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京都大学
Canonical Linear Transformation

on Fock Space with an Indefinite Metric

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Abstract: We first construct a Fock space with an indefinite metric $\langle , \rangle = ( , \Theta )$, where $\Theta$ is a unitary and hermitian operator. We define a $\Theta$-selfadjoint (Segal's) field $\phi_\varphi(f)$ which obeys the canonical commutation relations (CCR) with an indefinite metric. We consider a transformation $\phi_\varphi(f) \rightarrow \phi_\varphi(Tf)$ ($T$=real linear) which leaves the CCR invariant. We investigate the implementability of $T$ by an operator on the Fock space.

Let $\mathcal{H}_1 (1=+,-)$ be Hilbert spaces equipped with usual positive definite hermitian inner product $( , )_1$. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a Hilbert space equipped with the inner product $( , ) = \mathbb{I}_1 ( , )_1$. Let $P_\pm$ be selfadjoint projections onto $\mathcal{H}_\pm$. Then the Hilbert space equipped with an hermitian inner product $\langle , \rangle \equiv ( , \varphi )$ with $\varphi = P_+ - P_-$ is called a "Hilbert space with an indefinite metric".

Let $S_n$ be the usual $(n$-fold) symmetrization operator on the $n$-fold tensor product space $\bigotimes_n \mathcal{H}$, and let

$$\mathcal{F}^{(n)} \equiv S_n [ \bigotimes_n \mathcal{H} ]$$

be the $n$-particle (Fock) space. The total Fock space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}$$

is...
is also given by

$$\mathcal{F}(\mathcal{H}_+ \otimes \mathcal{H}_-)$$,

where $\mathcal{F}(\mathcal{H}_+)$ and $\mathcal{F}(\mathcal{H}_-)$ are Fock spaces constructed from $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively. For an operator $A$ on $\mathcal{H}$, define $\Gamma(A)$ by

$$\Gamma(A) |n\rangle = A |n\rangle$$

Then $\Theta \equiv \Gamma(\psi)$ is again an unitary and hermitian operator on $\mathcal{F}$.

We define an indefinite sesquilinear form in $\mathcal{F}$ by

$$\langle \cdot , \cdot \rangle = ( \cdot , \cdot )_\Theta.$$  

The adjoint of $A$ with respect to $\langle , \rangle$ is denoted by $A^{(\Theta)}$ and equals $\Theta A^* \Theta$.

Definition 1: (1) For $f \in \mathcal{H}$, the creation operator $a^*(f)$ is defined by

$$a^*(f) : \mathcal{F}(n) \rightarrow \mathcal{F}(n+1)$$

$$\phi \mapsto \sqrt{n+1} S_{n+1} [f \otimes \phi].$$

(2) For $f \in \mathcal{H}$, define the $\Theta$-selfadjoint (Segal's) field by

$$\Phi^\varphi(f) = \frac{1}{\sqrt{2}} [a^*(f) + [a^*(f)]^{(\Theta)}]^*$$

where $^*$ denotes the closure.

Since $[a^*(f)]^{(\Theta)} = [a^*(\varphi f)]^*$ with $\varphi = \mathcal{P}_+ - \mathcal{P}_-$, $\Phi^\varphi$ is a normal operator. \{\Phi^\varphi(f)\} obey the CCR with an indefinite metric:

$$[\Phi^\varphi(f), \Phi^\varphi(g)] = i \text{ Im} \langle f, g \rangle = -i \text{ Re} \langle \overline{f}, \varphi g \rangle$$

where $\overline{\cdot}$ is the complex conjugation of $f$ and $J = \sqrt{-1}$ is a multiplication operator of $i$. 

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Definition 2: (1) An invertible real linear transformation \( T \) is called \( \varphi \)-symplectic if it satisfies
\[
T(\varphi)JT = J
\]
where \( T(\varphi) = \varphi T^* \varphi \) and \( T^* \) is the adjoint of \( T \) with respect to \( \text{Re} \ (\ , \ ) \) in \( \mathcal{H} \). (If \( T \) is complex linear, then this adjoint is equivalent to the usual adjoint with respect to \( (\ , \ ) \) in \( \mathcal{H} \).

(2) \( T_+ = \frac{1}{2} [T + JTJ^{-1}] \). Especially anti-linear part \( T_- \) is called the off-diagonal part of \( T \).

Our purpose is to investigate an operator which is expected to implement \( U_T \Phi_\varphi(f)U_T^{-1} = \Phi_\varphi(Tf) \), and to investigate the new vacuum \( \Omega_T = U_T^{-1} \Omega \). Here \( \Omega \in F(0) = C \) is the Fock vacuum. Since \( \Phi_\varphi(f) \rightarrow \Phi_\varphi(Tf) \) leaves the CCR invariant, one may expect that \( U_T \) is a \( \Theta \)-unitary (bijeuctive \( \Theta \)-isometric) operator.

