

Permutation groups of some special degrees

A preliminary report on some joint work with P.M. Neumann

by Jan Saxl

In an impressive series of papers some fifteen years ago Noboru Ito considered transitive permutation groups of degree $p = 2q+1$, where p and q are prime numbers and $p > 11$, and proved that such groups are either soluble (and therefore well known) or very nearly 4-transitive. In this paper we want to use this remarkable result and a theorem of R. Brauer to obtain an extension to groups of degree kp with $k > 1$.

Throughout G will be a primitive permutation group on a set Ω , where $|\Omega| = n = kp = k(2q+1)$ with p and q prime numbers, $p > 11$ and $k > 1$. Let P be a Sylow p -subgroup of G . Our first result brings q into play.

Proposition 1. If $k < 10$ and $k \neq 8$ then either q divides the order of $N(P)$ or $\text{PSL}(2, n-1) \leq G \leq \text{P}\Gamma\text{L}(2, n-1)$.

Once we know that q does divide the order of G we can start the work on the proof of our main result.

Theorem. If $k = 2$ then G is 7-transitive.

If $k = 3$ then G is 10-transitive.

If $k = 4$ then either G is A_n , S_n or $\text{PSL}(2, n-1) \leq G \leq \text{P}\Gamma\text{L}(2, n-1)$.

Let $\Gamma_1, \dots, \Gamma_k$ be the P -orbits on Ω , and let H be the setwise stabilizer of each of these. The main step in the proof is to show that if $k \leq 4$ then H is insoluble. Then we know by Ito's theorem that H is 3-transitive (and in fact nearly 4-transitive) on each of the Γ_1 , which enables us to obtain very high transitivity of G on Ω . It should be possible to deduce then that

G is alternating or symmetric, however we have not been able to do this yet when k is 2 or 3. The case $k = 4$ is easier since we can find a non-trivial element in G fixing a large number of points in Ω so that an old theorem of W.A. Manning can be applied.

We obtain the following corollaries:

Corollary 1. If G is a primitive group of degree $n = 2p = 4q+2 = r+3$ where r is also a prime number then G is A_n or S_n . Similarly for

$$n = 2p = 4q+2 = 5r+3 \quad (\text{eg. } n = 118),$$

$$n = 2p = 4q+2 = r+5 \quad (\text{eg. } n = 94),$$

$$n = 3p = 6q+3 = r+4 \quad (\text{eg. } n = 141),$$

etc., etc.

Corollary 2. If G is 2-transitive of degree $3p+1$ with $p = 2q+1$ then G is alternating or symmetric.

Here the group is 2-primitive by a result of M.D. Atkinson [1, Cor.C], so G is 11-transitive by the theorem. An argument similar to that in the last section in the case $n = 4p$ then shows that G contains the alternating group.

Corollary 3. If any insoluble group of degree $p = 2q+1$ contains the alternating group then this is also true of any primitive group of degree $2p$ and $3p$. This holds for instance if $q = 2r+1$ with r prime [15] or if $p \leq 4079$ [16].

It should be noted that some of the results in this paper have been also previously obtained by Izumi Miyamoto in [13]. In particular, the case $k = 2$ of the assertion in Section 2 as well as the case $k = 2$ of Corollary 3 are due to him.

1. Some preliminaries and proof of Proposition 1

We shall assume throughout that $k < 10$. Then we can clearly suppose that P is cyclic of order p and semi-regular on Ω . Whenever convenient, we shall assume that G is a simple group; this is justified by the following

Lemma. If X is a minimal normal subgroup of G then X is simple and primitive on Ω .

Proof. Since p divides the order of X but p^2 does not, the simplicity of X is clear. Suppose that X is imprimitive. Let B be a block of maximal size, say $|B| = m$, and let Σ be the corresponding system of imprimitivity. If $p \nmid m$ then P lies in the kernel of X on Σ ; but X is simple, so X must act trivially on Σ , which is impossible. Thus m divides k and $|\Sigma| = \frac{k}{m}p$.

Now G is primitive on Ω , so for some g in G we have $Bg \neq \Sigma$. Then Σg is another system of imprimitivity for X . Now X acts on Σ and Σg , and by induction (cf. the Theorem) together with a theorem of Ito [9, Satz 3], the actions of X on Σ and Σg are at least 2-transitive and are similar to each other. Therefore X_B stabilizes some block in Σg , and so has an orbit on $\Omega - B$ of size at most m . On the other hand, X_B is transitive on $\Sigma - \{B\}$, so all its orbits on $\Omega - B$ have size at least $p-1$. This is a contradiction.

