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京都大学
Canonical Linear Transformation
on Fock Space with an Indefinite Metric

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Abstract: We first construct a Fock space with an indefinite metric \( \langle , \rangle \) where \( \Theta \) is a unitary and hermitian operator. We define a \( \Theta \)-selfadjoint (Segal's) field \( \Phi_\rho(f) \) which obeys the canonical commutation relations (CCR) with an indefinite metric. We consider a transformation \( \Phi_\rho(f) \mapsto \Phi_\rho(Tf) \) (\( T \) = real linear) which leaves the CCR invariant. We investigate the implementability of \( T \) by an operator on the Fock space.

Let \( \mathcal{H}_1 \) \((1=+,-)\) be Hilbert spaces equipped with usual positive definite hermitian inner product \( \langle , \rangle \). Let \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) be a Hilbert space equipped with the inner product \( \langle , \rangle = \mathcal{I}_1( , ) \). Let \( P_\pm \) be selfadjoint projections onto \( \mathcal{H}_\pm \). Then the Hilbert space equipped with an hermitian inner product \( \langle , \rangle = ( , \varphi) \) with \( \varphi = P_+ - P_- \) is called a "Hilbert space with an indefinite metric".

Let \( S_n \) be the usual (n-fold) symmetrization operator on the n-fold tensor product space \( \otimes_n \mathcal{H} \), and let

\[
\mathcal{F}(n) = S_n[ \otimes_n \mathcal{H}]
\]

be the n-particle (Fock) space. The total Fock space

\[
\mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{F}(n)
\]

is...
is also given by
\[ \mathcal{F}(\mathcal{H}_+) \otimes \mathcal{F}(\mathcal{H}_-) , \]
where \( \mathcal{F}(\mathcal{H}_+) \) and \( \mathcal{F}(\mathcal{H}_-) \) are Fock spaces constructed from \( \mathcal{H}_+ \)
and \( \mathcal{H}_- \) respectively. For an operator \( A \) on \( \mathcal{H} \), define \( \Gamma(A) \)
by
\[ \Gamma(A) \mathcal{F}(n) \subset \mathcal{F}(n) , \]
\[ \Gamma(A) \mathcal{F}(n) = A \mathcal{F}(n) \quad (n\text{-times}). \]
Then \( \Theta = \Gamma(\varphi) \) is again an unitary and hermitian operator on \( \mathcal{F} \).
We define an indefinite sesquilinear form in \( \mathcal{F} \) by
\[ \langle , \rangle = (, \Theta) . \]
The adjoint of \( A \) with respect to \( \langle , \rangle \) is denoted by \( A^\Theta \)
and equals \( \Theta A^\ast \Theta \).

**Definition 1:** (1) For \( f \in \mathcal{H} \), the creation operator \( a^\ast(f) \)
is defined by
\[ a^\ast(f) : \mathcal{F}(n) \to \mathcal{F}(n+1) \]
\[ \phi \mapsto \sqrt{n+1} S_{n+1}[f \phi] . \]
(2) For \( f \in \mathcal{H} \), define the \( \Theta \)-selfadjoint (Segal's) field by
\[ \phi_\varphi(f) = \frac{1}{\sqrt{2}}[a^\ast(f)]^{(\Theta)} - [a^\ast(f)]^{(\Theta)} \]
where \( \cdot \) denotes the closure.

