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AUGMENTED TEICHMÜLLER SPACES AND PERIOD REPRODUCING DIFFERENTIALS

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Introduction.

It is of interest to investigate how some characteristic quantities attached to Riemann surfaces vary under quasiconformal deformations of the surfaces. Such sort of studies have been made by L. Ahlfors [4], L. Bers [5], and so on.

In the present paper we consider first the Teichmüller space T_g of marked closed Riemann surfaces of genus g, and show in § 1 some theorems about the continuity on T_g for the holomorphic differentials with fixed A-periods and the period reproducing differentials. These results can be extended over the Teichmüller spaces of certain classes of open Riemann surfaces.

In § 3 and 4 we examine specifically the case of squeezing deformations with respect to a non-dividing simple closed curve c. For this purpose we consider the augmented Teichmüller space $\hat{T}_g$ and its subset ${}_c\hat{T}_g$ determined by c. We introduce a topology into ${}_c\hat{T}_g$, which we call the fine topology (cf. § 4 for the precise definition). Then it is proved that the fine topology is finer than the conformal topology introduced by W. Abikoff [2]. The period reproducing differentials with a suitable normalization vary continuously on ${}_c\hat{T}_g$ with respect to the fine topology.

The proofs are almost sketchy or omitted, and the details will appear elsewhere.

§ 1. The continuity on T_g for holomorphic differentials .

1. Let R^* be a closed Riemann surface of genus $g \geq 2$ and $\Pi = \{ A_i, B_i \}_{i=1}^{g}$ be the standard set of generators of the fundamental group of R^* with the single relation $\prod_{i=1}^{g} A_i \cdot B_i \cdot A_i^{-1} \cdot B_i^{-1} = 1$. We denote by T_g the Teichmüller space with the base point $\bar{R}^* = (R^*, \Pi)$, which is equipped with the usual Teichmüller topology. On each point of T_g a canonical homology basis is induced by Π and is denoted again by $\{ A_i, B_i \}_{i=1}^{g}$. The 1-cycles and (free) homotopy classes etc. given on R^* induce also on every point of T_g the corresponding ones, which will be denoted by the same notations.

Now let $\bar{R}_0 \in T_g$ be fixed and take a holomorphic abelian differential $\Theta_{\bar{R}_0}$ on R_0. Then on every point $\bar{R}$ of T_g there exists the unique holomorphic differential on R, say $\Theta_{\bar{R}}$, which has the same A-periods as $\Theta_{\bar{R}_0}$, and we have the following

<u>Theorem 1</u>. Let $f_{\bar{R}}$ be the Teichmüller mapping of $\bar{R}_0$ to $\bar{R}$, and $K_{\bar{R}} = \dfrac{1 + k_{\bar{R}}}{1 - k_{\bar{R}}}$ be the maximal dilatation of $f_{\bar{R}}$. Then we have

$$(1) \qquad || \Theta_{\bar{R}} \circ f_{\bar{R}} - \Theta_{\bar{R}_0} ||_{R_0} \leq \frac{2k_{\bar{R}}}{1 - k_{\bar{R}}} || \Theta_{\bar{R}_0} ||_{R_0} .$$

Hence, $\Theta_{\bar{R}} \circ f_{\bar{R}}$ converges to $\Theta_{\bar{R}_0}$ in the Dirichlet norm if $\bar{R}$ converges to $\bar{R}_0$ in T_g.

Proof. Noting that $\Theta_{\bar{R}} \circ f_{\bar{R}}$ is a closed differential on R_0 with the finite Dirichlet norm and $\omega = \Theta_{\bar{R}} \circ f_{\bar{R}} - \Theta_{\bar{R}_0}$ has vanishing A-periods, we have $(\omega, \omega^*) = 0$ by the bilinear relation, from which we can derive the desired inequality (1). (Cf. [4] and [12].) q.e.d.

