<table>
<thead>
<tr>
<th>Title</th>
<th>Canonical Linear Transformation on Fock Space with an Indefinite Metric (同型写像と非有界微分子)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>ITO, K.R.</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1978), 320: 52-61</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1978-03</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/104004">http://hdl.handle.net/2433/104004</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Canonical Linear Transformation

on Fock Space with an Indefinite Metric

K.R. Ito

Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606, Japan.

Abstract: We first construct a Fock space with an indefinite metric $\langle , \rangle = ( , \theta )$, where $\theta$ is a unitary and hermitian operator. We define a $\theta$-selfadjoint (Segal's) field $\phi_\theta(f)$ which obeys the canonical commutation relations (CCR) with an indefinite metric. We consider a transformation $\phi_\theta(f) \to \phi_\theta(Tf)$ ($T$=real linear) which leaves the CCR invariant. We investigate the implementability of $T$ by an operator on the Fock space.

Let $\mathcal{H}_1$ ($1=+,-$) be Hilbert spaces equipped with usual positive definite hermitian inner product $\langle , \rangle_1$. Let $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ be a Hilbert space equipped with the inner product $\langle , \rangle = \mathbb{I}_1( , )_1$. Let $P_\pm$ be selfadjoint projections onto $\mathcal{H}_\pm$. Then the Hilbert space equipped with an hermitian inner product $\langle , \rangle \equiv ( , \varphi )$ with $\varphi = P_+ - P_-$ is called a "Hilbert space with an indefinite metric".

Let $S_n$ be the usual (n-fold) symmetrization operator on the n-fold tensor product space $\otimes_n \mathcal{H}$, and let

$\mathcal{F}^{(n)} \equiv S_n[ \otimes_n \mathcal{H} ]$

be the n-particle (Fock) space. The total Fock space

$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}^{(n)}$

-1-
is also given by
\[ \mathcal{F}(\mathcal{H}_+) \otimes \mathcal{F}(\mathcal{H}_-), \]
where \( \mathcal{F}(\mathcal{H}_+) \) and \( \mathcal{F}(\mathcal{H}_-) \) are Fock spaces constructed from \( \mathcal{H}_+ \)
and \( \mathcal{H}_- \) respectively. For an operator \( A \) on \( \mathcal{H} \), define \( \Gamma(A) \)
by
\[ \Gamma(A) \mathcal{F}(n) = \mathcal{F}(n), \]
\[ \Gamma(A)\mathcal{F}(n) = A \otimes \cdots \otimes A \text{ (n-times)}. \]
Then \( \Theta \equiv \Gamma(\varphi) \) is again an unitary and hermitian operator on \( \mathcal{F} \).

We define an indefinite sesquilinear form in \( \mathcal{F} \) by
\[ \langle , \rangle = ( , \Theta ). \]
The adjoint of \( A \) with respect to \( \langle , \rangle \) is denoted by \( A^{(\Theta)} \)
and equals \( \Theta A^* \Theta \).

Definition 1: (1) For \( f \in \mathcal{H} \), the creation operator \( a^*(f) \)
is defined by
\[ a^*(f) : \mathcal{F}(n) \rightarrow \mathcal{F}(n+1) \]
\[ \phi \rightarrow \sqrt{n+1} S_{n+1} [f \otimes \phi]. \]
(2) For \( f \in \mathcal{H} \), define the \( \Theta \)-selfadjoint (Segal's) field by
\[ \phi_\varphi(f) = \frac{1}{\sqrt{2}} [a^*(f) + [a^*(f)]^{(\Theta)}]. \]
where \( - \) denotes the closure.

