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UNBOUNDED DERIVATIONS IN COMMUTATIVE C*-ALGEBRAS

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§1. Closed *-derivations in a commutative C*-algebra and Silov algebras.

Let $\mathfrak{A} = C(K)$ be the C*-algebra of all complex valued continuous functions on a compact Hausdorff space K. A linear mapping δ in $\mathfrak{A}$ is said to be a derivation in $\mathfrak{A}$ if it satisfies the following conditions:

(1) $\mathcal{D}(\delta)$ is a subalgebra of $\mathfrak{A}$ and separates the points of K, where $\mathcal{D}(\delta)$ is the domain of δ.

(2) $\delta(ab) = \delta(a)b + a\delta(b)$ $\quad (a, b \in \mathcal{D}(\delta))$.

Let δ be a derivation in $\mathfrak{A}$ and define $|||a|||_\delta = \left\|\begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix}\right\|$ $(\mathfrak{A}, b \in \mathcal{D}(\delta))$, where $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ $(x, y, z, w \in \mathfrak{A})$ is the matrix of 2×2 over $\mathfrak{A}$. Then $\mathcal{D}(\delta)$ is a normed algebra with $|||\cdot|||_\delta$, for $a \to \begin{pmatrix} a & \delta(a) \\ 0 & a \end{pmatrix}$ is an isomorphism.

1.1. Proposition. Let δ be a derivation in $\mathfrak{A}$ and suppose that $\mathcal{D}(\delta)$ is a ~~Banach~~ Silov (4.1.2) algebra on K under some norm $\|\cdot\|_1$, (cf. Def. 1.2) then $|||a|||_\delta \leqq k\|a\|_1$ $(a \in \mathcal{D}(\delta))$, where k is a fixed positive number.[1]

Proof. By Johnson's theorem (Theorem 3 in [2]) there is a finite family of mutually orthogonal idempotents $e_0, e_1, \ldots, e_n$ in $\mathcal{D}(\delta)$ such that for $p \in$ the support of e_0,

1) Longo has proved independenly that this proposition can be extended to a $*$-derivation in a non-commutative C*-algebra.

$a \to \delta(a)(p) \quad (a \in \mathcal{D}(\delta))$ is continuous with respect to $\|\cdot\|_1$ and $\mathcal{D}(\delta)e_i \ (i = 1,2,\ldots,n)$ has a unique maximal proper ideal and $\sum_{i=1}^{n} e_i = 1$. Since $\mathcal{D}(\delta)e_i \ (i = 1,2,\ldots,n)$ is semi-simple, it is one-dimensional and so the support of $e_i \ (i = 1,2,\ldots,n)$ consists of one point p_i.

Then $\delta(a)(p_i) = \delta(a)(p_i)e_i(p_i) = \delta(a\ e_i)(p_i) = 0$ $(a \in \mathcal{D}(\delta))$; hence $a \to \delta(a)(p) = f_p(a)$ is continuous with respect to $\|\cdot\|_1$ for each $p \in K$. Since $\{f_p | p \in K\}$ is compact in $\mathcal{D}(\delta)^*$, where $\mathcal{D}(\delta)^*$ is the dual of $\mathcal{D}(\delta)$ with respect to $\|\cdot\|_1$, $\sup_{p \in K} \|f_p\| < +\infty$.

Since $a \to a(p) \quad (a \in \mathcal{D}(\delta))$ is a character of the Banach algebra $\mathcal{D}(\delta)$ for each $p \in K$, $\|a\| \leqq \|a\|_1$ for $a \in \mathcal{D}(\delta)$. Hence $|||a|||_\delta = \sup_{p \in K} \left\| \begin{pmatrix} a(p) & \delta(a)(p) \\ 0 & a(p) \end{pmatrix} \right\| \leqq k\|a\|_1$ $(a \in \mathcal{D}(\delta))$. This completes the proof.