Let c denote a non-dividing simple closed curve on (surfaces of) T_g , and $\Theta_{c,\bar{R}}$ be the holomorphic reproducing differential for c on $\bar{R} \in T_g$. Namely, it is characterized by the relation

$$\text{(2)} \qquad \operatorname{Im} \int_{\gamma} \Theta_{c,\bar{R}} = c \times \gamma$$

for every 1-cycle γ on R, equivalently by $\int_c \omega = (\omega , \operatorname{Re} \Theta_{c,\bar{R}})$ for every harmonic differential ω with the finite Dirichlet norm on R. Then similarly as Theorem 1 we have

Theorem 1'. Under the same assumption as in Theorem 1, we have

$$\text{(3)} \qquad \| \Theta_{c,\bar{R}} \circ f_{\bar{R}} - \Theta_{c,\bar{R}_0} \|_{R_0} \leqq \frac{2k_{\bar{R}}}{1 - k_{\bar{R}}} \| \Theta_{c,\bar{R}_0} \|_{R_0}$$

Hence, $\Theta_{c,\bar{R}} \circ f_{\bar{R}}$ converges to $\Theta_{c,\bar{R}_0}$ in the Dirichlet norm if $\bar{R}$ converges to $\bar{R}_0$ in T_g.

Remark. As is seen from the proof, the mapping $f_{\bar{R}}$ of $\bar{R}_0$ to $\bar{R}$ need not be the Teichmüller mapping, but it suffices to be an appropriately smooth quasiconformal mapping (with a suitable change of $k_{\bar{R}}$).

2. As the universal covering surface of any $\bar{R} \in T_g$, we take the unit disk $B = \{ |z| < 1 \}$. Then $f_{\bar{R}}$ can be lifted to a quasi-conformal self-mapping of B. It is extended continuously onto $\{ |z| = 1 \}$ and is uniquely determined by the normalization fixing three points 1, i, and -1.

We write this lift as $f_{\bar{R}}$ again. The differentials $\Theta_{\bar{R}}$ and $\Theta_{c,\bar{R}}$ can also be lifted to holomorphic differentials, say $a_{\bar{R}}(z)dz$ and $a_{c,\bar{R}}(z)dz$, over B respectively.

Theorem 2. If $\bar{R}$ converges to $\bar{R}_0$ in T_g, then $a_{\bar{R}}(z)$ and $a_{c,\bar{R}}(z)$ converges uniformly on every compact set in B to $a_{\bar{R}_0}(z)$ and $a_{c,\bar{R}}(z)$ respectively.

Proof. The assertion follows from Theorem 1 and 1' as in [12]. Also see [4]. q.e.d.

Now since R is compact, the quadratic differential $\Theta^2_{c,\bar{R}}$ has closed trajectories ([3], [11]). Let $L(\Theta_{c,\bar{R}})$ be the admissible (i.e. homotopically independent) curve system determined by $\Theta^2_{c,\bar{R}}$, and set

$$S_c = \{ \bar{R} \in T_g \ : \ L(\Theta_{c,\bar{R}}) \text{ contains } c. \} .$$

Then we have the following result.

Theorem 3. If $\bar{R} \in T_g$ is sufficiently near $\bar{R}_0$, then $L(\Theta_{c,\bar{R}_0})$ is contained in $L(\Theta_{c,\bar{R}})$. Hence S_c is an open set in T_g.

Proof. For a holomorphic differential Θ on R, a trajectory arc of Θ^2 is the curve along which Im $\Theta = 0$. By means of this fact, (2), and Theorem 2, we can prove this theorem. q.e.d.

Remark. Roughly speaking, the decomposition of R by critical trajectories of $\Theta^2_{c,\bar{R}}$ into doubly connected domains also varies continuously on T_g. (For the details see [12].)

3. We are able to extend the above results to the Teichmüller space T(R) for an open Riemann surface R of finite or infinite genus. Here, since the existence of the Teichmüller mappings is not known, we need to take another standard (suitably smooth) quasiconformal mappings for convergent sequences in T(R).

Now to extend Theorem 1 and 1' we need further the existence theorem of the holomorphic differentials with the finite Dirichlet norm which satisfies the given period condition, and also the (generalized) bilinear relation on R. Under these consideration we can extend Theorem 1 for any R belonging to the class O'' (, cf. [7]), and Theorem 1' for any R belonging to the class O_{HD}.

A similar continuity for the holomorphic reproducing differentials also holds for a wider class of open Riemann surfaces if the behavior of those differentials are appropriately restricted.

All the details will appear in [8].

§ 2. The open set S_c in T_g.

In the sequel we shall consider again the Teichmüller space T_g with genus g (≧2). Fix a non-dividing simple closed curve c. Here without loss of generality we may assume that c is freely homotopic to A_1.