Since \( [a^\ast(f)]^{(\Theta)} = [a^\ast(\varphi f)]^{(\Theta)} \)
with \( \varphi = P_+ - P_- \), \( \phi_\varphi \) is a normal
operator. \( \{ \phi_\varphi(f) \} \) obey the CCR with an indefinite metric:
\[ [\phi_\varphi(f), \phi_\varphi(g)] = \text{Im} \langle f, g \rangle = -i \text{Re} \langle \bar{f}, \varphi g \rangle \]
where \( \bar{f} \) is the complex conjugation of \( f \) and \( J = \sqrt{-1} \) is a multi-
plation operator of \( i \).
Definition 2: (1) An invertible real linear transformation $T$ is called $\varphi$-symplectic if it satisfies
\[ T(\varphi)J = J \]
where $T(\varphi) = \varphi^T \varphi$ and $T^*$ is the adjoint of $T$ with respect to $\text{Re}(\cdot, \cdot)$ in $\mathcal{H}$. (If $T$ is complex linear, then this adjoint is equivalent to the usual adjoint with respect to $(\cdot, \cdot)$ in $\mathcal{H}$.)

(2) $T_+ = \frac{1}{2} [T + JTJ^{-1}]$. Especially anti-linear part $T_-$ is called the off-diagonal part of $T$.

Our purpose is to investigate an operator which is expected to implement $U_T^* \varphi(f)U_T^{-1} = \varphi(Tf)$, and to investigate the new vacuum $\Omega_T = U_T^{-1} \Omega$. Here $\Omega(0) = C$ is the Fock vacuum. Since $\varphi(f) \to \varphi(Tf)$ leaves the CCR invariant, one may expect that $U_T$ is a $\Theta$-unitary (bijective $\Theta$-isometric) operator.

Definition 3: (1) $T$ is called $\Theta$-unitarily implementable if there is a $\Theta$-unitary (bijective $\Theta$-isometric) operator $U_T$ which implements $U_T^* \varphi(f)U_T^{-1} = \varphi(Tf)$.

(2) $T$ is called weakly $\Theta$-unitarily implementable if there exist a $\Theta$-isometric (not necessarily bounded) operator $U_T^{-1}$ and a cyclic vector $\Omega_T \in \mathcal{F}$ such that
\[ U_T^{-1} P(\varphi(f)) \Omega = P(\varphi(Tf)) \Omega_T, \]
where $P(\varphi(f)) = P(\varphi(f_1), \ldots, \varphi(f_n))$ is any polynomial of $\{\varphi(f_i)\}$.

(3) $T$ is called $\Theta$-unitarily quasi-implementable if the Fredholm determinant $\det[1 + T_-^*(\varphi)T_-]$ uniformly converges to a non-vanishing finite value in $(0, \infty)$.

When $\varphi = 1$ (namely when $\Theta = 1$), three notions in this definition coincide each other $[1, 3, 4]$. For the implementability, the author proved $[1]$: 

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Theorem 1: $T$ is $\Theta$-unitarily implementable if and only if $T_-$ is Hilbert-Schmidt and $[T, \varphi] = 0$. In this case $U_T^{-1} = \Omega_0 T \in \mathcal{F}$. Then

(1) $T_\in H.S.$ (H.S. denotes the Hilbert-Schmidt class),

(2) $[-\infty, 0]$ is in the resolvent set of $T_+^{(\varphi)} T_+ = 1 + T_-^{(\varphi)} T_-$. 

In order to obtain a sufficient condition, we propose a $\varphi$-polar decomposition of $T$, namely a decomposition of $T$ in terms of a $\varphi$-selfadjoint operator and a $\varphi$-unitary operator.

Theorem 2: Let $U_T^{-1} = \Omega_0 T \in \mathcal{F}$. Then

1. $T_\in H.S.$ (H.S. denotes the Hilbert-Schmidt class),
2. $[-\infty, 0]$ is in the resolvent set of $T_+^{(\varphi)} T_+ = 1 + T_-^{(\varphi)} T_-$. 

In order to obtain a sufficient condition, we propose a $\varphi$-polar decomposition of $T$, namely a decomposition of $T$ in terms of a $\varphi$-selfadjoint operator and a $\varphi$-unitary operator.

Theorem 3: Let a $\varphi$-symplectic operator $T$ satisfy the conditions in Theorem 2. Then $T$ has a decomposition

$$ T = U H, $$

where $U$ is a $\varphi$-unitary operator (which commutes with $J$) and $H$ is a $\varphi$-selfadjoint $\varphi$-symplectic operator with its spectrum in the right half plane.