Proof of Proposition 1. Suppose that q does not divide $|N(P)|$. Since P is cyclic of order p and since $p = 2q+1$, it follows that $|N(P)/C(P)|$ divides 2, and in fact is equal to 2, as we see from the Burnside transfer theorem. It now follows by a theorem of Brauer [4, Theorem 9.C] that every involution in G is conjugate to one in $N(P) - C(P)$. Therefore, if an involution of G fixes u points then $u \leq k$. Moreover, since $p \equiv 3 \pmod{4}$ and $G \leq A_n$, we have $n \equiv -k \pmod{4}$. Hence

for k equal to 2, 3, 4, 5, 6, 7, 8, 9,

the maximum possible value for u is 2, 1, 4, 3, 6, 5, 8, 7.

If $k \leq 5$ then we can use the theorems of Buekenhout and Rowlinson [5] and deduce that G is $\text{PSL}(2, n-1)$. If $|C(P)|$ is even then an involution in $C(P)$ must be fixed-point-free on Ω , since if it fixed a point then it would fix pointwise the whole P -orbit containing that point. Hence such an involution is an odd permutation unless k is divisible by 4. Thus for $k \not\equiv 0 \pmod{4}$ the order of $C(P)$ is odd, so that there is only one conjugacy class of involutions in G , and another theorem of Rowlinson [17] applies.

This implies that $k = 8$ and the proposition is proved. It is perhaps worth observing that since a Sylow 2-subgroup S of G is semi-regular on the set of ordered $(k+1)$ -subsets of Ω , the order of S divides $kp \cdot (kp-1) \cdots (kp-k)$. When $k = 2$ this implies that $|S|$ divides 8, while for $k = 8$ the Sylow 2-subgroups of G have order at most 2^{11} .

Some more notation. Let Q be a Sylow q -subgroup of $N(P)$; then $Q \leq H$, where H is the setwise stabilizer of all the P -orbits Γ_1 . We shall assume that Q is in fact a Sylow q -subgroup of G , because otherwise G is known to satisfy the conclusion of the theorem. Let Δ_0 be the set of fixed points of Q . We shall denote the k points of Δ_0 by α, β, \dots . Let \mathcal{O} be the set of all Q -orbits, and let $\mathcal{O}_0 = \mathcal{O} - \Delta_0$. Let $\mathcal{O}_0 = \{\Delta_1, \dots, \Delta_{2k}\}$, where $\Gamma_1 = \{\alpha\} \cup \Delta_1 \cup \Delta_2$, $\Gamma_2 = \{\beta\} \cup \Delta_3 \cup \Delta_4$, etc. Finally, let K, L be the kernel of the action of $N(Q)$ on Δ_0, \mathcal{O}_0 , respectively.

2. The insolubility of H

Assume, to obtain a contradiction, that H is soluble. Then $H \leq N(P)$, and since it fixes every P-orbit, H is metacyclic of order pq or $2pq$. Let $X = N(Q)/Q$ and $Y = C(Q)/Q$. Then X/Y is a cyclic group, which is non-trivial by the Burnside transfer theorem. Note also that 3 does not divide the order of X/Y , since $q \equiv 2 \pmod{3}$.

Let $g \in N(Q)$ and assume that g is trivial on \mathcal{Q} . Then $g \in H$, so that $g \in N(P)$, and since g fixes all the Δ_i , we have $g \in Q$. Thus X is faithful in its action on \mathcal{Q} , so that $X \leq S_k \times S_{2k}$. It is this observation which is the key to our proof of the insolubility of H - it restricts the structure of X to only very few possibilities. We should also remark that in fact $X \leq A_{3k}$, since $G \leq A_n$.

The case $k = 2$. Here $X \leq (Z_2 \times S_4) \cap A_6$, and so X/Y is a non-trivial cyclic 2-group.

If $|X/Y| = 2$ then by [4, Theorem 9.C] all involutions of G are conjugate to involutions in $N(Q) - C(Q)$. Hence all involutions of G fix precisely two points of \mathcal{Q} . But G is 2-transitive on \mathcal{Q} by a theorem of Wielandt [20, 31.1], so that this contradicts a theorem of Hering [8]. It is perhaps worth mentioning that since $2p$ is 6 modulo 8 we can also deduce that 8 is the highest possible power of 2 dividing $|G|$ and obtain a contradiction this way.