A derivation δ in $\mathfrak{A}$ is said to be a *-derivation if it satisfies:

(1) $\mathcal{D}(\delta)$ is a dense *-subalgebra of $\mathfrak{A}$,

(2) $\delta(ab) = \delta(a)b + a\delta(b) \qquad (a,b \in \mathcal{D}(\delta))$.

(3) $\delta(a^*) = \delta(a)^* \qquad (a \in \mathcal{D}(\delta))$.

1.2. Definition. A commutative Banach algebra A consisting of some of the continuous functions on a compact Hausdorff space X under a norm possibly larger than the sup norm is said to be a Silov algebra if for any point p of X and disjoint closed set S, A contains a function vanishing at S and not vanishing at p.

Let $\mathfrak{A} = C(K)$ and let δ be a closed *-derivation in $\mathfrak{A}$; then $\mathcal{D}(\delta)$ is a Banach *-algebra under the norm $|||\cdot|||_\delta$ with $|||a^*|||_\delta = |||a|||_\delta$ $(a \in \mathcal{D}(\delta))$.

1.3. Proposition. Let $\{\delta_\alpha | \alpha \in \Pi\}$ be a family of closed *-derivations in $\mathfrak{A}$ and let $\mathcal{D} = \bigcap_{\alpha\in\Pi} \mathcal{D}(\delta_\alpha)$. For $a \in \mathcal{D}$, define $|||a||| = \sup_{\alpha\in\Pi} |||a|||\delta_\alpha$ and let $\mathcal{D}_0 = \{a \mid |||a||| < +\infty,\ a \in \mathcal{D}\}$. Then $\mathcal{D}_0$ is a Banach *-algebra.

Proof. Let $\{a_n\}$ be a Cauchy sequence in $\mathcal{D}_0$ under $|||\cdot|||$; then it is Cauchy under $|||\cdot|||\delta_\alpha$ so that there is an element b_α such that $\|a_n - b_\alpha\| \to 0$ and $\|\delta_\alpha(a_n) - \delta_\alpha(b_\alpha)\| \to 0$. Therefore $b_\alpha = b_\beta = b$ for $\alpha, \beta \in \Pi$ and $b \in \mathcal{D}(\delta_\alpha)$ for each $\alpha \in \Pi$. For $\varepsilon > 0$, there is a positive number $n(\varepsilon)$ such that $|||a_m - a_n||| = \sup_{\alpha\in\Pi} |||a_m - a_n|||\delta_\alpha < \varepsilon$ for $m, n \geq n(\varepsilon)$. Hence $|||a_m - a_n|||\delta_\alpha < \varepsilon$ for $m, n \geq n(\varepsilon)$ and $\alpha \in \Pi$, and so $\sup_{\alpha\in\Pi} |||a_m - b|||\delta_\alpha = |||a_m - b||| \leqq \varepsilon$ for $n \geqq n(\varepsilon)$. This implies $|||b||| < +\infty$ and $a_m \to b$ in $\mathcal{D}_0$ and completes the proof.

1.4. Proposition. Suppose that $\mathcal{D}_0$ is dense in $\mathfrak{A}$; then $\mathcal{D}_0$ is a Silov algebra.

Proof. Let p_0 be a point of K and let S be a closed set in K such that $p_0 \notin S$. Take a positive element h in $\mathfrak{A}$ such that $h(p_0) = 1$ and $h(p) = 0$ for $p \in S$.

For $0 < \varepsilon < 1/3$ let $k \geqq 0$ with $\|h-k\| < \varepsilon$ and $k \in \mathcal{D}_0$; then $0 \leqq k(p) \leqq 1/3$ for $p \in S$ and $2/3 < k(p_0) < 4/3$. Let f be an infinitely differentiable function on the real line such that $f(t) = 0$ for $t \in [0,1/3]$ and $f(t) > 1$ for $t \in [\frac{2}{3},\frac{4}{3}]$; then $f(k) \in \mathcal{D}(\delta_\alpha)$ (cf. §3) for each $\alpha \in \Pi$ and $f(k)(p_0) = f(k(p_0)) \neq 0$, $f(k)(p) = f(k(p)) = 0$ for $p \in S$. Moreover $\delta_\alpha(f(k)) = f'(k) \bullet \delta_\alpha(k)$ (cf. §3) and so $\sup_{\alpha\in\Pi}|||f(k)|||\delta_\alpha \leqq \|f'(k)\| \sup_\alpha |||k|||\delta_\alpha + \|f(k)\| < +\infty$. This completes the proof.

