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Two-point boundary-value problems
with a discontinuous semilinear term

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1. Introduction

In this paper we consider the equation

\begin{equation}
\begin{cases}
\frac{d^2 v}{dx^2} + g(v) = 0 & \text{on } I = (0, L)
\\
\frac{dv}{dx}(0) = \frac{dv}{dx}(L) = 0,
\end{cases}
\end{equation}

(1.1)

where \( g \) is a function with a discontinuity point of the first kind. Our purpose is to present all the solutions of (1.1) under some appropriate conditions on \( g \) and \( v \). At the same time we show that the cardinality of the set of all the solutions is \( \aleph_0 \) (countably infinite).

The equation (1.1) appears in the following situation. Consider the degenerate parabolic system of \( u(x,t) \) and \( v(x,t) \),

\begin{equation}
\begin{cases}
\frac{\partial u}{\partial t} = f(u,v) & \text{in } I \times (0, +\infty)
\\
\frac{\partial v}{\partial t} = \frac{d^2 v}{dx^2} + g(u,v) & \text{in } I \times (0, +\infty)
\\
\frac{\partial v}{\partial x}(0,t) = \frac{\partial v}{\partial x}(L,t) = 0 & t > 0
\\
\text{initial conditions.}
\end{cases}
\end{equation}

(1.2)

In a biological point of view, it may be considered that \( u \) (resp. \( v \)) represents the density of a plant (resp. a herbivore) for example. Consider steady-state solutions of (1.2). Then the first equation of (1.2) reduces an algebraic equation. Hence, \( u \) can be expressed as a multi-valued function of \( v \). If we fix \( v_s \) and assign each of two different parts of the multi-valued function on each side of \( v_s \), \( u \) becomes a function of \( v \) with discontinuity at \( v_s \). Substituting the function \( u \) of \( v \) into the second
equation of (1.2) and rewriting the composed function with the same symbol \( g \), we obtain the equation (1.1). The existence of such a steady-state solution is recognized by some numerical experiments. (See Mimura [1]. Mimura also proved that stable steady-state solutions of (1.2) have a unique \( v_* \) determined by \( f \) and \( g \) under an appropriate assumption.)

2. Presentation of results

The semilinear term \( g \) we deal with in this paper is restricted to the one satisfying the following condition.

Condition 1.

(i) \( g \) is a function defined on \((v_1, v_2)\) with a discontinuity point of the first kind \( v_* \) and satisfies
\[-\infty < g(v_* - 0) < 0 < g(v_* + 0) < +\infty.\]

(ii) \( g < 0 \) in \((v_1, v_*), v_2 \) in \((v_*, v_2)\), and \( g \) is Lipschitz continuous in \((v_1, v_* - 0)\) and \((v_* + 0, v_2)\). (i.e., \( g \) is Lipschitz continuous in \([v_1 + \varepsilon, v_*]\) for any \( \varepsilon > 0 \), where \( g = g(v) \) for \( v \in (v_1, v_*) \) and \( g(v_* - 0) \) at \( v = v_* \). Similarly \((v_* + 0, v_2)\) is considered.)

(iii) \( g \) is monotone decreasing in \((v_1, v_*)\) and \((v_*, v_2)\).

We rewrite (1.1) by the weak form:

\[
\begin{align*}
(2.1) \quad \begin{cases} 
\text{Find } v \in H^1(I) \text{ such that} \\
( dv/\partial x, d\phi/\partial x ) = ( g(v), \phi ) \\
\text{for all } \phi \in H^1(I),
\end{cases}
\end{align*}
\]

where \((,\,\)\) is the inner product in \( L^2(I) \). Our aim is to find all the solutions of (2.1) satisfying

\[
(2.2) \quad v_1 < v(x) < v_2 \quad (0 < x < l_0).
\]

Remark 1. (2.2) makes sense since \( v \in H^1(I) \) implies \( v \in C(I) \) by Sobolev's lemma. Furthermore, for the solution \( v \) of (2.1), we have \( v \in C(I) \), since Condition 1 gives \( g(v) \in L^2(I) \) which yields \( v \in H^2(I) \).