Definition 4: $\varphi$-selfadjoint $\varphi$-symplectic operator $S$ is called a generalized $\varphi$-scaling if $S$ leaves $K$ and $JK$ invariant where $K = K \in JK$ and $\Theta$ refers to the orthogonality with respect to both $\text{Re}(\ ,\ )$ and $\text{Re}<\ ,\ >$.

A generalized $\varphi$-scaling $S$ takes the following form on $K \in JK$:

$$
\begin{pmatrix}
  h & 0 \\
  0 & h^{-1}
\end{pmatrix}
$$

Here $ChC = h$, where $C$ is a complex conjugation operator:

$$ K = \{ x \in \mathcal{H} ; Cx = x \} . $$

Is $H$ in Theorem 3 always similar to a generalized $\varphi$-scaling $S$
through suitable $\mathcal{F}$-unitary operator $V$? (This holds if $\mathcal{F}=1$ [1,3,4].)

$$H=VSV^{-1}.$$ 

If this is the case, we have a decomposition

$$T=V_{1}SV_{2}$$ 

under the conditions of Theorem 2, where $V_{1}$ are $\mathcal{F}$-unitary. But $V$ seems unbounded in general.

For a generalized $\mathcal{F}$-scaling $S$, we can obtain rather concrete theorems [1]. It sometimes suffices to consider generalized $\mathcal{F}$-scalings for physical applications [1,2].

Theorem 4: For a generalized $\mathcal{F}$-scaling $S$, if $S \in H.S.$, and if $\alpha = \text{selfadjoint part of } \mathcal{F}$-selfadjoint operator $h^{-2} \succ 0$, then

(i) both $S$ and $S^{-1}$ are weakly $\Theta$-unitarily implementable.

(ii) The overlap between $\Omega$ and $\Omega_{S}$ is given by

$$|\langle \Omega, \Omega_{S} \rangle| = \text{det}^{-1/4}[1+S(\mathcal{F})S_{-}]$$

$$= \text{det}^{-1/4}[1 + \frac{1}{4}(h^{-1})^{2}]$$

This is non-vanishing finite.

Theorem 5: In Theorem 4, if $\inf \text{spec}(\alpha) < 0$, then the vector $\Omega_{S}$ which satisfies

$$\langle \Omega_{S}, P(\phi_{\mathcal{F}}(f))\Omega_{S} \rangle = \langle \Omega, P(\phi_{\mathcal{F}}(Sf))\Omega \rangle$$

cannot be in the Fock space: $\|\Omega_{S}\| = \infty$.

As is well known, when $\mathcal{F}=1$, the necessary and sufficient condition for $T$ to be unitarily implementable is $T \in H.S.$ Then for $\mathcal{F}=1$, the overlap of the vacua does not vanish if and only
if $T$ is unitarily implemented. In fact when $\varphi=1$, we have

$$T = U_1 S U_2$$

where $U_1$ are unitaries commuting with $J$. Further since

transformations $\phi_{\varphi=1}(f) + \phi_{\varphi=1}(U_1 f)$ are implemented by unitaries

$\Gamma(U_1)$ on the Fock space, we have $U_T = \Gamma(U_1) U_S \Gamma(U_2)$. Then

$$\Omega_T = \Gamma(U_2)^{-1} \Omega_S$$

and $(\Omega, \Omega_T) = (\Omega, \Omega_S)$.