Definition 3: (1) \( T \) is called \( \Theta \)-unitarily implementable if there is a \( \Theta \)-unitary (bijeuctive \( \Theta \)-isometric) operator \( U_T \) which implements \( U_T \Phi_\varphi(f)U_T^{-1} = \Phi_\varphi(Tf) \).

(2) \( T \) is called weakly \( \Theta \)-unitarily implementable if there exist a \( \Theta \)-isometric (not necessarily bounded) operator \( U_T^{-1} \) and a cyclic vector \( \Omega_T \in \mathcal{F} \) such that
\[
U_T^{-1}P(\Phi_\varphi(f))\Omega = P(\Phi_\varphi(Tf))\Omega_T,
\]
where \( P(\Phi_\varphi(f)) = P(\Phi_\varphi(f_1), \ldots, \Phi_\varphi(f_n)) \) is any polynomial of \( \{\Phi_\varphi(f_i)\} \).

(3) \( T \) is called \( \Theta \)-unitarily quasi-implementable if the Fredholm determinant \( \det[1 + T_-^*(\varphi)T_-] \) uniformly converges to a non-vanishing finite value in \((0, \infty)\).

When \( \varphi = 1 \) (namely when \( \Theta = 1 \)), three notions in this definition coincide each other \([1, 3, 4]\). For the implementability, the author proved \([1]\):
Theorem 1: $T$ is $\Theta$-unitarily implementable if and only if $T_-$ is Hilbert-Schmidt and $[T, \phi]=0$. In this case $U_T$ is a unitary operator commuting with $\Theta$.

Theorem 2: Let $U_T^{-1}\Omega=\Omega_\Theta \in \mathcal{F}$. Then

(i) $T_- \in H.S.$ (H.S. denotes the Hilbert-Schmidt class),

(ii) $(-\infty, 0]$ is in the resolvent set of $T_+(\phi)T_+-T_-(\phi)T_-$.

In order to obtain a sufficient condition, we propose a $\phi$-polar decomposition of $T$, namely a decomposition of $T$ in terms of a $\phi$-selfadjoint operator and a $\phi$-unitary operator.

Theorem 3: Let a $\phi$-symplectic operator $T$ satisfy the conditions in Theorem 2. Then $T$ has a decomposition

$$T=UH,$$

where $U$ is a $\phi$-unitary operator (which commutes with $J$) and $H$ is a $\phi$-selfadjoint $\phi$-symplectic operator with its spectrum in the right half plane.

Definition 4: $\phi$-selfadjoint $\phi$-symplectic operator $S$ is called a generalized $\phi$-scaling if $S$ leaves $K$ and $JK$ invariant where $K=K_{\phi}JK$ and $\Theta$ refers to the orthogonality with respect to both $\text{Re}(\cdot, \cdot)$ and $\text{Re}<\cdot, \cdot>$.

A generalized $\phi$-scaling $S$ takes the following form on $K_{\phi}JK$:

$$
\begin{pmatrix}
 h & 0 \\
 0 & h^{-1}
\end{pmatrix}
$$

Here $ChC=\overline{h}=h$, where $C$ is a complex conjugation operator:

$$K=\{x \in \mathcal{H} ; \text{C}x=x\}.$$

Is $H$ in Theorem 3 always similar to a generalized $\phi$-scaling $S$.
through suitable $\varphi$-unitary operator $V$? (This holds if $\varphi = 1$ [1,3,4].)

$$H = VSV^{-1}.$$ 

If this is the case, we have a decomposition

$$T = V_1SV_2$$

under the conditions of Theorem 2, where $V_1$ are $\varphi$-unitary. But $V$ seems unbounded in general.

For a generalized $\varphi$-scaling $S$, we can obtain rather concrete theorems [1]. It sometimes suffices to consider generalized $\varphi$-scalings for physical applications [1,2].

Theorem 4: For a generalized $\varphi$-scaling $S$, if $S \in \text{H.S.}$, and if $\alpha_\varphi$ is selfadjoint part of $\varphi$-selfadjoint operator $H^{-2} > 0$, then

(i) both $S$ and $S^{-1}$ are weakly $\Theta$-unitarily implementable.

(ii) The overlap between $\Omega$ and $\Omega_S$ is given by

$$|\langle \Omega, \Omega_S \rangle| = \det^{-1/4}[1 + S_{\varphi}(\varphi) S_{\varphi}]$$

$$= \det^{-1/4}[1 + \frac{1}{4}(H^{-1})^2].$$

This is non-vanishing finite.