Theorem 1. In the case \( g(v_*) \neq 0 \), all the solutions of (2.1) and (2.2)
are $v_n^i$, $i=1, 2$ and $n \geq n_0$, where $v_n^i(x)$ are functions defined in the subsequent section (see (3.9)) and $n_0$ is a positive integer depending on $g$ and $x_0$.

In the case $g(v_*) = 0$, $v_*$ is added to the solutions.

Remark 2. If $g(v_1^* + 0) = g(v_2^* - 0) = 0$, and if $g$ is Lipschitz continuous in $[v_1^* + 0, v_2^* - 0]$ and $[v_2^* + 0, v_2^* - 0]$, then $n_0 = 1$ for any $x_0 > 0$.

Remark 2 as well as Theorem 1 is proved at the next section.

3. Proof of Theorem 1.

Let $v(x; c)$ be a solution of the initial value problem:

$$\begin{cases}
  d^2v/dx^2 = -\tilde{g}(v) & (x > 0) \\
  dv/dx(0) = 0 \\
  v(0) = c,
\end{cases}$$

where $c > v_1$ and

$$\tilde{g}(v) = \begin{cases}
  g(v) & (v_1 < v < v_*) \\
  g(v_* - 0) & (v_* \leq v).
\end{cases}$$

Such $v(x; c)$ exists uniquely since $\tilde{g}$ is Lipschitz continuous in $[c - \varepsilon, +\infty)$ for some $\varepsilon > 0$ where any solution of (3.1) lies since $\tilde{g} < 0$.

Lemma 1. Let $v(x; c)$ be as above. Then, we have

(3.2) $v(x; c_1) > v(x; c_2)$ 

for $v_1 < c_1 < c_2$.

Proof. Let $x_0 > 0$ be the first intersecting point of $v(x; c_1)$ and $v(x; c_2)$.

Obviously it holds that

$$dv/dx(x_0; c_1) \geq dv/dx(x_0; c_2).$$

On the other hand, by (iii) of Condition 1, we have

$$d^2v/dx^2(x; c_1) = \tilde{g}(v(x; c_1)) \leq \tilde{g}(v(x; c_2)) = d^2v/dx^2(x; c_2) \quad (0 < x < x_0).$$

Integrating both sides from 0 to $x_0$, we have

$$dv/dx(x_0; c_1) \leq dv/dx(x_0; c_2).$$

Hence we obtain
\[(d/dx)^i v(x_0; \sigma_1) = (d/dx)^i v(x_0; \sigma_2) \quad (i = 0, 1).\]

By the uniqueness of the initial value problem, we have
\[v(x; \sigma_1) = v(x; \sigma_2) \quad (x \geq 0).\]

This is a contradiction. Hence we obtain (3.2). Q.E.D.

Define a mapping \(\phi\) from \((v_1, v_\ast)\) into \((0, +\infty)\) by
\[v(\phi(\sigma); \sigma) = v_\ast.\]

Since \(d^2v/dx^2(x; \sigma) \geq -g(c) > 0\), \(\phi(c)\) is well-defined. By Lemma 1, \(\phi(c)\) is monotone decreasing. Set
\[\bar{c} = \lim_{\sigma \to v_1} \phi(\sigma).\]

Lemma 2. \(\phi\) is homeomorphic from \((v_1, v_\ast)\) onto \((0, \bar{c})\) and strictly decreasing.

Proof. \(\phi\) is strictly decreasing and therefore injective by Lemma 1. We show \(\phi\) is continuous. Let \(\{c_j\}\) be any sequence in \((v_1, v_\ast)\) converging to \(c \in (v_1, v_\ast)\). It is sufficient for us to show that there exists a subsequence \(\{c_{j_k}\}\) such that \(\phi(c_{j_k})\) converges to \(\phi(c)\). Let \(\tilde{c}\) (resp. \(\underline{c}\)) be the supremum (resp. infimum) of \(\{c_j\}\). Since \(\phi(c_j) \in [\phi(\tilde{c}), \phi(\underline{c})]\), there exists a subsequence \(\{c_{j_k}\}\) which converges to some \(\tilde{c} \in [\phi(\tilde{c}), \phi(\underline{c})]\). Then it holds that
\[|v(d_j; \sigma) - v_\ast| \leq |v(d_j; \sigma) - v(\phi(c_{j_k}); \sigma)| + |v(\phi(c_{j_k}); \sigma) - v(\phi(c_{j_k}); \sigma)| + \max_{0 \leq x \leq \phi(\underline{c})} |v(x; \sigma) - v(x; \sigma)| \to 0 \quad \text{as } j_k \to +\infty,
\]