A homeomorphism of S_c. Let $T_{g-1,2}$ be the Teichmüller space of marked Riemann surfaces of type (g-1,2). We shall construct a mapping F_1 from S_c into $T_{g-1,2}$ as follows: Let $\bar{R} \in S_c$ and $W_{\bar{R}}$ be the characteristic ring domain of $\Theta^2_{c,\bar{R}}$ for c (, that is, the union of all closed trajectories of $\Theta^2_{c,\bar{R}}$ which are freely homotopic to c). This $W_{\bar{R}}$ can be mapped conformally onto a ring domain $\{ 1<|z|<r^2 \}$. Let $C_{\bar{R}}$ be the closed trajectory in $W_{\bar{R}}$ of $\Theta^2_{c,\bar{R}}$ corresponding to the circle $\{ |z| = r \}$. Then R - $C_{\bar{R}}$ becomes a bordered Riemann surface with two contours. Adding two regions D_1 and D_2 corresponding to $\{ 0<|z|<r \}$ and $\{ r<|z|<+\infty \}$ along each contours of R - $C_{\bar{R}}$ respectively, we get a Riemann surface R' of type (g-1,2).

Now we may assume that $\{ A_i, B_i \}_{i=1}^{g}$ of $\bar{R}$ lie in R - $C_{\bar{R}}$ except for B_1, and that $B_1 \cap C_{\bar{R}}$ consists of a single point. As the generators of the fundamental group $\pi_1(R', p)$ of R', we choose loops $\{ A'_i, B'_i, C_1, C_2 \}_{i=1}^{g-1}$ so that

(i) $A'_i = A_{i+1}$, and $B'_i = B_{i+1}$ ($i=1,\cdots, g-1$),

(ii) C_1 and C_2 are closed curves belonging to $\pi_1(R', p)$ which run along B_1 in R - $C_{\bar{R}}$ (considered as a subregion of R') from the base point p, arround the punctures in D_1 and D_2, respectively, and back to p along B_1 in R - $C_{\bar{R}}$.

(iii) They satisfy the single relation

$$\prod_{i=1}^{g-1} [A_i^! \cdot B_i^! \cdot A_i^{!-1} \cdot B_i^{!-1}] \cdot C_1 \cdot C_2 = 1.$$

With the marking induced by these generators we get a point $\bar{R}'$ of $T_{g-1,2}$, and defining F_1 by $F_1(\bar{R}) = \bar{R}'$ for every $\bar{R} \in S_c$, we have a mapping from S_c into $T_{g-1,2}$.

Next we define a mapping F_2 from S_c into the upper half plane $U = \{ \operatorname{Im} z > 0 \}$ so that for every $\bar{R} \in S_c$

$$\operatorname{Re} F_2(\bar{R}) = \frac{2}{\| \Theta_{c,\bar{R}} \|^2} \cdot \operatorname{Re} \int_{B_1} \Theta_{c,\bar{R}} , \quad \text{and}$$

$$\operatorname{Im} F_2(\bar{R}) = m_{\bar{R}} ,$$

where $m_{\bar{R}}$ denotes the modulus of $W_{\bar{R}}$.

Thus we have a mapping $F = (F_1, F_2)$ from S_c into the product space $T_{g-1,2} \times U$.

Theorem 4. The mapping F is a homeomorphism of S_c onto $T_{g-1,2} \times U$. In particular S_c is simply connected.

Proof is omitted (cf. [12]).

Remark. Generally S_c does not coincide with the whole space T_g. However we can see that $S_c = T_{1,1}$ for every non-dividing simple closed curve c on $T_{1,1}$.

§ 3. The augmented Teichmüller spaces.

As for the boundaries of the Teichmüller spaces many new investigations have been made since a series of studies by Bers [5], Maskit [10], and Abikoff [1]. In view of studying the limits in deformations of Riemann surfaces, here we shall consider as the boundary of T_g the set of marked closed Riemann surfaces with nodes of (arithmetic genus g) defined by Bers [6], which correspond to regular b-groups (cf. [1]). We denote by $\hat{T}_g$ the set obtained from T_g by adding such points, and call it the augmented Teichmüller space for genus g.