Theorem 5: In Theorem 4, if $\inf \text{spec}(\alpha_\varphi) < 0$, then the vector $\Omega_S$ which satisfies

$$\langle \Omega_S, P(\phi(f)) \Omega_S \rangle = \langle \Omega, P(\phi(Sf)) \Omega \rangle$$

cannot be in the Fock space: $\| \Omega_S \| = \infty$.

As is well known, when $\varphi = 1$, the necessary and sufficient condition for $T$ to be unitarily implementable is $T \in \text{H.S.}$ Then for $\varphi = 1$, the overlap of the vacua does not vanish if and only
if $T$ is unitarily implemented. In fact when $\varphi=1$, we have

\[ T = U_1 S U_2 \]

where $U_1$ are unitaries commuting with $J$. Further since

transformations $\phi_{\varphi=1}(f) + \phi_{\varphi=1}(U_1^* f)$ are implemented by unitaries

$\Gamma(U_1)$ on the Fock space, we have $U_T = \Gamma(U_1) U_S \Gamma(U_2)$. Then

$\Omega_T = \Gamma(U_2)^{-1} \Omega_S$ and $(\Omega, \Omega_T) = (\Omega, \Omega_S)$.

For given $S$, let $T = V_1 S V_2$ where $V_1$ are $\varphi$-unitaries. Then

\[ S \in H.S. \quad \leftrightarrow \quad T \in H.S. \]

and

\[ \det[1+S_{\varphi} S_{\varphi}] = \det[1+T_{\varphi} T_{\varphi}]. \]

Since $\Gamma(V_1)$ are not bounded operators, $T$ is not necessarily weakly $\theta$-unitarily implementable even if $S$ is weakly $\theta$-unitarily implementable. But the above equation means that the formal overlap $\det^{-1/4}[1+T_{\varphi} T_{\varphi}]$ is an invariant quantity under $\varphi$-unitaries. Furthermore if $\varphi \neq 1$, $\det^{-1/4}[1+S_{\varphi} S_{\varphi}]$ can converge to a non-vanishing (finite) quantity even if $S \notin H.S$. Then Definition 3 (3) implies that the formally defined overlap is non-vanishing (finite), which is equivalent to the unitarily implementability of $S$ when $\varphi=1$.

(Sketch of the proof of Theorem 4)

Let $K = K_+ \oplus K_-$ ($K_+ = P_+ K$) and let $\{e_\downarrow\}$ be complete orthonormal basis in $K$ with respect to both $\text{Re}(\cdot, \cdot)$ and $\text{Re}<\cdot, \cdot>$. We use the following unitary transformation $W$:

\[ W = L^2(Q; d\mu_0), \]

\[ Q = \mathbb{R}^\infty, \quad d\mu_0 = \prod_{i=1}^\infty \exp[-q_i^2] \frac{dq_i}{\sqrt{\pi}}, \]

\[ W \Omega = 1, \]

\[ W[\phi_{\varphi}(e_\downarrow)] W^{-1} = \begin{cases} \cdot q_i & e_\downarrow \in K_+ \\ -q_i & e_\downarrow \in K_- \end{cases}, \]
$$W[\phi_{\varphi}(J_{e_1})]W^{-1} = \{-i\varphi/a_{q_1} + i\varphi_1, e_1 \in K_+ \}
{-a/\varphi q_1 + q_1, e_1 \in K_- .}$$

Note that
$$[a^*(e_1)](\Theta) = \frac{1}{\sqrt{2}}[\phi_{\varphi}(e_1) + i\phi_{\varphi}(J_{e_1})].$$

Since the transformed vacuum should satisfy
$$[\phi_{\varphi}(S^{-1}e_1) + i\phi_{\varphi}(S^{-1}J_{e_1})]n_S = 0,$$
$$\langle n_S, n_S \rangle = 1,$$
we have [1]
$$\Omega_S = [\det (\alpha)]^{1/4} \exp[-\frac{1}{2}(q, (\alpha^{-1}q))]

where
$$(q, \alpha q) = \sum_{ij} q_i \alpha_{ij} q_j$$
and
$$\alpha_{ij} = (e_1, \psi h^{-2}\psi_{e_j}), \psi = p_+ + i p_-.$$  

Remark that $\alpha$ is a $\varphi$-selfadjoint symmetric matrix.