since \(v(x; \sigma)\) is continuous and \(\tilde{g}\) is Lipschitz continuous in \([\underline{c}, v_\ast - 0]\).

We show \(\phi\) is surjective. It is sufficient for us to show that
\[\phi(\sigma) \to 0 \quad \text{as } \sigma \to v_\ast.\]

Suppose that \(\lim_{\sigma \to v_\ast} \phi(\sigma) = d > 0\). Fix \(x_0 \in (0, d)\). Then it holds that
\[v(x_0; \sigma) \to v(x_0; v_\ast) > v_\ast \quad \text{as } \sigma \to v_\ast,\]
since \( v(x; c) \) converges to \( v(x; v_\ast) \) uniformly in \([0, d]\). On the other hand, by

\[ v(x_0; c) < v(d; c) < v_\ast, \]

we have

\[ \limsup_{\sigma \to v_\ast} v(x_0; \sigma) \leq v_\ast. \]

This is a contradiction. Hence, \( d = 0 \).

That \( \phi^{-1} \) is continuous is easily proved. Q.E.D.

For any \( \ell \in (0, 1) \), \( v(x; \phi^{-1}(\ell)) \) is a unique solution of

\[
\begin{cases}
  \frac{d^2 v}{dx^2} = -g(v) & (0 < x < \ell) \\
  \frac{dv}{dx}(0) = 0 \\
  v(\ell) = v_\ast
\end{cases}
\]

(3.3)

satisfying

\[
v_\ell < v(x) < v_\ast & (0 \leq x \leq \ell).
\]

(3.4)

Uniqueness is proved easily by making use of Lemma 1. Define a mapping \( a \) from

\((0, 1) \) into \((0, +\infty) \) by

\[
a(\ell) = \frac{dv}{dx}(\ell; \phi^{-1}(\ell)).
\]

Lemma 3. \( a \) is strictly increasing and homeomorphic from \((0, 1) \) onto

\((0, \bar{a}) \) and satisfies

\[
0 < a(\ell) \leq -g(v_\ast - 0) \ell & (0 < \ell \leq 1),
\]

(3.5)

where \( \bar{a} = \limsup_{\ell \to 1} a(\ell) \).

Proof. We first show that \( a \) is strictly increasing. Fix \( \ell_1, \ell_2, \) such that

\[
0 < \ell_1 < \ell_2 < 1 \quad \text{and} \quad \ell_2 < 2\ell_1.
\]

Set for \( s \in [0, \ell_2] \)

\[
w_i(s) = v(\ell_i - s; \phi^{-1}(\ell_i)) \quad \text{and} \quad w_i(s) = v(\ell_2 - s; \phi^{-1}(\ell_2)),
\]

where \( v(x; \phi^{-1}(\ell)) \) is assumed to be extended in \( x < 0 \) to the even function.

Then \( w_i, i = 1, 2, \) satisfies

\[
\begin{cases}
  \frac{d^2 w_i}{ds^2} = -g(w_i(s)) & (0 < s < \ell_i) \\
  w_i(0) = v_\ast
\end{cases}
\]