Now let R be a Riemann surface with nodes, and N(R) be the set of nodes of R. Put $R' = R - N(R)$. For two marked Riemann surfaces $\bar{R}_1$ and $\bar{R}_2$ (possibly with nodes), a deformation $< \bar{R}_1, \bar{R}_2, f >$ is, by definition, a continuous surjection f from R_1 to R_2 which preserves the marking, and satisfies the following conditions ;

(i) $f^{-1}|_{R_2'}$ (the restriction of f^{-1} on R_2') is a homeomorphism into R_1, and

(ii) for any node p in $N(R_2)$, $f^{-1}(p)$ is either a node of R_1 or a simple closed curve on R_1.

In terms of deformations we can say that the augmented Teichmüller space $\hat{T}_g$ is the set of marked closed Riemann surfaces $\bar{R}$ (possibly with nodes) for which there exists a deformation $< \bar{R}^*, \bar{R}, f >$ from the base point $\bar{R}^*$ of T_g.

Definition. Following Abikoff [2] we introduce on $\hat{T}_g$ the conformal topology as follows: First for every $\bar{R} \in \hat{T}_g$ we set

$$D(\bar{R}) = \{ \bar{S} \in \hat{T}_g : \text{For two deformations } < \bar{R}^*, \bar{R}, f > \text{ and } < \bar{R}^*, \bar{S}, f_1 >, \text{ there is a deformation } < \bar{S}, \bar{R}, f_2 > \text{ such that } f = f_2 \circ f_1. \}.$$

Next given a neighbourhood K of N(R) in R and a positive ε, we define a (K,ε)-conformal neighbourhood $N_{K,\varepsilon}$ of $\bar{R}$ in $\hat{T}_g$ by the set

$$\{ \bar{S} \in D(\bar{R}) : \text{There is a deformation } < \bar{S}, \bar{R}, f > \text{ such that } f^{-1}|_{(R-K)} \text{ is a } (1+\varepsilon)\text{-quasiconformal mapping into S. } \} .$$

Taking the system of $N_{K,\varepsilon}$ for arbitrary K and ε as above as a fundamental neighbourhood system of $\bar{R}$, we have a topology on $\hat{T}_g$, which we call the conformal topology on $\hat{T}_g$.

The conformal topology restricted on T_g is equivalent with the usual Teichmüller topology, and it satisfies the first countability axiom.

§ 4. The fine topology.

1. Let c be a non-dividing simple closed curve on T_g, which is fixed once for all. And set

$$\partial_c T_g = \{ \bar{R} \in \hat{T}_g : \text{N(R) consists of one point p and for the deformation} < \bar{R}^*, \bar{R}, f >, f^{-1}(p) \text{ is freely homotopic to c.} \},$$

and

$${}_c\hat{T}_g = T_g \cup \partial_c T_g .$$

Obviously $D(\bar{R}) = {}_c\hat{T}_g$ for every $\bar{R} \in \partial_c T_g$. Also note that this boundary space $\partial_c T_g$ for c is naturally identified with $T_{g-1,2}$. We denote this identifying mapping from $\partial_c T_g$ onto $T_{g-1,2}$ by J. Then J induces a topology on $\partial_c T_g$ from the Teichmüller topology on $T_{g-1,2}$, which we call also the Teichmüller topology.

Remark. The conformal topology restricted on $\partial_c T_g$ is equivalent with the Teichmüller topology.

Now by using the homeomorphism F defined in § 2, we can introduce a new topology on ${}_c\hat{T}_g$.

Definition. Let $\hat{U} = U \cup \{\infty\}$. The fundamental neighbourhood system of ∞ is defined by $\{ \hat{U}_n \}_{n=1}^{\infty}$, where

$$\hat{U}_n = \{ z \in \hat{U} : z = \infty, \text{ or } \operatorname{Im} z > n. \}.$$

For every $\bar{R} \in \partial_c T_g$ define

$$F(\bar{R}) = (J(\bar{R}), \infty),$$

and we can extend F so that it gives a bijection from $\hat{S}_c$ to

$T_{g-1,2} \times \hat{U}$, where $\hat{S}_c = S_c \cup \partial_c T_g$. Then we can introduce a topology on ${}_c\hat{T}_g$ so that this extended mapping $\hat{F}$ gives a homeomorphism from $\hat{S}_c$ onto $T_{g-1,2} \times \hat{U}$ and it is equivalent with the Teichmüller topology on T_g (cf. Theorem 4). We call this topology the fine topology on ${}_c\hat{T}_g$.