Let δ be a *-derivation in $\mathfrak{A}$ and suppose that for some positive integer n, $\mathcal{D}(\delta^n)$ is dense in $\mathfrak{A}$; then $\mathcal{D}(\delta^n)$ is a dense *-subalgebra of $\mathfrak{A}$. It is clear that $\mathcal{D}(\delta^m) \supset \mathcal{D}(\delta^n)$ $(m \leqq n)$. For $a \in \mathcal{D}(\delta^n)$, define

$$|||a|||\delta^n = \left\| \begin{pmatrix} a & \delta(a) & \frac{\delta^2(a)}{2!} & \cdots\cdots & \frac{\delta^n(a)}{n!} \\ 0 & a & \delta(a) & \frac{\delta^2(a)}{2!} \cdots & \frac{\delta^{n-1}(a)}{(n-1)!} \\ 0 & 0 & a & 0 \cdots & \vdots \\ \vdots & 0 & 0 & a \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \delta(a) \\ 0 & 0 & 0 & 0 \cdots 0 & a \end{pmatrix} \right\| .$$

Then $\mathcal{D}(\delta^n)$ becomes a normed *-algebra under the norm

$$|||a|||_{\delta^n}, \text{ for } a \to \begin{pmatrix} 0 & \delta(a) & \cdots & \frac{\delta^n(a)}{n!} \\ 0 & a & \delta(a) \cdots & \vdots \\ 0 & 0 & a \cdots & \vdots \\ 0 & & 0 \ddots & \delta(a) \\ 0 & 0 & 0 & a \end{pmatrix} \quad \text{is an isomorphism.}$$

Suppose that δ is closed; then $\mathcal{D}(\delta^n)$ is a Banach *-algebra. Denote that

$$\Phi(a) = \begin{pmatrix} a & \delta(a) & \frac{\delta^2(a)}{2!} & \cdots & \frac{\delta^n(a)}{n!} \\ 0 & a & \delta(a) & \cdots & \vdots \\ \vdots & 0 & a & \cdots & \vdots \\ \vdots & 0 & \cdots & a \cdots & \frac{\delta^2(a)}{2!} \\ 0 & 0 & 0 & 0 & \delta(a) \\ 0 & 0 & 0 & 0 & a \end{pmatrix};$$

then $\Phi(f(a)) = f(\Phi(a))$ for $f \in C^\infty(R)$ (cf. [ ]). - In particular $f(a) \in \mathcal{D}(\delta^n)$ if $a \in \mathcal{D}(\delta^n)$ and $f \in C^\infty(R)$.

1.5. Proposition. Let δ be a closed *-derivation in $\mathfrak{A}$ and suppose that $\mathcal{D}(\delta^n)$ is dense in $\mathfrak{A}$ for some positive integer n; then $\mathcal{D}(\delta^n)$ is a Silov algebra under the norm $|||\cdot|||_{\delta^n}$. The proof is similar with the proof of Proposition 1.4.

Let A be a Silov algebra on K. Call an ideal I primary if I is contained in exactly one maximal ideal. Given a maximal ideal M_p, consisting of all functions vanishing at $p(\in K)$, there exists a unique smallest closed primary ideal attached to M_p; it is the closure of the set of all functions vanishing in a neighbourhood of p (the neighbourhood depending on the function). Let us write $J(p)$ for this ideal.

1.6. Proposition. Under the assumptions of Proposition 1.4, consider the Silov algebra $\mathcal{D}_0$; then $J(p) \subseteqq \{a | a(p) = \delta_\alpha(a)(p) = 0$ for $a \in \mathcal{D}_0$ and all $\alpha \in \Pi\}$.