For given $S$, let $T = V_1 S V_2$ where $V_1$ are $\varphi-$unitaries. Then

$$S \in \text{H.S.} \iff T \in \text{H.S.}$$

and

$$\det[1 + S(\varphi)S] = \det[1 + T(\varphi)T].$$

Since $\Gamma(V_1)$ are not bounded operators, $T$ is not necessarily

weakly $\Theta-$unitarily implementable even if $S$ is weakly $\Theta-$unitarily

implementable. But the above equation means that the formal

overlap $\det^{-1/4}[1 + T(\varphi)T]$ is an invariant quantity under

$\varphi$-unitaries. Furthermore if $\varphi \neq 1$, $\det^{-1/4}[1 + S(\varphi)S]$ can converge

to a non-vanishing (finite) quantity even if $S \notin \text{H.S.}$ Then

Definition 3 (3) implies that the formally defined overlap

is non-vanishing (finite), which is equivalent to the uni-

tarily implementability of $S$ when $\varphi=1$.

(Sketch of the proof of Theorem 4)

Let $K = K_+ \oplus K_-$ ($K_i = P_i K$) and let $\{e_i\}$ be complete orthonormal

basis in $K$ with respect to both $\text{Re}(\cdot, \cdot)$ and $\text{Re} <\cdot,\cdot>$. We use

the following unitary transformation $W$:

$$W \mathcal{K} = L^2(Q; \text{d} \mu_0),$$

$$Q = \mathbb{R}^\infty, \quad \text{d} \mu_0 = \prod_{i=1}^{\infty} \exp[-q_i^2] \frac{\text{d} q_i}{\sqrt{\pi}},$$

$$W \Omega = 1,$$

$$W[\phi_{\varphi}(e_i)] W^{-1} = \begin{cases} q_i & e_i \in K_+ \\ -q_i & e_i \in K_- \end{cases},$$

$$-\delta-$$
\[ W[\phi_\theta(Je_1)]W^{-1} = \left\{ \begin{array}{ll} -i\alpha/q_1 + iq_1 & e_1 \in K_+ \\ -\alpha/q_1 + q_1 & e_1 \in K_- \end{array} \right. \]

Note that
\[ [a^*(e_1)](\theta) = \frac{1}{\sqrt{2}}[\phi_\theta(e_1) + i\phi_\theta(Je_1)]. \]

Since the transformed vacuum should satisfy
\[ [\phi_\theta(S^{-1}e_1) + i\phi_\theta(S^{-1}Je_1)]\Omega_S = 0, \]
\[ <\Omega_S, \Omega_S> = 1, \]
we have [1]
\[ \Omega_S = [\det(a)]^{1/4} \exp[-\frac{1}{2}(q,(a-1)q)] \]
where
\[ (q,aq) = \Sigma_{ij} q_i \alpha_{ij} q_j \]
and
\[ \alpha_{ij} = (e_1, \psi h^{-2} \psi), \psi = P_+ + iP_. \]

Remark that \( a \) is a \( \theta \)-selfadjoint symmetric matrix.

Under the conditions of Theorem 4, we can prove that
\[ \Omega_S = \Omega_S(q) \in L^2(q, d\mu_0) \in \mathcal{F} \text{ and the cyclicity of } \Omega_S \text{ [1]. Further} \]
\[ \Omega_{S^{-1}} = [\det(a^{-1})]^{1/4} \exp[-\frac{1}{2}(q,(a^{-1}-1)q)]. \]
Let \( a = a_r + ia_1 \) where \( a_r \) and \( a_1 \) are selfadjoint real matrices
( this follows from the properties of \( a \)). If \( a_r \) is positive
( then strictly positive since \( a_r^{-1} \) is H.S. ), since
\[ (a^{-1})_r = (a_r + a_1 a_r^{-1} a_1)^{-1}, \]
\[ (a^{-1})_1 = -a_r^{-1} a_1 (a_r + a_1 a_r^{-1} a_1)^{-1}, \]
then \( (a^{-1})_r \) is again a (strictly) positive operator. Thus \( \Omega_{S^{-1}} \in \mathcal{F} \). The \( \Theta \)-isometricity of \( U_S^{-1} \) follows from
\[ <\Omega_S, P(\phi_\theta(f))\Omega_S> = <\Omega, P(\phi_\theta(Sf))\Omega> \]

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which is proved in [1]. Finally
\[
\langle \Omega, \Omega_S \rangle = \langle \Omega_S, \Omega \rangle = \int \Omega_S(q) d\nu_0 = \det^{-1/4} [1 + S(\mathcal{F}) S_+].
\]

From the above proof, the reader can guess that \( \alpha > 0 \) is needed to ensure \( \| \Omega_S \| < \infty \).

