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Canonical Linear Transformation on Fock Space with an Indefinite Metric

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Abstract: We first construct a Fock space with an indefinite metric $\langle\ ,\ \rangle=(\ ,\Theta\ )$, where Θ is a unitary and hermitian operator. We define a Θ-selfadjoint (Segal's) field $\Phi_{\varphi}(f)$ which obeys the canonical commutation relations (CCR) with an indefinite metric. We consider a transformation $\Phi_{\varphi}(f)\to\Phi_{\varphi}(Tf)$ (T=real linear) which leaves the CCR invariant. We investigate the implementability of T by an operator on the Fock space.

Let $\mathcal{H}_i$ (i=+,-) be Hilbert spaces equipped with usual positive definite hermitian inner product $(\ ,\ )_i$. Let $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$ be a Hilbert space equipped with the inner product $(\ ,\ )=\Sigma_i(\ ,\ )_i$. Let $P_\pm$ be selfadjoint projections onto $\mathcal{H}_\pm$. Then the Hilbert space equipped with an hermitian inner product $\langle\ ,\ \rangle\equiv(\ ,\varphi\ )$ with $\varphi=P_+-P_-$ is called a "Hilbert space with an indefinite metric".

Let S_n be the usual (n-fold) symmetrization operator on the n-fold tensor product space $\otimes_n\mathcal{H}$, and let

$$\mathcal{F}^{(n)}\equiv S_n[\otimes_n\mathcal{H}]$$

be the n-particle (Fock) space. The total Fock space

$$\mathcal{F}=\oplus_{n=0}^{\infty}\mathcal{F}^{(n)}$$

is also given by

$$\mathcal{F}(\mathcal{H}_+)\otimes\mathcal{F}(\mathcal{H}_-) ,$$

where $\mathcal{F}(\mathcal{H}_+)$ and $\mathcal{F}(\mathcal{H}_-)$ are Fock spaces constructed from $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively. For an operator A on $\mathcal{H}$, define $\Gamma(A)$ by

$$\Gamma(A)\ \mathcal{F}^{(n)} \subset \mathcal{F}^{(n)} ,$$

$$\Gamma(A)\restriction\mathcal{F}^{(n)}=A\otimes\ldots\otimes A \quad \text{(n-times)}.$$

Then $\Theta\equiv\Gamma(\varphi)$ is again an unitary and hermitian operator on $\mathcal{F}$. We define an indefinite sesquilinear form in $\mathcal{F}$ by

$$\langle\ ,\ \rangle=(\ ,\Theta\ ).$$

The adjoint of A with respect to $\langle\ ,\ \rangle$ is denoted by $A^{(\Theta)}$ and equals $\Theta A^*\Theta$.

Definition 1: (1) For $f\in\mathcal{H}$, the creation operator $a^*(f)$ is defined by

$$\begin{array}{rccl} a^*(f)\ : & \mathcal{F}^{(n)} & \to & \mathcal{F}^{(n+1)} \\ & \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ & \phi & \mapsto & \sqrt{n+1}\ S_{n+1}[f\otimes\phi]. \end{array}$$

(2) For $f\in\mathcal{H}$, define the Θ-selfadjoint (Segal's) field by

$$\Phi_\varphi(f)=\frac{1}{\sqrt{2}}[a^*(f)+[a^*(f)]^{(\Theta)}]^-$$

where - denotes the closure.

Since $[a^*(f)]^{(\Theta)}=[a^*(\varphi f)]^*$ with $\varphi=P_+-P_-$, Φ_φ is a normal operator. $\{\Phi_\varphi(f)\}$ obey the CCR with an indefinite metric:

$$[\Phi_\varphi(f),\ \Phi_\varphi(g)]=i\ \mathrm{Im}\langle\bar{f},g\rangle=-i\ \mathrm{Re}\ (\bar{f},\varphi Jg)$$

where $\bar{f}$ is the complex conjugation of f and $J=\sqrt{-1}$ is a multiplication operator of i.

Definition 2: (1) An invertible real linear transfomation T is called φ-symplectic if it satisfies

$$T^{(\varphi)}JT=J$$

where $T^{(\varphi)}=\varphi T^{*}\varphi$ and T^{*} is the adjoint of T with respect to Re (,) in $\mathcal{H}$. (If T is complex linear, then this adjoint is equivalent to the usual adjoint with respect to (,) in $\mathcal{H}$.)
(2) $T_{\pm} = \frac{1}{2}[T \pm JTJ^{-1}]$. Especially anti-linear part T_{-} is called the off-diagonal part of T.