(3.6)
\[
\frac{d\omega_2}{ds}(0) = -a(t_2').
\]
For the purpose of an indirect proof, assume that \(a(t_1) \geq a(t_2)\). If \(a(t_1) = a(t_2)\), (3.6) implies that \(v_1(s) = v_2(s)\) \((0 \leq s \leq t_2)\), which contradicts that
\[
v_1(t_2') > \phi^{-1}(t_1) > \phi^{-1}(t_2') = v_2(t_2').
\]
Assume \(a(t_1) > a(t_2)\). Let \(s_0\) be the first intersecting point of \(v_1\) and \(v_2\). Obviously it holds that
\[
\frac{d\omega_1}{ds}(s_0') \geq \frac{d\omega_2}{ds}(s_0).
\]
On the other hand, we have
\[
d^2\omega_2/ds^2(s) = -g(\omega_2(s)) = -g(\omega_1(s)) = d^2\omega_1/ds^2(s) \quad (0 \leq s \leq s_0),
\]
so \(v_2 \geq v_1\) on \([0, s_0]\) and \(g\) is monotone decreasing. Hence we obtain
\[
\frac{d\omega_2}{ds}(s_0) = -a(t_2) + \int_0^{s_0} d^2\omega_2/ds^2 \, ds
\]
\[
\geq -a(t_2) + \int_0^{s_0} d^2\omega_1/ds^2 \, ds
\]
\[
> \frac{d\omega_1}{ds}(s_0),
\]
which is a contradiction. Therefore we get \(a(t_1) < a(t_2)\), which also shows \(a\) is injective.

(3.5) is proved easily by making use of
\[
a(t) = \int_0^t d^2v/\xi^2(x; t; \phi^{-1}(t)) \, dx
\]
\[
= -\int_0^t g(v(x; \phi^{-1}(t)) \, dx.
\]
We next show that \(a\) is continuous. We have
\[
|a(t_1) - a(t_2)| \leq |d\omega/\xi_2(t_1; \phi^{-1}(t_1)) - d\omega/\xi_2(t_2; \phi^{-1}(t_2))|
\]
\[
+ |d\omega/\xi_2(t_2; \phi^{-1}(t_2)) - d\omega/\xi_2(t_2; \phi^{-1}(t_2'))|
\]
\[
\leq |d\omega/\xi_2(t_1; \phi^{-1}(t_1)) - d\omega/\xi_2(t_2; \phi^{-1}(t_2))|
\]
\[
+ \max_{0 \leq t \leq t_1+\varepsilon} |d\omega/\xi_2(t; \phi^{-1}(t)) - d\omega/\xi_2(t; \phi^{-1}(t_2))|,
\]
for \(0 < t_1 < t_2 < t_1+\varepsilon < \bar{t}\).
Letting $l_2 + l_1$, we obtain $a(l_2) + a(l_1)$ since $\phi^{-1}(l_2) + \phi^{-1}(l_1)$.

That $a$ is surjective and that $a^{-1}$ is continuous are easily proved. Q.E.D.

Define $s_1 = a^{-1}$. Then, $s_1$ is homeomorphic from $(0, \bar{a})$ onto $(0, \tilde{l})$ and strictly increasing and satisfies

$$-a/g(v_+0) \leq s_1(a) \quad (0 < a < \bar{a})$$

$v(x; \phi^{-1}(s_1(a)))$ is a unique solution of

$$\begin{cases}
\frac{d^2 v}{dx^2} = -g(v) & (0 < x < s_1(a)) \\
\frac{dv}{dx}(0) = 0 \\
v(s_1(a)) = v_+ \\
v_1 < v(x) < v_+ & (0 \leq x \leq s_1(a)).
\end{cases}$$

(3.7)

Similarly we can define $s_2$ such that $s_2$ is homeomorphic from $(0, \bar{a})$ onto $(0, \tilde{l})$ and strictly increasing and satisfies

$$a/g(v_+0) \leq s_2(a) \quad (0 < a < \bar{a}).$$

For any $a \in (0, \bar{a})$ there exists a unique solution of

$$\begin{cases}
\frac{d^2 v}{dx^2} = -g(v) & (0 < x < s_2(a)) \\
\frac{dv}{dx}(0) = 0 \\
v(s_2(a)) = v_+ \\
v_2 < v(x) < v_+ & (0 \leq x \leq s_2(a)).
\end{cases}$$

(3.8)