(Sketch of the proof of Theorem 5)

Since \( \alpha \) is a \( \mathcal{F} \)-selfadjoint operator, \( \alpha \) takes the following form on \( JK_+ \otimes JK_- \):
\[
\begin{pmatrix}
(\alpha_+^+ & \mathbf{1} & \alpha_+^-) \\
\mathbf{1} & (\alpha_-^- & \alpha_-^+)
\end{pmatrix},
\]
\[
\alpha_{ij} = P_1 \alpha P_j.
\]

First assume that \( f \in JK_+ \) be an eigenvector of \( \alpha \) belonging to the eigenvalue \( -\lambda < 0 \). Since \( \Phi_\mathcal{F}(f) \) is selfadjoint,
\[
\| \exp[i \Phi_\mathcal{F}(f)] \| = 1.
\]

Note
\[
\langle \Omega_S, \exp[i \Phi_\mathcal{F}(f)] \Omega_S \rangle = \langle \Omega, \exp[i \Phi_\mathcal{F}(Sf)] \Omega \rangle
\]
\[
= \exp[- \frac{1}{4} \langle Sf, Sf \rangle] = \exp[- \frac{\lambda}{4} \| f \|^2].
\]

If \( \lambda > 0 \), the right hand side can be made arbitrarily large, which contradicts
\[
|\langle \Omega_S, \exp[i \Phi_\mathcal{F}(f)] \Omega_S \rangle| \leq \| \Omega_S \|^2 \leq \infty.
\]

The case of \( f \in JK_- \) is similarly discussed.

Our theory can be applied for quantum electrodynamics-type models where \( \mathcal{F}(\mathcal{H}) \) is the Fock space of the gaugeon (ghost particle which has a negative norm) and \( \mathcal{F}(\mathcal{H}_+) \) is the Fock space of physical particles (photon, etc.). In these...
models, the Hamiltonian $H$ is expected to be $\theta$-selfadjoint
(namely $H\theta$ is selfadjoint). As a simple example, let $H$ be
$\theta$-selfadjoint and bilinear with respect to creation and annihi-
lation operators. Let $H$ be diagonalized $[1,2]$ by a transforma-
tion defined by $\phi_\theta(f) \mapsto \phi_\theta(Tf)$ for any $f \in \mathcal{H}$. Then $\Omega_T$ is the
physical vacuum of the Hamiltonian. If $T$ is weakly $\theta$-unitarily
implementable, then $\rho_T(\cdots) = \langle \Omega_T, \cdots \Omega_T \rangle$ is a normalized $\theta$-self-
adjoint linear functional on the field algebra, which typically
appears in QED-type models. $\rho_T$ is called a Lorentz state in

Theorem 5 implies that the linear functional $\rho_T$ defined
by
$$\rho_T(P(\phi_\theta(f)) = \langle \Omega, P(\phi_\theta(Tf)) \Omega \rangle$$
cannot be a continuous state in general on the $C^*$-algebra gene-
rated by $\{\exp[i\phi(f)]; f \in \mathcal{H}\}$, where $\phi(f)$ is the selfadjoint
Segal's field.

The converse problem, namely to obtain a representation
(or $T$) from the expectation values, is the problem which
must be solved to construct a QED-type model in a mathematically
rigorous way $[2]$. This corresponds to a generalization of the
GNS-construction. This will be discussed someday.

-References-

[1] K.R.Ito, "Canonical Linear Transformation on Fock Space
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