This shows that $W^{-1}(\mathcal{F}_1\otimes\mathcal{F}_1)$ is in $D(\overline{S\otimes I}) = D(T_2\otimes I)$.

Next we shall show that $(S\otimes I)W^{-1}(\mathcal{F}_1\otimes\mathcal{F}_1)$ is in $D(I\otimes T_2)$. Indeed by Corollary 4 and the fact $Sf_1\in\mathcal{G}_1\subset L^1(G)\cap L_c^\infty(G)$, if we select the C.O.N.S $\{\varphi_\alpha\}$ as $\varphi_\alpha\in\mathcal{F}_1$, $<R_{g^{-1}}(Sf_1), \varphi_\alpha>\in D_\infty$ and hence $\varphi_\alpha(g_1)<R_{g_2^{-1}}(Sf_1), \varphi_\alpha>f_2(g_2)\in D(I\otimes T_2)$.

Using (4), $(I\otimes T_2)(\varphi_\alpha(g_1)\langle R_{g_2^{-1}}(Sf_1),\ \varphi_\alpha\rangle f_2(g_2)) =$

$\varphi_\alpha(g_1)\langle R_{g_2} S\overline{\varphi_\alpha}, \overline{Sf_1}\rangle(Sf_2)(g_2) = \varphi_\alpha(g_1)\langle S(\overline{R_{g_2}\varphi_\alpha}), \overline{Sf_1}\rangle(Sf_2)(g_2) =$

$= \varphi_\alpha(g_1)\langle R_{g_2}\overline{\varphi_\alpha}, \overline{f_1}\rangle(Sf_2)(g_2) = \varphi_\alpha(g_1)\langle R_{g_2^{-1}}f_1,\ \varphi_\alpha\rangle(Sf_2)(g_2)$.

Obviously $\sum^N \varphi_\alpha(g_1)\langle R_{g_2^{-1}}(Sf_1), \varphi_\alpha\rangle f_2(g_2)$ and

$\sum^N \varphi_\alpha(g_1)\langle R_{g_2^{-1}}f_1, \varphi_\alpha\rangle (Sf_2)(g_2)$ $(=(I\otimes T_2)(\sum^N \varphi_\alpha(g_1)\langle R_{g_2^{-1}}(Sf_1), \varphi_\alpha\rangle\times$ $\times f_2(g_2)))$ converge to $(S\otimes I)W^{-1}(f_1\otimes f_2)$ and $W^{-1}(f_1\otimes Sf_2)$ in $L^2(G)\otimes L^2(G)$ respectively. That is, $(S\otimes I)W^{-1}(f_1\otimes f_2)\in D(I\otimes T_2)$ and $(I\otimes T_2)(S\otimes I)W^{-1}(f_1\otimes f_2) = W^{-1}(f_1\otimes Sf_2)$. Combining these, we get the wanted results.

Summalizing Lemmata 16 and 17, we conclude,

<u>Proposition 4.</u> The closed operator T_2 is admissible.

Now we are in the step to apply Proposition 3 together with Lemma 13 to our operator T_2, and get,

<u>Lemma 18.</u> There exist an element g_0 with order 1 or 2 and an one-parameter subgroup $g(t)$ in the centre $Z(G)$ of G such that $T_2 = R_{g_0} e^H$, Here iH is the infinitesimal generator of $R_{g(t)}$.

<u>Proof.</u> The existence of g_0 and $g(t)$ are an direct results of the above arguments, so we must show that g_0 and $g(t)$ are in $Z(G)$ and the order of g_0 is atmost two.

But because (7) gives the polar decomposition of T_2, and by Lemma 13, T_2 hence R_{g_0} and e^H must commute with R_g ($\forall g\in G$). The relation (3) and the definitions of $\mathcal{F}_1$, $\mathcal{G}_1$ and S give $\langle f_1, Sf_2\rangle = \langle Sf_1, f_2\rangle$ for $f_1, f_2\in \mathcal{F}_1$. This concludes T_2 is symmetric. But since R_{g_0} is unitary and e^H is positive definite without kernel, R_{g_0} must be the form $P-(I-P)$ for some projection P.

Hence $(R_{g_0})^2 = I$, and $g_0{}^2 = e$.

The assertion of Lemma 18 talks about only operators on $L^2(G)$. However using [Assumption 2], we can extend this to the whole space as follows. That is, consider the operators on $\mathcal{H}$,

$H_0 \equiv (1/i)(d/dt)U_{g(t)}\big|_{t=0}$, $V \equiv e^{H_0}$ and $T^1 \equiv U_{g_0}V$ in which g_0 ,$g(t)$ are elements of G given in Lemma 18.