Proof. Let $I_\alpha = \{a | a(p) = \delta_\alpha(a)(p) = 0, \ a \in \mathcal{D}_0\}$; then I_α is a closed set of $\mathcal{D}_0$. For $y \in \mathcal{D}_0$, $(ya)(p) = 0$ and $\delta_\alpha(ya) = \delta_\alpha(y)(p)a(p) + y(p)\delta_\alpha(a)(p) = 0$; hence I_α is an ideal. Since $I_\alpha = \{a | \begin{pmatrix} a(p) & \delta_\alpha(a)(p) \\ 0 & a(p) \end{pmatrix} = 0$, $a \in \mathcal{D}_0\}$ and $a \to \begin{pmatrix} a(p) & \delta_\alpha(a)(p) \\ 0 & a(p) \end{pmatrix}$ is a homomorphism, $\mathcal{D}_0/I_\alpha$ is at most two-dimensional, a unit element together with an element of square 0. If $I_\alpha \subset M_q (p \neq q)$, then $I_\alpha \subset M_p \cap M_q$ and so $\mathcal{D}_0/I_\alpha$ is two-dimenisonal, semi-simple, a contradiction. Hence I_α is primary so that $J(p) \subset I_\alpha$ and so $J(p) \subset \bigcap_{\alpha \in \Pi} I_\alpha = I$. This completes the proof.

1.7. Proposition. Under the assumption of Proposition 1.5, consider the Silov algebra $\mathcal{D}(\delta^n)$; then $J(p) \subseteqq \{a | a(p) = \delta(a)(p) = \ldots = \delta^n(a)(p) = 0, \ a \in \mathcal{D}(\delta^n)\}$.

The proof is similar with the proof of Proposition 1.6.

1.8. Definition. Let A be a Silov algebra on X. A is said to be of type C if the norm in A is equivalent to the sup, taken over x $(x \in X)$ of norms in the quotient algebra $A/J(x)$.

1.9. Proposition. Let δ be a closed *-derivation in $\mathfrak{A}$; then the Silov algebra $\mathcal{D}(\delta)$ with the norm $|||\cdot|||_\delta$ is of type C.

Proof. $|||a|||_\delta = \sup_{p\in K} \left\| \begin{pmatrix} a(p) & \delta(p) \\ 0 & a(p) \end{pmatrix} \right\|$ and $\left\| \begin{pmatrix} a(p) & \delta(p) \\ 0 & a(p) \end{pmatrix} \right\|$ $\leqq$ the norm of a in the quotient algebra $\mathcal{D}(\delta)/J(p)$; hence $|||a|||_\delta \leqq \sup_{p\in K}$ { the norms of a in the quotient algebra $\mathcal{D}(\delta)/J(p)$ } $\leqq |||a|||_\delta$. This completes the proof.

1.10. Proposition. Let δ be a closed *-derivation in $\mathfrak{A}$ and suppose that $\mathcal{D}(\delta^n)$ is dense in $\mathfrak{A}$ for some positive integer n; then the Silov algebra $\mathcal{D}(\delta^n)$ with the norm $|||\cdot|||_{\delta^n}$ is of type C.

The proof is similar with the proof of Proposition 1.9.

1.11. Proposition. Let $\mathfrak{A} = C(K)$ with a totally disconnected compact Hausdorff space K. Then every closed *-derivation δ in $\mathfrak{A}$ is identically zero so that $\mathcal{D}(\delta) = \mathfrak{A}$.

