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Dimension Subgroups

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Let G be a group, $\mathbb{Z}G$ the group ring of G over $\mathbb{Z}$, and $I(G)$ be the augmentation ideal of $\mathbb{Z}G$. Then we define the nth dimension subgroup of G as follows:

$$D_n(G) = \{ g \in G \mid g - 1 \in I^n(G) \} ,$$

where $I^n(G) = (I(G))^n$. Let $G = G_1 \geqq G_2 \geqq \cdots \geqq G_n \geqq G_{n+1} \geqq \cdots$ be the lower central series of G. Then we have easily $D_n(G) \geqq G_n$, $n \geqq 1$. Now <u>the dimension subgroup problem</u> is to investigate if $D_n(G)$ is equal to G_n for what group G, more generally to determine the structure of $D_n(G)$ for any group G. To do so we consider the following additive groups

$$Q_n(G) = I^n(G)/I^{n+1}(G), \quad n \geqq 1.$$

In this connection, the first result is the following classical one:

<u>Theorem 1</u>. $Q_1(G) \cong G_1/G_2$ for any group G.

As its corollary we have

<u>Corollary 2</u>. $D_2(G) = G_2$ for any group G.

This equality is given by Passi [7] through cohomological methods of groups.

The second one is the following given by Bachmann-Grünenfelder [1], Losey [4], Passi [7] and Sandling [11]:

<u>Theorem 3</u>. If G is a finitely generated group, then $Q_2(G) \cong sp^2(G_1/G_2) \oplus G_2/G_3$, where $sp^n(G_1/G_2)$ is the nth symmetric power of the abelian group G_1/G_2.

As its corollary we have

<u>Corollary 4</u>. $D_3(G) = G_3$ for any group G.

This equality is also given by Passi [7] through cohomological methods of groups.

Let G be a finite group. Then for any $k \geqq 1$ the finite abelian group G_k/G_{k+1} has a basis $\{\overline{x_{ki}}\}_{1 \leqq i \leqq \lambda(k)}$ with $x_{ki} \in G_k$ and $\overline{x_{ki}} = x_{ki}G_{k+1}$ $(1 \leqq i \leqq \lambda(k))$. In particular we denote by $d(i)$ and $d'(i)$ the order of $\overline{x_{1i}}$ and $\overline{x_{2i}}$, respectively. Here we may assume that $d(i) \mid d(i+1)$ and $d'(i) \mid d'(i+1)$. Then we have (for example cf. [17]).

<u>Lemma 5</u>. $I^3(G)$ has a $\mathbb{Z}$-basis consisting of

(1) $(x_{1i}-1)^{d(i)}$, $d(i) \geqq 3$, $1 \leqq i \leqq \lambda(1)$

(2) $(x_{2i}-1)^{d'(i)}$, $1 \leqq i \leqq \lambda(2)$

(3) $d(i)(x_{1i}-1)(x_{1j}-1)$, $1 \leqq i \leqq j \leqq \lambda(1)$

(4) $P(\alpha)$, α:basic, $W(\alpha) \geqq 3$.

The third one is the following given by ourselves:

Theorem 6. If G is a finite group, then

$$Q_3(G) \cong \{ sp^3(G_1/G_2) \oplus (G_1/G_2 \otimes G_2/G_3) \oplus G_3/G_4\} / R,$$

where R is the submodule of $sp^3(G_1/G_2) \oplus (G_1/G_2 \otimes G_2/G_3) \oplus G_3/G_4$ generated by the following:

$$\{\binom{d(j)}{2}(\overline{x_{1i}} \vee \overline{x_{1j}} \vee \overline{x_{1j}}) - \frac{d(j)}{d(i)}\binom{d(i)}{2}(\overline{x_{1i}} \vee \overline{x_{1i}} \vee \overline{x_{1j}})\}$$
$$+\{\frac{d(j)}{d(i)}(\overline{x_{1j}} \otimes \overline{x_{1i}^{d(i)}}) - (\overline{x_{1i}} \otimes \overline{x_{1j}^{d(j)}})\} + \frac{d(j)}{d(i)}\overline{[x_{1i}^{d(i)}, x_{1j}]} \qquad (1 \leqq i < j \leqq \lambda(1)).$$

This isomorphism Ψ is given by

$$\Psi(\overline{(x_{1i}-1)^{d(i)}}) = R, \quad d(i) \geqq 4$$

$$\Psi(\overline{(x_{2i}-1)^{d'(i)}}) = R$$

$$\Psi(\overline{d(i)(x_{1i}-1)(x_{1j}-1)}) = -\binom{d(i)}{2}(\overline{x_{1i}} \vee \overline{x_{1i}} \vee \overline{x_{1j}}) + (\overline{x_{1j}} \otimes \overline{x_{1i}^{d(i)}}) + \overline{[x_{1i}^{d(i)}, x_{1j}]} + R$$

$$\Psi(\overline{(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)}) = \overline{x_{1i}} \vee \overline{x_{1j}} \vee \overline{x_{1k}} + R$$

$\Psi(\overline{(x_{1i}-1)(x_{2j}-1)}) = \overline{x_{1i}} \otimes \overline{x_{2j}} + R$

$\Psi(\overline{(x_{3i}-1)}) = \overline{x_{3i}} + R$

$\Psi(\overline{P(\alpha)}) = R$, α:basic, $W(\alpha) \geqq 4$.