Lemma 19. $\langle \pi(B)\Omega, U_g\, \pi(A^*)\Omega\rangle = \langle U_{gg_0} V^{1/2}\pi(A)\Omega, V^{1/2}\pi(B^*)\Omega\rangle$ for $\forall A, B \in \mathfrak{A}_0$.

Proof. Let $\varphi(t) \equiv e^{-t^2}$ and for $c \in (0,\infty)$ and $A \in \mathfrak{A}$,

$$A_{\varphi,c} \equiv (2c/\sqrt{\pi}) \int_{-\infty}^{\infty} \alpha_{g(t)}(A)\, \varphi(ct)\, dt \; .$$

Then it is easy to see $\pi(A_{\varphi,c})\Omega \in D(T^1)$ and $A_{\varphi,c} \xrightarrow{c\to\infty} A$ in $\mathfrak{A}$, hence $\pi(A_{\varphi,c})\Omega \to \pi(A)\Omega$ and $\pi(A^*_{\varphi,c})\Omega \to \pi(A^*)\Omega$ in $\mathcal{H}$. Denote $\mathfrak{A}_1 \equiv \{A_{\varphi,c}\,;\, c\in(0,\infty),\, A\in\mathfrak{A}_0\}$. Then direct calculations lead us to

$$R_{g_0}\Big(\sum_n \frac{1}{n!}\Big(\frac{1}{i}\frac{d}{dt}\Big)^n \big(R_{g(t)}f\big)\Big|_{t=0}\Big)(g) = \langle U_g T^1 \pi(A)\Omega, \pi(B^*)\Omega\rangle - \langle \pi(A)\Omega, \Omega\rangle\langle \pi(B)\Omega, \Omega\rangle \; .$$

Since the convergences of $H_0\pi(A)\Omega = (1/i)\lim_{t\to 0} t^{-1}(U_{g(t)}\pi(A)\Omega - \pi(A)\Omega)$ and $V\pi(A)\Omega = \sum_n (1/n!)H_0{}^n \pi(A)\Omega$ are in norm sense, the convergence of the left hand side is uniform in g. Generally $\mathfrak{A}_1$ is not contained in $\mathfrak{A}_0$, but all elements of $\mathfrak{A}_1$ are norm limits of elements of $\mathfrak{A}_0$ and vice versa. Hence $f_{A,B}$, $g_{A,B}$ $(A, B \in \mathfrak{A}_0 \cup \mathfrak{A}_1)$ are uniform limits of f_{A_j,B_j}, g_{A_j,B_j} $(A_j, B_j \in \mathfrak{A}_0)$ And by (2)

(16) $$\int (f_{AB}(g) f_1(g)) f_2(g)\, dg = \int (g_{AB}(g) Sf_1(g)) Sf_2(g)\, dg$$ for $\forall A, B \in \mathfrak{A}_1 \cup \mathfrak{A}_0$, $\forall f_1, f_2 \in \mathcal{F}_1$.

Now $\mathcal{F}_1 \ni f_{A_jB_j}f_1 \to f_{AB}f_1$, $T_2(f_{A_jB_j}f_1)=(Sf_{A_jB_j})(Sf_1)=g_{A_jB_j}(Sf_1)$
$\to g_{AB}(Sf_1)$ in $L^2(G)$, therefore $f_{AB}f_1 \in D_2$ and $T_2(f_{AB}f_1)=g_{AB}Sf_1$
for $\forall A,B \in \mathfrak{A}_1 \cup \mathfrak{A}_0$. And Lemma 18 assures $T_2(f_{AB}f_1)=R_{g_0}(\sum_n (1/n!)$
$(-i)^n(d/dt)^n(R_{g(t)}f_{AB}))(Sf_1)$ for $\forall A \in \mathfrak{A}_1$, thus $g_{AB}=R_{g_0}(\sum_n (1/n!)$
$(-i)^n(d/dt)^n(R_{g(t)}f_{AB}))$ (converges in $L^\infty(G)$). Therefore
$\langle U_gT^1\pi(A)\Omega, \pi(B^*)\Omega\rangle = \langle \pi(B)\Omega, U_g\pi(A^*)\Omega\rangle$ for $\forall A \in \mathfrak{A}_1$,
$\forall B \in \mathfrak{A}$. Especially for $A,B \in \mathfrak{A}_0$, $A_{\varphi,c}B_{\varphi,c} \in \mathfrak{A}_1$, hence
$\langle U_{gg_0}V^{1/2}\pi(A_{\varphi,c_1})\Omega, V^{1/2}\pi(B^*_{\varphi,c_2})\Omega\rangle = \langle\pi(B_{\varphi,c_2})\Omega, U_g\pi(A^*_{\varphi,c_1})\rangle$.
Put $g = g_0^{-1}$. When c tends to ∞, $A_{\varphi,c} \to A$ in $\mathfrak{A}$, $A^*_{\varphi,c} \to A^*$,
$\pi(A_{\varphi,c})\Omega \to \pi(A)\Omega$, and $\pi(A^*_{\varphi,c})\Omega \to \pi(A^*)\Omega$. Taking the limit,
we obtain $\lim_{\substack{c_1 \to \infty \\ c_2}} \langle V^{1/2}\pi(A_{\varphi,c_1})\Omega, V^{1/2}\pi(B^*_{\varphi,c_2})\rangle =$