Under the conditions of Theorem 4, we can prove that
$$\Omega_S = \Omega_S(q) \in L^2(q, d\mu_0) \in \mathcal{F}$$
and the cyclicity of $\Omega_S$ [1]. Further
$$\Omega_{S^{-1}} = [\det(\alpha^{-1})]^{1/4} \exp[-\frac{1}{2}(q, (\alpha^{-1}q))].$$

Let $\alpha = \alpha_r + i\alpha_1$ where $\alpha_r$ and $\alpha_1$ are selfadjoint real matrices
( this follows from the properties of $\alpha$). If $\alpha_r$ is positive
( then strictly positive since $\alpha_r - 1$ is H.S. ), since
$$(\alpha^{-1})_r = (\alpha_r + i\alpha_1^{-1}\alpha_1^{-1}),$$
$$(\alpha^{-1})_i = -\alpha^{-1}_1\alpha_1(\alpha_r + i\alpha_1^{-1}\alpha_1^{-1}),$$
then $(\alpha^{-1})_r$ is again a (strictly) positive operator. Thus
$\Omega_{S^{-1}} \in \mathcal{F}$. The $\Theta$-isometricity of $U_{S^{-1}}$ follows from
$$\langle \Omega_S, P(\phi_{\varphi}(f))\Omega_S \rangle = \langle \Omega, P(\phi_{\varphi}(Sf))\Omega \rangle.$$
which is proved in [1]. Finally
\[ \langle \Omega_0, \Omega_S \rangle = \int \Omega_0(q) d\nu_0 = \text{det}^{-1/4} [l + S(\varphi)] S_0. \]

From the above proof, the reader can guess that \( \alpha_r > 0 \) is needed to ensure \( \| \Omega_S \| < \infty \).

(Sketch of the proof of Theorem 5)

Since \( \alpha \) is a \( \varphi \)-selfadjoint operator, \( \alpha \) takes the following form on \( JK_+ @ JK_- \):
\[
\begin{pmatrix}
(\alpha_r)_{++} & 1(\alpha_1)_{+-} \\
1(\alpha_1)_{-+} & (\alpha_r)_{++}
\end{pmatrix}, \quad \alpha ij = P_i \alpha P_j.
\]

First assume that \( f \in JK_+ \) be an eigenvector of \( \alpha_r \) belonging to the eigenvalue \( -\lambda \in 0 \). Since \( \Phi^\varphi(f) \) is selfadjoint,
\[ \| \exp[i \Phi^\varphi(f)] \| = 1. \]

Note
\[ \langle \Omega_0, \exp[i \Phi^\varphi(f)] \Omega_S \rangle = \langle \Omega, \exp[i \Phi^\varphi(Sf)] \Omega \rangle \]
\[ = \exp[-\frac{1}{4} \langle Sf, Sf \rangle] = \exp[-\frac{\lambda}{4} \| f \|^2] . \]

If \( \lambda > 0 \), the right hand side can be made arbitrarily large, which contradicts
\[ | \langle \Omega_0, \exp[i \Phi^\varphi(f)] \Omega_S \rangle | \leq \| \Omega_S \|^2 \infty . \]

The case of \( f \in JK_- \) is similarly discussed.

Our theory can be applied for quantum electrodynamics-type models where \( \mathcal{F}(\mathcal{H}) \) is the Fock space of the gaugeon (ghost particle which has a negative norm) and \( \mathcal{F}(\mathcal{H}_+) \) is the Fock space of physical particles (photon, etc.). In these...
models, the Hamiltonian $H$ is expected to be $\Theta$-selfadjoint (namely $H\Theta$ is selfadjoint). As a simple example, let $H$ be $\Theta$-selfadjoint and bilinear with respect to creation and annihilation operators. Let $H$ be diagonalized $[1,2]$ by a transformation defined by $\phi_p(f)\rightarrow\phi_p(Tf)$ for any $f\in\mathcal{K}$. Then $\Omega_T$ is the physical vacuum of the Hamiltonian. If $T$ is weakly $\Theta$-unitarily implementable, then $\rho_T(\ldots)=\langle\Omega_T,\ldots\Omega_T\rangle$ is a normalized $\Theta$-self-adjoint linear functional on the field algebra, which typically appears in QED-type models. $\rho_T$ is called a Lorentz state in $[2]$.

Theorem 5 implies that the linear functional $\rho_T$ defined by

$$\rho_T(P(\phi_p(f))=\langle\Omega,P(\phi_p(Tf))\Omega\rangle$$

cannot be a continuous state in general on the $C^*$-algebra generated by $\{\exp[i\phi(f)]; f\in\mathcal{K}\}$, where $\phi(f)$ is the selfadjoint Segal's field.

The converse problem, namely to obtain a representation (or $T$) from the expectation values, is the problem which must be solved to construct a QED-type model in a mathematically rigorous way $[2]$. This corresponds to a generalization of the GNS-construction. This will be discussed someday.

References