Hence X/Y is a cyclic 2-factor of $(Z_2 \times S_4) \cap A_6$ of order at least 4. The normalizer of a Sylow 3-subgroup of $Z_2 \times S_4$ is $Z_2 \times S_3$, so by the Frattini argument we see that $3 \nmid |X|$. The Sylow 2-subgroup of A_6 is D_8 , which does not have Z_4 as a factor. Thus $C(Q) = Q$ and $N(Q) = Q \cdot Z_4$. But then the normalizer of Q in G has order $2q$. If all the involutions of G are conjugate then they fix precisely 2 points of \mathcal{Q} and we obtain a contradiction as before. Hence by [4, Theorem 9C] we have $O_q(G) \neq 1$. Since Q is self-central-

izing, a q -element of G acts as a fixed-point-free automorphism on $O_q(G_\alpha)$. Therefore $O_q(G_\alpha)$ is nilpotent by a theorem of J.G. Thompson [18]. On the other hand, G_α is transitive and hence primitive of degree $4q+1$. It follows that $4q+1$ is a power of a prime. Now 3 divides $4q+1$, so $4q+1$ is an even power of 3, say $4q+1 = 3^{2s}$. Then $4q = (3^s-1)(3^s+1)$, which is impossible. Hence if $k = 2$ then H is insoluble.

The case $k = 3$. Here $X \leq (S_3 \times S_6) \cap A_9$. If g is in the kernel L of X on Δ_0 then either g is a 2-element and therefore is not in A_9 , or g is a 3-element and so lies in $Y \cap L$. But $Y \cap L = 1$, so that $L = 1$ and $X \leq S_6$.

Now X is transitive on Δ_0 by the Jordan lemma, so X^{Δ_0} is a factor of X isomorphic to Z_3 or S_3 . The normalizer in S_6 of a Sylow 5-subgroup has order prime to 3, so by the Frattini argument 5 does not divide the order of X . Hence X is a $\{2,3\}$ -subgroup of S_6 , and X/Y is a cyclic 2-group.

Let T be a Sylow 3-subgroup of Y . Then $T \neq 1$, since we have already noticed that 3 divides $|X|$ but does not divide $|X/Y|$. Hence $|T|$ is 3 or 9. If $|T|$ is 9 let T' be a Sylow 3-subgroup of the kernel K of X on Δ_0 , otherwise let $T' = T$. Then the normalizer of T' inside S_6 is either $S_3 \times S_3$ or $(Z_3 \text{ wr } Z_2) \cdot Z_2$, neither of which has a subgroup with a 2-factor of order greater than 2. Hence $|X/Y| = 2$ by the Frattini argument. Then, using the theorem of Brauer [4] again, all involutions in G are conjugate to those in $N(Q) - C(Q)$, whence they fix at most five points of Ω .

If now $|C(Q)|$ is odd then G has only one class of involutions and we obtain a contradiction from [17]. Assume then that $|C(Q)|$ is even. If an involution in $C(Q)$ fixed a Δ_1 setwise then it would fix it pointwise, which is not possible since $q > 5$. Hence the involutions of Y are semi-regular on Δ_0 . It follows that $|Y|$ is twice an odd number, and so $|X|$ is 12 or 36.

Since the order of the normalizer of T in X is even by the Frattini argument, we have $T \triangleleft X$, and so also $T' = T \cap K \triangleleft X$. Note also that the semi-regularity of the involutions of Y on \mathcal{O}_0 now implies that X is transitive on \mathcal{O}_0 . Thus X has index 1 or 3 in $(Z_3 \text{ wr } Z_2) \cdot Z_2$. Let t be an involution in K . Since t is even on \mathcal{O} , it cannot be semi-regular on \mathcal{O}_0 . This forces K to be of order 6 with two orbits of size 3 on \mathcal{O}_0 , and therefore $K = S_3$ and $X = (Z_3 \text{ wr } Z_2) \cdot Z_2$. But now an inspection of $(Z_3 \times Z_2) \cdot Z_2$ shows that K cannot be a normal subgroup of X , a contradiction. 12

The case $k = 4$. First we shall show that, quite independently of the assumption on H , there is a subdegree of G which is 3 modulo q . Suppose that there is a G_α -orbit of size $aq+1$. Assume first that $a \geq 3$. Using the theorem of Weiss [19] extensively we see that the only possibilities are