Since \( [a^*(f)]^{(\Theta)} = [a^*(\varphi f)]^* \) with \( \varphi = p_+ - p_- \), \( \phi_\varphi \) is a normal operator. \( \{\phi_\varphi(f)\} \) obey the CCR with an indefinite metric:
\[ [\phi_\varphi(f), \phi_\varphi(g)] = \text{Im} \langle \bar{f}, g \rangle = -1 \text{ Re} \langle \bar{f}, J f g \rangle \]
where \( \bar{f} \) is the complex conjugation of \( f \) and \( J = \sqrt{-1} \) is a multiplication operator of \( i \).
Definition 2: (1) An invertible real linear transformation $T$ is called $\varphi$-symplectic if it satisfies
\[ T(\varphi)J = J \]
where $T(\varphi) = \varphi T^* \varphi$ and $T^*$ is the adjoint of $T$ with respect to $\text{Re}(\ , \ )$ in $\mathcal{H}$. (If $T$ is complex linear, then this adjoint is equivalent to the usual adjoint with respect to $(\ , \ )$ in $\mathcal{H}$.)
(2) $T_\pm = \frac{1}{2} [T \pm i JT^{-1}].$ Especially anti-linear part $T_-$ is called the off-diagonal part of $T$.

Our purpose is to investigate an operator which is expected to implement $U_T \Phi_\varphi(f) U_T^{-1} = \Phi_\varphi(Tf)$, and to investigate the new vacuum $\Omega_T = U_T^{-1} \Omega$. Here $\Omega \in \mathcal{F}^{(0)} = \mathcal{C}$ is the Fock vacuum. Since $\Phi_\varphi(f) \Phi_\varphi(Tf)$ leaves the CCR invariant, one may expect that $U_T$ is a $\Theta$-unitary (bijective $\Theta$-isometric) operator.

Definition 3: (1) $T$ is called $\Theta$-unitarily implementable if there is a $\Theta$-unitary (bijective $\Theta$-isometric) operator $U_T$ which implements $U_T \Phi_\varphi(f) U_T^{-1} = \Phi_\varphi(Tf)$.
(2) $T$ is called weakly $\Theta$-unitarily implementable if there exist a $\Theta$-isometric (not necessarily bounded) operator $U_T^{-1}$ and a cyclic vector $\Omega_T \in \mathcal{F}$ such that
\[ U_T^{-1} P(\Phi(f)) \Omega = P(\Phi_\varphi(Tf)) \Omega_T, \]
where $P(\Phi(f)) = P(\Phi_\varphi(f_1), \ldots, \Phi_\varphi(f_n))$ is any polynomial of $\{\Phi_\varphi(f_i)\}$.
(3) $T$ is called $\Theta$-unitarily quasi-implementable if the Fredholm determinant $\text{det}[1 + T_-(\varphi) T_-]$ uniformly converges to a non-vanishing finite value in $(0, \infty)$.

When $\varphi = 1$ (namely when $\Theta = 1$), three notions in this definition coincide each other $[1, 3, 4]$. For the implementability, the author proved [1]:

-3-
Theorem 1: T is \( \Theta \)-unitarily implementable if and only if \( T_- \) is Hilbert-Schmidt and \( [T, \varphi] = 0 \). In this case \( U_T \) is a unitary operator commuting with \( \Theta \).

Theorem 2: Let \( U_T^{-1} \Omega = \Omega_T \epsilon \varphi \). Then

1. \( T_- \in \text{H.S.} \) (H.S. denotes the Hilbert-Schmidt class),
2. \( (\infty, 0] \) is in the resolvent set of \( T_+^{(\varphi)} T_+ = 1 + T_-^{(\varphi)} T_- \).

In order to obtain a sufficient condition, we propose a \( \varphi \)-polar decomposition of \( T \), namely a decomposition of \( T \) in terms of a \( \varphi \)-selfadjoint operator and a \( \varphi \)-unitary operator.

Theorem 3: Let a \( \varphi \)-symplectic operator \( T \) satisfy the conditions in Theorem 2. Then \( T \) has a decomposition

\[
T = UH,
\]

where \( U \) is a \( \varphi \)-unitary operator (which commutes with \( J \)) and \( H \) is a \( \varphi \)-selfadjoint \( \varphi \)-symplectic operator with its spectrum in the right half plane.