Proof. Consider the Banach algebra $\mathcal{D}(\delta)$ with the norm $|||\cdot|||_\delta$. The space K of all maximal ideals of $\mathcal{D}(\delta)$ is totally disconnected (cf. § 3) so that by Silov's theorem any idempotent in $\mathfrak{A}$ belongs to $\mathcal{D}(\delta)$. Suppose that e is an idempotent; then $\delta(e) = \delta(e^2) = e\delta(e) + \delta(e)e = 2\delta(e)e$ and so $\delta(e)e = \delta(e) = 0$. Let $\mathfrak{A}_0$ be the set of all finite linear combinations of all idempotents in $\mathfrak{A}$; then $\delta(a) = 0$

for $a \in \mathfrak{A}_0$ and $|||a|||_\delta = \|a\|$ $(a \in \mathfrak{A}_0)$. Since $\mathfrak{A}_0$ is dense in $\mathfrak{A}$, $\mathcal{D}(\delta) = \mathfrak{A}$ and $\delta(x) = 0$ $(x \in \mathfrak{A})$. This completes the proof.

Problem 1.1. Suppose that K is not totally disconnected. Then can we conclude that C(K) has a non-trivial closed *-derivation?[1] (The answer is no. If K has a totally disconnected open dense subset, then C(K) has no non-trivial closed *-derivation.[2])

Now let $\mathfrak{A} = C([0,1])$ with the unit interval $[0,1]$ and $\delta_0 = \frac{d}{dx}$ with $\mathcal{D}(\delta_0) = C^{(1)}([0,1])$, where $C^{(1)}([0,1])$ is the algebra of all continuously differentiable functions on $[0,1]$. Then δ_0 is a closed *-derivation in $\mathfrak{A}$.

For $p \in [0,1]$, it is well known that $J(p) = \{a | a(p) = a'(p) = 0,\ a \in \mathcal{D}(\delta_0)\}$.

Hence $J(p) = \{a \,|\begin{pmatrix} a(p) & \delta_0(a)(p) \\ 0 & a(p) \end{pmatrix} = 0,\ a \in \mathcal{D}(\delta_0)\}$ and so $\mathcal{D}(\delta_0)/J(p)$ is a two-dimensional algebra, a unit element together with an element of square 0.

Now let δ_1 be another derivation in $\mathfrak{A} = C([0,1])$ with $\mathcal{D}(\delta_1) = \mathcal{D}(\delta_0)$. Then by Proposition 1.1, $|||a|||_{\delta_1} \leqq k|||a|||_{\delta_0}$ $(a \in \mathcal{D}(\delta_0))$. Let
$I_p = \{a \,|\begin{pmatrix} a(p) & \delta_1(a)(p) \\ 0 & a(p) \end{pmatrix} = 0,\ a \in \mathcal{D}(\delta_0)\}$; then I_p is a closed primary ideal in $\mathcal{D}(\delta_0)$. Since $I_p \subset M_p$, $J(p) \subset I_p$. Hence

1) The following problem is interesting: Suppose that C(K) has a closed *-derivation. Then can we conclude that K contains [0,1] topologically?

2) This remark is duo to Johnson

$$\left\| \begin{pmatrix} a(p) & \delta_1(a)(p) \\ 0 & a(p) \end{pmatrix} \right\| \leqq k_p \left\| \begin{pmatrix} a(p) & \delta_0(a)(p) \\ 0 & a(p) \end{pmatrix} \right\|$$

$(a \in \mathcal{D}(\delta_1)$ and $p \in [0,1])$ where k_p is a positive number depending on p.

$$\left\| \begin{pmatrix} a(p)-a(p)1 & \delta_1(a)(p) \\ 0 & a(p)-a(p)1 \end{pmatrix} \right\| \leqq k_p \left\| \begin{pmatrix} a(p)-a(p)1 & \delta_0(a) \\ 0 & a(p)-a(p)1 \end{pmatrix} \right\|$$

and so $|\delta_1(a)(p)| \leqq k_p|\delta_0(a)(p)|$ $(a \in \mathcal{D}(\delta_0))$. Hence there is a number $\lambda(p)$ such that $\delta_1(a)(p) = \lambda(p)\delta_0(a)(p)$ $(a \in \mathcal{D}(\delta_0))$.

Put $a_0(p) = p$ $(p \in [0,1])$; then $\delta_0(a_0) = 1$ and so $\delta_1(a_0)(p) = \lambda(p)$. Therefore we have the following theorem.