$$7q+1, \quad q+2,$$

$$6q+1, \quad 2q+2,$$

$$6q+1, \quad q+1, \quad q+1,$$

$$5q+1, \quad 3q+2,$$

$$4q+1, \quad 3q, \quad q+2,$$

$$\text{and } 3q+1, \quad 2q, \quad 2q, \quad q+2.$$

We shall consider just the third case - all the other cases can be ruled out in the same way. Let Γ be the G_α -orbit of size $6q+1$, and let Δ be one of the G_α -orbits of length $q+1$. Let $\delta \in \Delta$. Since the greatest common divisor of $q+1$ and $6q+1$ divides 5, the $G_{\alpha\delta}$ -orbits on Γ have size a multiple of $(6q+1)/5$. On the other hand q divides $|G_{\alpha\delta}|$. This implies that $G_{\alpha\delta}$ is transitive on Γ , which contradicts the primitivity of G (cf. the second part of the proof of Theorem 1 in [19]). This contradiction shows that $a \leq 2$, and in fact $a = 1$, because $2q+1 = p$. Let us consider then the case where Γ is a G_α -orbit of

length $q+1$. Then G_x^{Γ} is 2-transitive, so by [6] there is a G_x -orbit Σ of size cq with $3 \leq c \leq 6$. It also follows from [6] that G_x^{Γ} is not 3-transitive; hence $G_x^{\Gamma} = \text{PSL}(2, q)$. But then the action of G_x^{Γ} on Σ implies that $q = 11$; this possibility is easily excluded by an ad hoc argument.

Hence we have shown that no non-trivial subdegree of G is 1 modulo q . This implies that one of the subdegrees is 3 modulo q , whence $N(Q)$ is 2-transitive on Δ_0 by Witt's lemma. Hence $C(Q)^{\Delta_0} \cong A_4$, since $3 \nmid |N(Q)/C(Q)|$.

Let us return now to the proof of the insolubility of H in the case $k = 4$. Assume that the kernel L of X on \mathcal{O}_0 is non-trivial. Then L is transitive on Δ_0 . But $L \cap Y = 1$, so L is cyclic and therefore contains an odd permutation. Hence $L = 1$ and $X \leq S_8$. Moreover, we see as before that 5 and 7 do not divide the order of X by the Frattini argument. So X is a $\{2, 3\}$ -subgroup of S_8 , and X/Y is a cyclic 2-group. Let T be a Sylow 3-subgroup of Y ; then $|T| \leq 9$.

Assume first that $|T| = 9$. Then $T' = K \cap T$ has order 3. If $|X/Y| \geq 4$ then the Frattini argument shows that $N_{S_8}(T')$ has a 2-factor of order at least 4. But the normalizer of a group of order 3 in S_8 is either $S_3 \times S_5$ or $Z_2 \times (Z_3 \text{ wr } Z_2) \cdot Z_2$, so that $|X/Y| = 4$ and $N_X(T')$ is a subgroup of $S_3 \times S_4$ of order divisible by 9. However there is no subgroup in S_4 of order divisible by 3 with Z_4 as a factor, since the Sylow 3-normalizer in S_4 has order 6.

So $|X/Y| = 2$. Then, as before, a Sylow 2-subgroup S of Y is semi-regular on \mathcal{O}_0 . Hence X has order 72 or 144. Then $|K \cap Y|$ is 3 or 6, and so T' is characteristic in the normal subgroup $K \cap Y$, so that $T' \triangleleft X$ and $X \leq N_{S_8}(T')$. But $N_{S_8}(T')$ has orbits on \mathcal{O}_0 of size 3 and 5 or 2 and 6, whereas Y is semi-regular of order at least 4.

Hence $|T| = 3$. Suppose first that $|X/Y| \geq 4$. Then by the Frattini argument, $N_X(T)$ has Z_4 as a factor. Thus T has 5 fixed points on \mathcal{O}_0 . Consider an element x of order 4 in $N_X(T)$. Then x either inverts T and therefore is

of type 2,1,1 on Δ_0 and of type 4,2,1,1 on \mathbb{C}_0 , or it centralizes T and acts as a 4-cycle on \mathbb{C}_0 . In either case x is an odd permutation, which is not possible.

Hence $|X/Y| = 2$. Then [4, 9C] implies that all involutions fix at most 8 points of Ω . Notice that since $4p$ is 12 modulo 16, we see that 2^{10} does not divide $|G|$, so that G is known by various recent results. But let us argue directly.

We see again that a Sylow 2-subgroup of Y is semi-regular on \mathbb{C}_0 , so that $|Y|$ is 12 or 24 and $|X|$ is 24 or 48. Assume first that $|X| = 24$, $|Y| = 12$. Then Y has two orbits of size 4 on \mathbb{C}_0 and therefore acts as A_4 on each. If X preserves the Y -orbits then X acts as S_4 on each of these and on Δ_0 . But then an odd permutation in S_4 acts as an odd permutation on each of these and hence is odd on \mathbb{C}_0 . Hence X is transitive on \mathbb{C}_0 . But then any 2-element of X is semi-regular on \mathbb{C}_0 , so that involutions in X fix at most 4 points of \mathbb{C}_0 . Hence the involutions in G fix at most 4 points of Ω and so G is known [17], which leads to a contradiction. In fact, since $4p$ is 12 modulo 16, we see that 64 is the highest possible power of 2 dividing $|G|$, which gives an alternative argument.