Set $\hat{a} = \min(\bar{a}, \bar{b})$ and $\hat{l} = \lim_{a \to \hat{a}} (s_1 + s_2)(a)$. For $a \in (0, \hat{a})$ we denote the unique solution of (3.7) (resp. (3.8)) by $v(x; a, 1)$ (resp. $v(x; a, 2)$). Then, $s_1 + s_2$ is homeomorphic from $(0, \hat{a})$ onto $(0, \hat{l})$ and strictly increasing and satisfies

$$(-1/g(v_+0) + 1/g(v_+0))a \leq (s_1 + s_2)(a).$$

Let $n_0$ be the smallest positive integer greater than $l_0/\hat{l}$. Define $a_n$ and $v_i^n$, $i=1, 2$ and $n \geq n_0$, by
where \( v(x; a_n, 3) \) is equal to \( v(x; a_n, 1) \). It is easy to observe that \( v_{n}^{i}, i=1,2 \) and \( n \geq n_0 \), satisfy (2.1) and (2.2). To complete the proof of Theorem 1 we must show that there exist no other solutions of (2.1) and (2.2). Let \( v \neq v_{*} \) be a solution of (2.1) and (2.2). By Remark 1 and (2.1), we have

\[
(3.10) \quad v \in C^1(0, l_0) \quad \text{and} \quad \int_{0}^{l_0} g(v(x)) \, dx = 0.
\]

We first show that there exists \( x_0 \in (0, l_0) \) such that

\[
(3.11) \quad v(x_0) = v_{*} \quad \text{and} \quad \frac{dv}{dx}(x_0) \neq 0.
\]

Take \( z_0 \) such that \( v(z_0) \neq v_{*} \). Let \( x_0 \) be the nearest point to \( z_0 \) satisfying \( v(x_0) = v_{*} \). Such \( x_0 \) is well-defined since \( (x; x \in (0, l_0), v(x) = v_{*}) \) is not empty by (3.10). Without loss of generality, we may assume that

\[
x_0 < z_0 \quad \text{and} \quad v(z_0) < v(x_0) = v_{*}.
\]

From (2.1), we observe \( v \) satisfies the first equation of (1.1) in \((x_0, z_0)\). Integrating the equation from \( x_0 \) to \( y_0 \), where \( y_0 \in (x_0, z_0) \) is a point satisfying \( dv/dx(y_0) < 0 \), we have

\[
\frac{dv}{dx}(x_0) = \frac{dv}{dx}(y_0) + \int_{x_0}^{y_0} g(v(s)) \, ds < 0.
\]

Hence \( x_0 \) satisfies (3.11). Set \( \alpha = -\frac{dv}{dx}(x_0) > 0 \). While \( v \) is lying in \((v_1, v_{*})\), \( v \) satisfies the first equation of (1.1). Therefore, \( v \) can be extended until \( v \) reaches \( v_{*} \) or \( x = l_0 \). In the former case there exists
\( x_1 = x_0 + 2s_1(a) \varepsilon I \) and satisfies
\[ v(x_1) = v_* \quad \text{and} \quad \frac{dv}{dx}(x_1) = a. \]
Since \( v \in C^1(I) \) and \( a > 0 \), \( v(x) \) transverses \( v = v_* \). While \( v \) is lying in \( (v_*, v_2) \), \( v \) satisfies the first equation of (1.1). Hence \( v \) can be extended until \( v \) reaches \( v_* \) or \( x = l_0 \). In the former case there exists \( x_2 = x_1 + 2s_2(a) \varepsilon I \) and satisfies
\[ v(x_2) = v_* \quad \text{and} \quad \frac{dv}{dx}(x_2) = -a. \]
Repeating this process on both sides of \( x_0 \), and noting the boundary condition, we observe that \( a \) must be equal to some \( a_n \) and that \( v = v_1 \) or \( v_2 \). This completes the proof of Theorem 1.

**Proof of Remark 2.** Under the conditions of Remark 2, the equation (3.1) with \( c = v_1 \) and \( g(v_1) = 0 \) has a unique solution \( v = v_1 \). Since \( g \) is Lipschitz continuous in \( [v_1, v_*] \), we have \( \overline{t} = +\infty \). Similarly \( \overline{t} = +\infty \) is obtained.

Therefore, we get \( n_0 = 1 \). Q.E.D.

**Reference**