1.12. Theorem. Let δ be a derivation in $C([0,1])$ such that $\mathcal{D}(\delta) = C^{(1)}([0,1])$; then there is a unique continuous function λ on $[0,1]$ such that $\delta = \lambda \cdot \frac{d}{dx}$ on $C^{(1)}([0,1])$.

1.13. Theorem. Any derivation δ defined on $C^{(1)}([0,1])$ is closable.

Proof. By Theorem 1.12, $\delta = \lambda\frac{d}{dx}$. Suppose that $\|a_n\| \to 0$ and $\|\delta(a_n)-b\| \to 0$ with $b \in C([0,1])$. Let $0_\varepsilon=\{p \mid |\lambda(p)| > \varepsilon\}$ for $\varepsilon > 0$ and let $p \in 0_\varepsilon$. Since $C^{(1)}([0,1])$ is a Silov algebra, there is an element c in $C^{(1)}([0,1])$ such that $c(p) \neq 0$ and $C(q) = 0$ for $q \in 0_{\varepsilon}^{c}$. Then $a_n c^2 \to 0$ and $\delta(a_n c^2) = \delta(a_n)c^2 + a_n\delta(c^2) \to bc^2$. On the other hand, $\delta(a_n c^2)(r) = \lambda(r)\delta_0(a_n c^2)(r) = \lambda(r)\{\delta_0(a_n)(r)c^2(r) + a_n(r)(2c)(r)\delta_0(c)(r)\}$; hence $\lambda(r) = 0$ implies $\delta(a_n c^2)(r) = 0$ and so $\delta_0(a_n c^2)(r) = \dfrac{\delta(a_n c^2)(r)}{\lambda(r)}$. Therefore $\delta_0(a_n c^2)(r) \to b(r)c^2(r) \cdot \dfrac{1}{\lambda(r)}$ for $r \in [0,1]$, where if $\lambda(r) = 0$, define $\dfrac{b(r)c^2(r)}{\lambda(r)} = 0$. Since $b(r)c^2(r) \cdot \dfrac{1}{\lambda(r)}$ is a continuous function d on $[0,1]$, $\delta_0(a_n c^2)(r) \to d(r)$ $(r \in [0,1])$. Since δ_0 is closed, $d = 0$; hence $bc^2 = 0$. Since ε is arbitrary, $b(r) = 0$, when $\lambda(r) \neq 0$. It is clear that $b(r) = 0$, where $\lambda(r) = 0$. Hence $b = 0$. This completes the proof.

1.14. Theorem. Let δ be a derivation in $C([0,1])$ such that $\mathcal{D}(\delta) = C^{(n)}([0,1])$ for some positive integer n, where $C^{(n)}([0,1])$ is the algebra of all n-times continuously differentiable functions on $[0,1]$. Then there is a unique continuous function λ on $[0,1]$ such that $\delta = \lambda\frac{d}{dx}$.

The proof is similar with the proof of Proposition 1.12.

1.15. Theorem. Any derivation defined on $C^{(n)}([0,1])$ for some positive integer n is closable.

Let $C^{(\infty)}([0,1]) = \bigcap_{n=1}^{\infty} C^{(n)}([0,1])$; then there is no norm on $C^{(\infty)}([0,1])$ under which $C^{(\infty)}([0,1])$ becomes a Banach algebra, for if there were such a norm, then $C^{(\infty)}([0,1])$

becomes a semi-simple Banach algebra so that $\delta_0 \equiv 0$, a contradiction.

Problem 1.2. Is there a non-closable derivation on $C^{(\infty)}([0,1])$? (The answer is no any derivation of $C^{\infty}[[0,1])$ into $C([0,1])$ is closable.[1])

Problem 1.3. Can we extend Theorem 1.13 to general cases? Namely let δ_0 be a closed $*$-derivation in a commutative C*-algebra $\mathfrak{A}$ and let δ be a derivation defined on $\mathcal{D}(\delta)$. Then can we conclude that δ is closable?