$$= \langle \pi(B)\Omega, U_{g_0^{-1}}\pi(A^*)\Omega\rangle.$$

Hence

$$\lim_{\substack{c_1 \to \infty \\ c_2}} \|V^{1/2}\pi(A_{\varphi,c_1})\Omega - V^{1/2}\pi(A_{\varphi,c_2})\Omega\|^2 = \lim \sum_{i,j=1,2} (-1)^{i+j} \times$$

$$\times \langle V^{1/2}\pi(A_{\varphi,c_i})\Omega, V^{1/2}\pi(A_{\varphi,c_j})\Omega\rangle =$$

$$= \sum_{i,j=1,2} (-1)^{i+j} \langle \pi(A)\Omega, U_{g_0}^{-1}\pi(A)\Omega\rangle = 0.$$

That is, $\{V^{1/2}\pi(A_{\varphi,c})\Omega\}_{c\to\infty}$ is a Cauchy sequence in $\mathcal{H}$, so
$\pi(A)\Omega \in D(V^{1/2})$, $V^{1/2}\pi(A)\Omega = \lim_{c\to\infty} V^{1/2}\pi(A_{\varphi,c})\Omega$, and
$\langle U_{gg_0}V^{1/2}\pi(A)\Omega, V^{1/2}\pi(B^*)\Omega\rangle =$

$$= \lim_{c\to\infty} \langle U_{gg_0}V^{1/2}\pi(A_{\varphi,c})\Omega, V^{1/2}\pi(B^*_{\varphi,c})\Omega\rangle$$

$$= \langle \pi(B)\Omega, U_g\pi(A^*)\Omega\rangle = g_{AB}(g).$$

$\mathfrak{A}_0$ is norm dense in $\mathfrak{A}$, therefore $\mathfrak{A}_1$ is norm dense in $\mathfrak{A}$, and $\{\pi(A)\Omega\ ;\ A \in \mathfrak{A}_1\}$ is dense in $\mathcal{H}$ too. Thus,

<u>Corollary 6.</u> There exists a norm dense subalgebra $\mathfrak{A}_1$ in $\mathfrak{A}$, and a closed operator T^1 on $\mathcal{H}$ such that $\{\pi(A)\Omega;\ A\in\mathfrak{A}_1\} \subset D(T^1)$,

and $\langle U_g T^1 \pi(A)\Omega, \pi(B^*)\Omega\rangle = \langle \pi(B)\Omega, U_g \pi(A^*)\Omega\rangle$

for $\forall g \in G$, $\forall A \in \mathfrak{A}_1$, $\forall B \in \mathfrak{A}$.

Lemma 20. In Lemma 18 , the element g_0 is equal to e.

Proof. Consider two positive definite functions

$\psi_1(g) = \langle U_g \pi(B)\Omega, \pi(B)\Omega\rangle$,

$\psi_2(g) = \langle U_g V^{1/2}\pi(B^*)\Omega, V^{1/2}\pi(B^*)\Omega\rangle$.

In Lemma 19, putting $A = B^*$, we obtain

(17) $\psi_1(g) = \psi_2(gg_0)$ for $\forall g \in G$.

But, $\psi_1(g_0^{-1}) = \psi_2(e) = \|\psi_2\|_\infty = \|\psi_1\|_\infty = \psi_1(e)$, therefore $\langle U_{g_0} \pi(B)\Omega, \pi(B)\Omega\rangle = \|\pi(B)\Omega\|^2$ and $U_{g_0}\pi(B)\Omega = \pi(B)\Omega$ for $\forall B \in \mathfrak{A}_0$. That is $U_{g_0} = I$, hence g_0 is in $K = \{e\}$.

Thus the results of Lemmata 18 - 20 give a proof of our Main theorem.

[Acknowledgement] . Professors H.Araki, D.Kastler, M.Takesaki gave to the author their idea to use general duality theorem for proving the KMS-property of a C*-system. The author wishes to express his thanks to thire hospitality and kind discussion, especially to Prof. D. Kastler who gave the opportunity of discussion in Ellmau.

Reference

[1] N. Tatsuuma : A duality theorem for locally compact groups, J. Math. Kyoto Univ. , 6 (1967), 187 - 293.