Definition 4: A \( \varphi \)-selfadjoint \( \varphi \)-symplectic operator \( S \) is called a generalized \( \varphi \)-scaling if \( S \) leaves \( K \) and \( JK \) invariant where \( \mathcal{K} = K \otimes JK \) and \( \Theta \) refers to the orthogonality with respect to both \( \text{Re} \langle , \rangle \) and \( \text{Re} \langle , \rangle^\ast \).

A generalized \( \varphi \)-scaling \( S \) takes the following form on \( K \otimes JK \):

\[
\begin{pmatrix}
\begin{bmatrix} h & 0 \\
0 & h^{-1}
\end{bmatrix}
\end{pmatrix}.
\]

Here \( ChC = h = h \), where \( C \) is a complex conjugation operator:

\[
K = \{ x \in \mathcal{H} ; Cx = x \}.
\]

Is \( H \) in Theorem 3 always similar to a generalized \( \varphi \)-scaling \( S \).
through suitable \( \Phi \)-unitary operator \( V \) ? (This holds if \( \Phi = 1 \) \([1,3,4]\).)

\[ H = V S V^{-1}. \]

If this is the case, we have a decomposition

\[ T = V_1 S V_2 \]

under the conditions of Theorem 2, where \( V_1 \) are \( \Phi \)-unitary. But \( V \) seems unbounded in general.

For a generalized \( \Phi \)-scaling \( S \), we can obtain rather concrete theorems \([1]\). It sometimes suffices to consider generalized \( \Phi \)-scalings for physical applications \([1,2]\).

**Theorem 4**: For a generalized \( \Phi \)-scaling \( S \), if \( S \in \mathcal{H} \cdot S_1 \), and if \( \alpha_r \) is selfadjoint part of \( \Phi \)-selfadjoint operator \( h^{-2} > 0 \), then

(1) both \( S \) and \( S^{-1} \) are weakly \( \Theta \)-unitarily implementable.

(2) The overlap between \( \Omega \) and \( \Omega_S \) is given by

\[
|<\Omega, \Omega_S>| = \det^{-1/4}[1 + S_\Phi S_-]
= \det^{-1/4}[1 + \frac{1}{4}(h^{-1})^2].
\]

This is non-vanishing finite.

**Theorem 5**: In Theorem 4, if \( \inf \text{spec}(\alpha_r) < 0 \), then the vector \( \Omega_S \) which satisfies

\[
<\Omega_S, P(\varphi(f))\Omega_S> = <\Omega, P(\varphi(Sf))\Omega>
\]

cannot be in the Fock space: \( \|\Omega_S\| = \infty \).

As is well known, when \( \Phi = 1 \), the necessary and sufficient condition for \( T \) to be unitarily implementable is \( T \in \mathcal{H} \cdot S_1 \). Then for \( \Phi = 1 \), the overlap of the vacua does not vanish if and only
if $T$ is unitarily implemented. In fact when $\varphi = 1$, we have $T = U_1 S U_2$ where $U_1$ are unitaries commuting with $J$. Further since transformsions $\phi_{\varphi = 1}(f) + \phi_{\varphi = 1}(U_1 f)$ are implemented by unitaries $\Gamma(U_1)$ on the Fock space, we have $U_T = \Gamma(U_1) U_S \Gamma(U_2)$. Then $\Omega_T = \Gamma(U_2)^{-1} \Omega_S$ and $(\Omega, \Omega_T) = (\Omega, \Omega_S)$.