1.16. Proposition. Let δ be a closed $*$-derivation in $C([0,1])$ and suppose that $\mathcal{D}(\delta)$ contains a self-adjoint elemnet h such that the C*-algebra generated by h is $C([0,1])$.

Then there exists a $*$-automorphism ξ on $C([0,1])$ such that $\xi^{-1}C^{(1)}([0,1]) \subset \mathcal{D}(\delta)$ and $\xi\delta\xi^{-1}f = \lambda\frac{d}{dx}f$ $(f \in C^{(1)}([0,1]))$ where λ is a continuous real valued function on $[0,1]$.

Proof. Let $k = \dfrac{\|h\|1+h}{\|\,\|h\|1+h\,\|}$; then $k(t) \neq k(s)$ if $t \neq s$. Let $k(t_0) = \inf\limits_{t\in[0,1]} k(t)$ and let $\eta = \dfrac{k-k_0(t)}{\|k-k_0(t)1\|}$; then the spectrum of $\eta = [0,1]$ and $t \to \eta(t)$ is a homeomorphism on $[0,1]$. Moreover $\eta \in \mathcal{D}(\delta)$ and $\delta(f(\eta)) = \delta(\eta)f'(\eta)$ for $f \in C^{(1)}([0,1])$.

Consider the mapping $f(\eta) \to f$ of $C([0,1])$ onto $C([0,1])$; then it is a $*$-isomorphism ξ of $C([0,1])$ onto

[1] This remark is due to Johnson.

$C([0,1])$. Moreover under this isomorphism

$$\xi\delta(f(\eta)) = \xi\delta\xi^{-1}\xi f(\eta) = \xi\delta\xi^{-1}f = \lambda\cdot f'$$

for $f \in C^{(1)}([0,1])$. Hence $\xi\delta\xi^{-1}f = \lambda\cdot\frac{d}{dt}f$.
This completes the proof.

Problem 1.4. Can we conclude that a Silov algebra $\mathcal{D}(\delta)$ for a closed *-derivation in $C([0,1])$ has a single self-adjoint element h such that the C*-algebra generated by h is $C([0,1])$?

Now suppose that a closed derivation δ in $C([0,1])$ is an extension of $\frac{d}{dx}$ - i.e. $\delta = \frac{d}{dx}$ on $\mathcal{D}(\frac{d}{dx}) = C^{(1)}([0,1])$. Since $\frac{d}{dx}\mathcal{D}(\frac{d}{dx}) = C([0,1])$, for any $a \in \mathcal{D}(\delta)$, there is an element b in $\mathcal{D}(\frac{d}{dx})$ such that $\delta(b) = \delta(a)$ and so $\delta(a-b) = 0$. Let $\mathcal{L} = \{x \mid \delta(x) = 0,\ x \in \mathcal{D}(\delta)\}$; then $\mathcal{L}$ is a subalgebra of $\mathcal{D}(\delta)$. Moreover, $|||x|||_\delta = \|x\|$; hence $\mathcal{L}$ is a norm closed subalgebra of $C([0,1])$.

Moreover $\mathcal{D}(\delta) = C^{(1)}([0,1]) + \mathcal{L}$ and $C^{(1)}([0,1]) \cap \mathcal{L} = \mathbb{C}1$ and $\delta(\mathcal{L}) = 0$.

Problem 1.5. Is there a closed derivation δ in $C([0,1])$ such that $\mathcal{D}(\delta) \supsetneqq C^{(1)}([0,1])$ and $\delta = \frac{d}{dx}$ on $C^{(1)}([0,1])$?

Remark. R. Herman communicates to the author that there is a non-closable *-derivation δ_1 such that $\mathcal{D}(\delta_1) \supset C^{(1)}([0,1])$

and $\delta_1 = \frac{d}{dx}$ on $C^{(1)}([0,1])$.