In other words, we may define this topology by taking, as a fundamental neighbourhood system of each $\bar{R} \in \partial_c T_g$, the system $\{ V_n \}_{n=1}^{\infty}$, where

$$V_n = \{ \bar{S} \in \hat{S}_c : d(F_1(\bar{S}), F_1(\bar{R})) < \frac{1}{n}, \text{ and } \bar{S} \in \partial_c T_g \text{ (, that is, } F_2(\bar{S}) = \infty) \text{ or } \operatorname{Im} F_2(\bar{S}) > \log n. \}.$$

Here d(,) denotes the Teichmüller distance on $T_{g-1,2}$.

Theorem 5. The fine topology is finer than the conformal topology restricted on ${}_c\hat{T}_g$.

Proof. From the definition and the above remark, it suffices to prove that a sequence $\{ \bar{R}_n \}_{n=1}^{\infty}$ in T_g converging to $\bar{R}_0 \in \partial_c T_g$ in the sense of the fine topology converges to $\bar{R}_0$ also in the sense of the conformal topology. But for such a sequence $\{ \bar{R}_n \}$ we can easily find sequences $\{ c_n \}$ and $\{ f_n \}$ satisfying the condition (C) below, hence the assertion follows from the next lemma.

q.e.d.

Lemma. A sequence $\{ \bar{R}_n \}$ in T_g converges to $\bar{R}_0 \in \partial_c T_g$ in the sense of the conformal topology if and only if for every n there exists an M-quasiconformal mapping f_n from $R_n - c_n$ into R_0 (, where c_n is a simple closed curve on R_n freely homotopic to c,) which preserves the marking and satisfies the following condition (C) :

(C) For any neighbourhood K of $N(R_0)$ and positive ε, there is an N such that for every $n \geq N$ we have

(i) $f_n(R_n - c_n) \supset R_0 - K$, and

(ii) $f_n^{-1}|_{(R_0 - K)}$ is $(1+\varepsilon)$-quasiconformal.

Here M is a positive constant independent of n.

Proof. Using the extension theorem on quasiconformal mappings ([9] Theorem II -8-1) we can show the assertion. q.e.d.

Remark. We can see that the conformal topology and the fine topology are equivalent to each other also when they are restricted on the foliage

$$\hat{U}_{\bar{R}'} = \{ \bar{R} \in \hat{S}_c : F_1(\bar{R}) = \bar{R}' .\}$$

for each $\bar{R}' \in T_{g-1,2}$. Also note that these topologies restricted on such a foliage corresponds to the Schiffer's variations by attaching a handle.

2. Let p_1 and p_2 be the punctures of $\bar{R}' = J(\bar{R}) \in T_{g-1,2}$ corresponding to C_1 and C_2 respectively for each $\bar{R} \in \partial_c T_g$, and $\phi_{\bar{R}}'$ be the elementary differential of the third kind of $\bar{R}'$ with singularities p_1 and p_2 (cf. [12]). We define

$$\Theta_{\bar{R}} = \frac{2\cdot\Theta_{c,\bar{R}}}{\|\Theta_{c,\bar{R}}\|^2} \qquad \text{for every } \bar{R}\in T_g, \quad \text{and}$$

$$\Theta_{\bar{R}} = \frac{i}{2\pi}\cdot\phi_{\bar{R}'}, \ (\ \bar{R}' = J(\bar{R})\ ) \quad \text{for every } \bar{R}\in \partial_c T_g.$$

Then we have the following

Theorem 6. Suppose that a sequence $\{\ \bar{R}_n\ \}_{n=1}^{\infty}$ in ${}_c\hat{T}_g$ converges to $\bar{R}_0 \in {}_c\hat{T}_g$ in the sense of the fine topology. Then there exists a sequence $\{ < \bar{R}_n, \bar{R}, f_n > \}_{n=1}^{\infty}$ of deformations satisfying the following condition : For every neighbourhood K of $N(R_0)$ and positive ε, we can find an N such that

(i) $\quad f_n^{-1}|_{(R_0 - K)}$ is $(1+\varepsilon)$-quasiconformal, and

(ii) $\quad \| \Theta_{\bar{R}_n} \circ f_n^{-1} - \Theta_{\bar{R}_0} \|_{(R_0 - K)} < \varepsilon$

for every $n \geq N$. Here if $N(R_0) = \phi$, then we assume that $K = \phi$.

Proof. We can show this theorem by (3) in § 1 and the homeomorphism F in § 2. The details will appear elsewhere. q.e.d.

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