Our purpose is to investigate an operator which is expected to implement $U_T\Phi_\varphi(f)U_T^{-1}=\Phi_\varphi(Tf)$, and to investigate the new vacuum $\Omega_T=U_T^{-1}\Omega$. Here $\Omega\in\mathcal{F}^{(0)}=C$ is the Fock vacuum. Since $\Phi_\varphi(f)\to\Phi_\varphi(Tf)$ leaves the CCR invariant, one may expect that U_T is a Θ-unitary (bijective Θ-isometric) operator.

Definition 3: (1) T is called Θ-unitarily implementable if there is a Θ-unitary (bijective Θ-isometric) operator U_T which implements $U_T\Phi_\varphi(f)U_T^{-1}=\Phi_\varphi(Tf)$.
(2) T is called weakly Θ-unitarily implementable if there exist a Θ-isometric (not necessarily bounded) operator U_T^{-1} and a cyclic vector $\Omega_T\in\mathcal{F}$ such that

$$U_T^{-1}P(\Phi_\varphi(f))\Omega=P(\Phi_\varphi(Tf))\Omega_T,$$

where $P(\Phi_\varphi(f))=P(\Phi_\varphi(f_1),\ldots,\Phi_\varphi(f_n))$ is any polynomial of $\{\Phi_\varphi(f_i)\}$.
(3) T is called Θ-unitarily quasi-implementable if the Fredholm determinant $\det[1+T_{-}^{(\varphi)}T_{-}]$ uniformly converges to a non-vanishing finite value in $(0,\infty)$.

When $\varphi=1$ (namely when Θ=1), three notions in this definition coincide each other [1,3,4]. For the implementability, the author proved [1]:

Theorem 1: T is Θ-unitarily implementable if and only if T_- is Hilbert-Schmidt and $[T,\varphi]=0$. In this case U_T is a unitary operator commuting with Θ.

Theorem 2: Let $U_T^{-1}\Omega=\Omega_T\in\mathcal{F}$. Then
(i) $T_-\in$ H.S. (H.S. denotes the Hilbert-Schmidt class),
(ii) $(-\infty,0]$ is in the resolvent set of $T_+^{(\varphi)}T_+=1+T_-^{(\varphi)}T_-$.

In order to obtain a sufficient condition, we propose a φ-polar decomposition of T, namely a decomposition of T in terms of a φ-selfadjoint operator and a φ-unitary operator.

Theorem 3: Let a φ-symplectic operator T satisfy the conditions in Theorem 2. Then T has a decomposition

$$T=UH,$$

where U is a φ-unitary operator (which commutes with J) and H is a φ-selfadjoint φ-symplectic operator with its spectrum in the right half plane.

Definition 4: φ-selfadjoint φ-symplectic operator S is called a generalized φ-scaling if S leaves K and JK invariant where $\mathcal{H}=K\oplus JK$ and $\oplus$ referes the orthogonality with respect to both Re(,) and Re< , > .

A generalized φ-scaling S takes the following form on $K\oplus JK$:

$$\begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} .$$

Here $ChC=\bar{h}=h$, where C is a complex conjugation operator:

$$K=\{x\in\mathcal{H}\ ;\ Cx=x\} .$$

Is H in Theorem 3 always similar to a generalized φ-scaling S through suitable φ-unitary operator V ? (This holds if $\varphi=1$ [1,3,4].)

$$H=VSV^{-1}.$$

If this is the case, we have a decomposition

$$T=V_1SV_2$$

under the conditions of Theorem 2, where V_i are φ-unitary. But V seems unbounded in general.

For a generalized φ-scaling S, we can obtain rather concrete theorems [1]. It sometimes suffices to consider generalized φ-scalings for physical applications [1,2].

Theorem 4: For a generalized φ-scaling S, if $S_-\in$ H.S., and if $\alpha_r \equiv$ selfadjoint part of φ-selfadjoint operator $h^{-2}>0$, then

(i) both S and S^{-1} are weakly Θ-unitarily implementable.

(ii) The overlap between Ω and Ω_S is given by

$$|\langle\Omega,\Omega_S\rangle| = \det{}^{-1/4}[1+S_-^{(\varphi)}S_-]$$
$$= \det{}^{-1/4}[1+\frac{1}{4}(h-h^{-1})^2].$$

This is non-vanishing finite.