For given $S$, let $T = V_1 S V_2$ where $V_1$ are $\varphi$-unitaries. Then $S \in H.S. \iff T \in H.S.$

and

$$\det[1 + S(\varphi) S^{-1}] = \det[1 + T(\varphi) T^{-1}]$$

Since $\Gamma(V_1)$ are not bounded operators, $T$ is not necessarily weakly $\theta$-unitarily implementable even if $S$ is weakly $\theta$-unitarily implementable. But the above equation means that the formal overlap $\det^{-1/4}[1 + T(\varphi) T^{-1}]$ is an invariant quantity under $\varphi$-unitaries. Furthermore if $\varphi \neq 1$, $\det^{-1/4}[1 + S(\varphi) S^{-1}]$ can converge to a non-vanishing (finite) quantity even if $S \notin H.S.$ Then Definition 3 (3) implies that the formally defined overlap is non-vanishing (finite), which is equivalent to the unitarily implementability of $S$ when $\varphi = 1$.

(Sketch of the proof of Theorem 4)

Let $K = K_+ \oplus K_-$ ($K_+ = P_+ K$) and let $\{e_1\}$ be complete orthonormal basis in $K$ with respect to both $\text{Re}(\ , \ )$ and $\text{Re}<\ , \ >$. We use the following unitary transformation $W$:

$$W \mathcal{K} = L^2(Q; d\mu_0),$$

$$Q = \mathbb{R}^\infty, \quad d\mu_0 = \prod_{i=1}^\infty \exp[\frac{-q_i^2}{2}] \frac{dq_i}{\sqrt{\pi}},$$

$$W \Omega = 1,$$

$$W[\phi_{\varphi}(e_1)] W^{-1} = \begin{cases} q_i & e_1 \in K_+ \\ -q_i & e_1 \in K_- \end{cases}$$
\[
W[\Phi_\psi(Je_1)]W^{-1} = \begin{cases}
-ia/q_1 + iq_1 & e_1 \in K_+ \\
-a/q_1 + q_1 & e_1 \in K_-
\end{cases}
\]

Note that
\[
[a^*(e_1)](\theta) = \frac{1}{\sqrt{2}}[\Phi_\psi(e_1) + i\Phi_\psi(Je_1)].
\]

Since the transformed vacuum should satisfy
\[
[\Phi_\psi(S^{-1}e_1) + i\Phi_\psi(S^{-1}Je_1)]\Omega_S = 0,
\]
\[
\langle \Omega_S, \Omega_S \rangle = 1,
\]
we have [1]
\[
\Omega_S = [\det (a)]^{1/4} \exp[-\frac{1}{2}(q,(a-1)q)]
\]
where
\[
(q,aq) = \sum_{ij} q_i a_{ij} q_j
\]
and
\[
a_{ij} = (e_1, \psi^* h^{-2} \psi_j), \quad \psi = p_+ + i p_-.
\]
Remark that \(
a\) is a \(\varphi\)-selfadjoint symmetric matrix.

Under the conditions of Theorem 4, we can prove that
\[
\Omega_S = \Omega_S(q) \in L^2(q, d\mu_0) = \mathcal{F}
\]
and the cyclicity of \(\Omega_S\) [1]. Further
\[
\Omega_{S^{-1}} = [\det(a^{-1})]^{1/4} \exp[-\frac{1}{2}(q,(a^{-1}-1)q)].
\]
Let \(a = a_r + ia_1\) where \(a_r\) and \(a_1\) are selfadjoint real matrices (this follows from the properties of \(a\)). If \(a_r\) is positive (then strictly positive since \(a_r - 1\) is H.S.), since
\[
(a^{-1})_r = (a_r + a_1 a_r^{-1} a_1)^{-1},
\]
\[
(a^{-1})_1 = -a_1^{-1} a_1 (a_r + a_1 a_r^{-1} a_1)^{-1},
\]
then \((a^{-1})_r\) is again a (strictly) positive operator. Thus \(\Omega_{S^{-1}} \in \mathcal{F}\). The \(\theta\)-isometricity of \(U_{S^{-1}}\) follows from
\[
\langle \Omega_S, P(\Phi_\psi(f)) \Omega_S \rangle = \langle \Omega, P(\Phi_\psi(Sf)) \Omega \rangle
\]
which is proved in [1]. Finally
\[<\Omega, \Omega_S> = <\Omega_S, \Omega> = \int \Omega_S(q) d\nu_0 = \det^{-1/4} [1 + \mathbf{S} - (\mathbf{S}^-)_-].\]

From the above proof, the reader can guess that \( \alpha > 0 \) is needed to ensure \( \|\Omega_S\| < \infty \).