As its corollary we can completely determine $D_4(G)$. We put for $1 \leqq i \leqq \lambda(1)$

$$x_{1i}^{d(i)} = x_{21}^{c_{i1}} x_{22}^{c_{i2}} \cdots x_{2\lambda(2)}^{c_{i\lambda(2)}} x_3, \quad x_3 \in G_3.$$

<u>Corollary 7</u>. If G is a finite group with $G_4=1$, then $D_4(G)$ is the subgroup of G_3 (written additively) generated by the elements

$$\sum_{1 \leqq i<j \leqq \lambda(1)} u_{ij} \frac{d(j)}{d(i)} [x_{1i}^{d(i)}, x_{1j}]$$

for all integers u_{ij} $(1 \leqq i<j \leqq \lambda(1))$ satisfying the following

(*) $u_{ij}\binom{d(j)}{2} \equiv 0 \pmod{d(i)}$

and for $1 \leqq i \leqq \lambda(1)$ and $1 \leqq k \leqq \lambda(2)$

(**) $\sum_{1 \leqq h<i} u_{hi} \frac{d(i)}{d(h)} c_{hk} - \sum_{i<j \leqq \lambda(1)} u_{ij} c_{jk} \equiv 0 \pmod{(d(i), d'(k))}$.

This corollary implies many results as follows:

1) If G is a finite p-group with $p \neq 2$, then $D_4(G) = G_4$.

This equality is first given by Passi [7] through cohomological methods.

2) $\mathrm{Exp}(D_4(G)/G_4) \leqq 2$ for any group G.

This one is given by Losey [4], Sjogren [12] and Tahara [13, 15].

3) There exist infinitely many counterexamples to $D_4(G) = G_4$ for finite 2-groups G.

Rips [10] constructed only one counterexample to $D_4(G) = G_4$ in 1972, but the interpretation of this example has been indistinct till now. However we can understand the meaning of it, and we can construct infinitely many counterexamples containing Rips' one as the smallest one in this family.

The last result is one for free groups [16].

<u>Theorem 8</u>. If F is a finitely generated free group, then

$$Q_n(F) \cong \Sigma \oplus \{sp^{a_1}(F_{b_1}/F_{b_1+1}) \otimes sp^{a_2}(F_{b_2}/F_{b_2+1}) \otimes \cdots \otimes sp^{a_s}(F_{b_s}/F_{b_s+1})\}$$

$n \geqq 1$, where Σ runs all natural integers $a_1, \cdots, a_s$, $b_1, \cdots, b_s$ with $b_1 < b_2 < \cdots < b_s$ and $a_1 b_1 + \cdots + a_s b_s = n$ for possible natural integers s.

The denominator submodule R in general case is trivial for these free groups. As its corollary we have

<u>Corollary 9</u>. If F is any free group, then $D_n(F) = F_n$, $n \geqq 1$.

This is the first result on dimension subgroups which is given by Grün [3] and Magnus [6] in 1937. Moreover we have

Corollary 10. If F is a finitely generated free abelian group, then $Q_n(F) \cong sp^n(F)$, $n \geqq 1$.

This is Passi's result [8].

Thus by considering of the structure of $Q_n(G)$, we can determine the structure of $D_4(G)$, and moreover we can understand all results on dimension subgroups, which are given separately by cohomological methods, Lie ring's methods and so on.

We want to determine the structure of $Q_n(G)$ for all $n \geqq 4$. To do so we have to determine $\mathbb{Z}$-basis of $I^n(G)$ for all $n \geqq 4$.

Finally we have the following conjecture:

Conjecture 11. Let G be a finite group and $o^*(x_{ki})$ be the order of $\overline{x_{ki}} \in G_k/G_{k+1}$. If for $1 \leqq k \leqq u-1$, $x_{ki}^{o^*(x_{ki})}$ is contained in G_{k+2}, $1 \leqq i \leqq \lambda(k)$, then we have $D_\ell(G) = G_\ell$ for $1 \leqq \ell \leqq u+2$.

For $u = 2, 3$, this conjecture is true. Moreover if this is proved, then this result is a generalization of both results of P. M. Cohn [2] and D. G. Quillen [9].

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