Theorem 5: In Theorem 4, if inf spec$(\alpha_r)<0$, then the vector Ω_S which satisfies

$$\langle\Omega_S,P(\Phi_\varphi(f))\Omega_S\rangle=\langle\Omega,P(\Phi_\varphi(Sf))\Omega\rangle$$

cannot be in the Fock space: $\|\Omega_S\|=\infty$.

As is well known, when $\varphi=1$, the necessary and sufficient condition for T to be unitarily implementable is $T_-\in$ H.S. Then for $\varphi=1$, the overlap of the vacua does not vanish if and only if T is unitarily implemented. In fact when $\varphi=1$, we have $T=U_1SU_2$ where U_i are unitaries commuting with J. Further since transformarions $\Phi'_{\varphi=1}(f)\to\Phi_{\varphi=1}(U_if)$ are implemented by unitaries $\Gamma(U_i)$ on the Fock space, we have $U_T=\Gamma(U_1)U_S\Gamma(U_2)$. Then $\Omega_T=\Gamma(U_2)^{-1}\Omega_S$ and $(\Omega,\Omega_T)=(\Omega,\Omega_S)$.

For given S, let $T=V_1SV_2$ where V_i are φ-unitaries. Then

$$S_-\in H.S. \quad \leftrightarrow \quad T_-\in H.S.$$

and

$$\det[1+S_-^{(\varphi)}S_-]=\det[1+T_-^{(\varphi)}T_-].$$

Since $\Gamma(V_i)$ are not bounded operators, T is not necessarily weakly Θ-unitarily implementable even if S is weakly Θ-unitarily implementable. But the above equation means that the formal overlap $\det^{-1/4}[1+T_-^{(\varphi)}T_-]$ is an invariant quantity under φ-unitaries. Furthermore if $\varphi\neq 1$, $\det^{-1/4}[1+S_-^{(\varphi)}S_-]$ can converge to a non-vanishing (finite) quantity even if $S_-\notin H.S.$ Then Definition 3 (3) implies that the formally defined overlap is non-vanishing (finite), which is equivalent to the unitarily implementability of S when $\varphi=1$.

(Sketch of the proof of Theorem 4)

Let $K=K_+\oplus K_-$ $(K_\pm=P_\pm K)$ and let $\{e_i\}$ be complete orthonormal basis in K with respect to both Re(,) and Re< , >. We use the following unitary transformation W :

$$W\mathcal{F}=L^2(Q;d\mu_0),$$

$$Q=R^\infty, \quad d\mu_0=\Pi_{i=1}^\infty \exp[-q_i^2]\frac{dq_i}{\sqrt{\pi}},$$

$$W\Omega=1,$$

$$W[\Phi_\varphi(e_i)]W^{-1}=\begin{cases} q_i & e_i\in K_+ \\ -iq_i & e_i\in K_- ,\end{cases}$$

$$W[\Phi_{\varphi}(Je_i)]W^{-1}=\begin{cases} -i\partial/\partial q_i \ +iq_i & e_i\in K_+ \\ -\partial/\partial q_i \ +q_i & e_i\in K_- \end{cases} .$$

Note that

$$[a^*(e_i)]^{(\Theta)}=\frac{1}{\sqrt{2}}[\Phi_{\varphi}(e_i)+i\Phi_{\varphi}(Je_i)] .$$

Since the transformed vacuum should satisfy

$$[\Phi_{\varphi}(S^{-1}e_i)+i\Phi_{\varphi}(S^{-1}Je_i)]\Omega_S=0,$$

$$<\Omega_S,\Omega_S>=1 ,$$

we have [1]

$$\Omega_S=[\det\ (\alpha)]^{1/4}\ \exp[-\ \frac{1}{2}(q,(\alpha-1)q)]$$

where

$$(q,\alpha q)=\Sigma_{ij}\ q_i\alpha_{ij}q_j$$

and

$$\alpha_{ij}=(e_i,\psi^*h^{-2}\psi e_j)\ ,\ \psi=P_++iP_-.$$

Remark that α is a φ-selfadjoint symmetric matrix.