(Sketch of the proof of Theorem 5)

Since \( \alpha \) is a \( \mathbf{S} \)-selfadjoint operator, \( \alpha \) takes the following form on \( \mathcal{J}_K \cap \mathcal{K}_- \):

\[
\begin{pmatrix}
(\alpha_{\mathbf{R}})_{++} & 1(\alpha_{\mathbf{L}})_{+-} \\
1(\alpha_{\mathbf{L}})_{-+} & (\alpha_{\mathbf{R}})_{-+}
\end{pmatrix}, \quad \alpha_{ij} = \mathbf{P}_i \mathbf{P}_j.
\]

First assume that \( f \in \mathcal{J}_K \) be an eigenvector of \( \alpha_{\mathbf{R}} \) belonging to the eigenvalue \( -\lambda < 0 \). Since \( \phi_{\mathbf{S}}(f) \) is selfadjoint,

\[\|\exp[i\phi_{\mathbf{S}}(f)]\| = 1.\]

Note
\[<\Omega, \exp[i\phi_{\mathbf{S}}(f)]\Omega_S> = <\Omega, \exp[i\phi_{\mathbf{S}}(Sf)]\Omega>\]
\[= \exp[-\frac{1}{4} \langle Sf, Sf \rangle] = \exp[-\frac{\lambda}{4} \|f\|^2].\]

If \( \lambda > 0 \), the right hand side can be made arbitrarily large, which contradicts

\[|<\Omega, \exp[i\phi_{\mathbf{S}}(f)]\Omega_S>| \leq \|\Omega_S\|^2 < \infty.\]

The case of \( f \in \mathcal{J}_K \) is similarly discussed.

Our theory can be applied for quantum electrodynamics-type models where \( \mathcal{F}(\mathcal{H}_-) \) is the Fock space of the gaugeon (ghost particle which has a negative norm) and \( \mathcal{F}(\mathcal{H}_+) \) is the Fock space of physical particles (photon, etc.). In these
models, the Hamiltonian $H$ is expected to be $\Theta$-selfadjoint
(namely $H\Theta$ is selfadjoint). As a simple example, let $H$ be
$\Theta$-selfadjoint and bilinear with respect to creation and annihi-
lation operators. Let $H$ be diagonalized $[1,2]$ by a transform-
ation defined by $\Phi_{\psi}(f) \rightarrow \Phi_{\psi}(Tf)$ for any $f \in \mathcal{H}$. Then $\Omega_T$ is the
physical vacuum of the Hamiltonian. If $T$ is weakly $\Theta$-unitarily
implementable, then $\rho_T(\ldots) = \langle \Omega_T \ldots \Omega_T \rangle$ is a normalized $\Theta$-self-
adjoint linear functional on the field algebra, which typically
appears in QED-type models. $\rho_T$ is called a Lorentz state in
[2].

Theorem 5 implies that the linear functional $\rho_T$ defined by
\[ \rho_T(P(\Phi_{\psi}(f)) = \langle \Omega, P(\Phi_{\psi}(Tf)) \Omega \rangle \]
cannot be a continuous state in general on the $C^*$-algebra gene-
rated by $\{ \exp[i\Phi(f)]; f \in \mathcal{K} \}$, where $\Phi(f)$ is the selfadjoint
Segal's field.

The converse problem, namely to obtain a representation
(or $T$) from the expectation values, is the problem which
must be solved to construct a QED-type model in a mathematically
rigorous way $[2]$. This corresponds to a generalization of the
GNS-construction. This will be discussed someday.

References

[1] K.R.Ito, "Canonical Linear Transformation on Fock Space
with an Indefinite Metric", RIMS-preprint (1977) to appear
in Publ.RIMS.