1.17. Proposition. Let $\mathfrak{A} = C(T)$, where T is a one-dimensional torus group and let δ be a closed derivation in $\mathfrak{A}$ such that $\mathcal{D}(\delta)$ is dense in $\mathfrak{A}$ and $\tau_t\delta = \delta\tau_t$ for all $t \in T$, where $\tau_t a(s) = a(t+s)$ $(a \in C(T))$. Then $\mathcal{D}(\delta) = C^{(1)}(T)$ with $\delta = k\frac{d}{dt}$ $(k \neq 0$, a constant) or $\mathcal{D}(\delta) = \mathfrak{A}$ with $\delta \equiv 0$.

Proof. $|||a^t|||_\delta = |||a|||_\delta$ $(t \in T)$ for $a \in \mathcal{D}(\delta)$, where $(a^t)(s) = a(t+s)$. Hence the mapping $t \to a^t$ is continuous on $\mathcal{D}(\delta)$ for each $a \in \mathcal{D}(\delta)$.

Put $a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}a^t dt$ $(a \in \mathcal{D}(\delta))$; then $a_n^s(x) = e^{ins}a_n(x)$. Hence $a_n(s+x) = e^{ins}a_n(x)$ and $a_n(s) = e^{ins}a_n(0)$ $a_n(0) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}a(t)dt$.

Since $\mathcal{D}(\delta)$ is dense in $C(T)$, there is an element a in $\mathcal{D}(\delta)$ such that $a_n(0) \neq 0$; hence $e^{ins} \in \mathcal{D}(\delta)$ $(n = 0,\pm1,\pm2,\ldots)$. $\tau_t\delta(e^{ins}) = \delta(e^{in(s+t)}) = e^{int}\delta(e^{ins})$. Put $\delta(e^{ins}) = f_n(s)$; then $f_n(t+s) = e^{int}f_n(s)$. Hence $f_n(t) = e^{int}f_n(0)$ and so $\delta(e^{ins}) = f_n(0)e^{ins}$.
$\delta(e^{it}) = f_1(0)e^{it} = \frac{f_1(0)}{i} ie^{it} = \frac{f_1(0)}{i}\frac{d}{dt}e^{it}$; hence $\delta(e^{int}) = \delta((e^{it})^n) = \frac{f_1(0)}{i}\frac{d}{dt}e^{int}$ $(n = 0,\pm1,\pm2,\ldots)$.
Let $g(t) \in C^{(\infty)}(T)$ and $g(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$; then $g'(t) = \sum_{n=-\infty}^{\infty} c_n ine^{int}$. Since $|c_n n^p| \to 0$ $(n \to \infty)$ for each

positive integer p (note $g \in C^{(\infty)}(T)$), $g'(t) = \sum_{n=-\infty}^{\infty} c_n ine^{int}$ is absolute convergent; hence $\delta(g) = \frac{f_1(0)}{i} \frac{d}{dt}(g)$ for $g \in C^{\infty}(T)$, and $C^{(\infty)}(T) \subset \mathcal{D}(\delta)$.

By Silov's theorem (cf. [4]) $\mathcal{D}(\delta) = C^{(n)}(T)$ for some non-negative integer n. By Theorem 1.14, $\mathcal{D}(\delta) = C^{(1)}(T)$ or $\mathcal{D}(\delta) = \mathfrak{A}$. If $\mathcal{D}(\delta) = C^{(1)}(T)$, then $\delta(g) = \lambda g'$ for $g \in C^{(1)}(T)$ and so $\lambda = \frac{f_1(0)}{i} = k$. If $\mathcal{D}(\delta) = C(T)$, then $\delta \equiv 0$. This completes the proof.

Problem 1.6. Let $\mathfrak{A} = C_0(R)$, where $C_0(R)$ is the algebra of all continuous functions on the real line R vanishing at infinity and let δ be a closed *-derivation in $C_0(R)$ such that $\tau_g \delta = \delta \tau_g$ $(g \in R)$. Then can we conclude that $\mathcal{D}(\delta) = C_0^{(1)}(R)$ and $\delta = k \cdot \frac{d}{dt}$ $(k \neq 0$, a constant) or $\mathcal{D}(\delta) = C_0(R)$ and $\delta \equiv 0$?

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