Under the conditions of Theorem 4, we can prove that $\Omega_S=\Omega_S(q)\in L^2(Q,d\mu_0)=\mathcal{H}$ and the cyclicity of Ω_S [1]. Further

$$\Omega_{S^{-1}}=[\det(\alpha^{-1})]^{1/4}\exp[-\frac{1}{2}(q,(\alpha^{-1}-1)q)].$$

Let $\alpha=\alpha_r+i\alpha_i$ where α_r and α_i are selfadjoint real matrices (this follows from the properties of α). If α_r is positive (then strictly positive since α_r-1 is H.S.) , since

$$(\alpha^{-1})_r=(\alpha_r+\alpha_i\alpha_r^{-1}\alpha_i)^{-1},$$

$$(\alpha^{-1})_i=-\alpha_r^{-1}\alpha_i(\alpha_r+\alpha_i\alpha_r^{-1}\alpha_i)^{-1},$$

then $(\alpha^{-1})_r$ is again a (strictly) positive operator. Thus $\Omega_S-1\in\mathcal{H}$. The Θ-isometricity of U_S^{-1} follows from

$$<\Omega_S,P(\Phi_{\varphi}(f))\Omega_S> =<\Omega,P(\Phi_{\varphi}(Sf))\Omega>$$

which is proved in [1]. Finally

$$\langle\Omega,\Omega_S\rangle=\langle\Omega_S,\Omega\rangle=\int\Omega_S(q)d\mu_0=\det^{-1/4}[1+S_-^{(\varphi)}S_-].$$

□

From the above proof, the reader can guess that $\alpha_r>0$ is needed to ensure $\|\Omega_S\|<\infty$.

(Sketch of the proof of Theorem 5)

Since α is a φ-selfadjoint operator, α takes the following form on $JK_+\oplus JK_-$:

$$\begin{pmatrix} (\alpha_r)_{++} & i(\alpha_i)_{+-} \\ i(\alpha_i)_{-+} & (\alpha_r)_{--} \end{pmatrix}, \quad \alpha_{ij}=P_i\alpha P_j.$$

First assume that $f\in JK_+$ be an eigenvector of α_r belonging to the eigenvalue $-\lambda<0$. Since $\Phi_\varphi(f)$ is selfadjoint,

$$\|\exp[i\Phi_\varphi(f)]\|=1.$$

Note

$$\langle\Omega_S,\exp[i\Phi_\varphi(f)]\Omega_S\rangle=\langle\Omega,\exp[i\Phi_\varphi(Sf)]\Omega\rangle$$

$$=\exp[-\frac{1}{4}\langle Sf,Sf\rangle]=\exp[\frac{\lambda}{4}\|f\|^2] .$$

If $\lambda>0$, the right hand side can be made arbitralily large, which contradicts

$$|\langle\Omega_S,\exp[i\Phi_\varphi(f)]\Omega_S\rangle|\leq\|\Omega_S\|^2<\infty .$$

The case of $f\in JK_-$ is similarly dicussed.

□

Our theory can be applied for quantum electrodynamics-type models where $\mathcal{F}(\mathcal{H}_-)$ is the Fock space of the gaugeon (ghost particle which has a negative norm) and $\mathcal{F}(\mathcal{H}_+)$ is the Fock space of physical particles (photon, etc.). In these models, the Hamiltonian H is expected to be Θ-selfadjoint (namely $H\Theta$ is selfadjoint). As a simple example, let H be Θ-selfadjoint and bilinear with respect to creation and annihilation operators. Let H be diagonalized [1,2] by a transformation defined by $\Phi_\varphi(f)\to\Phi_\varphi(Tf)$ for any $f\in\mathcal{H}$. Then Ω_T is the physical vacuum of the Hamiltonian. If T is weakly Θ-unitarily implementable, then $\rho_T(..)=\langle\Omega_T,..\Omega_T\rangle$ is a normalized Θ-selfadjoint linear functional on the field algebra, which typically appears in QED-type models. ρ_T is called a Lorentz state in [2].

Theorem 5 implies that the linear functional ρ_T defined by

$$\rho_T(P(\Phi_\varphi(f))=\langle\Omega,P(\Phi_\varphi(Tf))\Omega\rangle$$

cannot be a continuous state in general on the C^*-algebra generated by $\{\exp[i\Phi(f)];\ f\in\mathcal{H}\}$, where $\Phi(f)$ is the selfadjoint Segal's field.

The converse problem, namely to obtain a representation (or T) from the expectation values, is the problem which must be solved to construct a QED-type model in a mathematically rigorous way [2]. This corresponds to a generalization of the GNS-construction. This will be discussed someday.

-References-

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