Assume now finally that $|X| = 48$ and $|Y| = 24$. Here Y is transitive on \mathbb{C}_0 . If $|K| = 2$ then K has 4 orbits of size 2 on \mathbb{C}_0 , and X/K acts as S_4 on these and on Δ_0 . Hence any involution of X fixes at most 4 points of \mathbb{C}_0 , and we arrive at a contradiction as before. So $|K| = 4$, and $X/K \cong A_4$. Now A_4 has no subgroup of index 2, so K has four orbits of size 2 on \mathbb{C}_0 . Hence K is $Z_2 \times Z_2$ with two involutions of type $2^2, 1^4$ on \mathbb{C}_0 and one of type 2^4 . Since X/K has no subgroup of index 2 this forces K to be central in X , which is impossible since K is not semi-regular on \mathbb{C}_0 .

3. The high transitivity of G

By the theorem of Ito [10] mentioned in the introduction, the insolubility of H, which we established in Section 2, implies that H is 3-transitive on each of the Γ_i . Then we know by another theorem of Ito [9, Satz 3] that the H-actions on $\Gamma_1, \dots, \Gamma_k$ are isomorphic to each other. Our notation for the points of Ω will from now on be such that $\Gamma_1 = \{\alpha_1, \alpha_2, \dots\}$, $\Gamma_2 = \{\beta_1, \beta_2, \dots\}$, etc., with α_i, β_i etc. corresponding to each other under the action of H for each i. We shall also write α_i, β_i etc., for α_1, β_1 , etc. As before, α is chosen to be fixed by Q. Let R be a complement of Q in $H_{\alpha, \Delta, \Delta_2}$. Then any element in R other than 1 fixes precisely three points of Γ_1 ; one of these is α , the others will be α_2 and α_3 .

We shall prove the theorem only for $k = 2$ and $k = 4$; the proof in the case $k = 3$ is similar. Our original proof relied on the 4-transitivity of H on each Γ_i . Unfortunately, as Professor Ito has noticed recently, there is a mistake in the last part of [10, III], which so far remains uncorrected. In the later stages of the proof we therefore have to work harder, using the following result which pushes the character theory in [10] just one step further:

Lemma (P.M. Neumann, unpublished). Let X be an insoluble group of degree $p = 2q+1$, with p and q prime numbers and $p > 11$. If X is not 4-transitive then the stabilizer $X_{\alpha, \beta, \gamma}$ of three points has two orbits on $\Omega - \{\alpha, \beta, \gamma\}$, each of size $q-1$. Moreover, the normalizer of a Sylow q-subgroup in X' has order $\frac{1}{2}q(q-1)$.

The case $k = 2$.

Step 1. G is 3-transitive.

We have already ^{noticed} that G is 2-transitive by [20, 31.1], and the action of Q implies that G is 2-primitive. Now H_α fixes β and has two 2-transitive orbits $\Gamma_1 = \{\alpha\}$ and $\Gamma_2 = \{\beta\}$. Since $p \nmid |G_\alpha|$ and G_α does not fix β , the assertion follows.

Step 2. G is 4-transitive.

From the action of H_{α_1, α_2} we see that the possibilities for the length of the G_{α_1, α_2} -orbits are

$$\begin{aligned} & 1, 2q-1, 2q-1, \\ & 2q, 2q-1, \\ & 1, 4q-2, \\ \text{or} & 4q-1. \end{aligned}$$

Now the second is clearly impossible, since $q \nmid |G_{\alpha_1, \alpha_2}|$. Consider the first and third case. Here the stabilizer of any 3 points in G fixes exactly 4 points. Hence we obtain a Steiner system $S(3, 4, n)$ on Ω . Clearly $\{\alpha_i, \beta_i, \alpha_j, \beta_j\}$ is a line for any pair i, j , and it is the unique line containing any triple in it. Hence the fourth point of the line on $\alpha_1, \alpha_2, \alpha_3$ is not one of $\beta_1, \beta_2, \beta_3$, and so it is α_4 or β_4 . Hence $H_{\alpha_1, \alpha_2, \alpha_3}$ fixes α_4 , which contradicts the semi-regularity of R on $\Gamma_1 = \{\alpha_1, \alpha_2, \alpha_3\}$. Thus this is impossible, and so G is 4-transitive.

Step 3. G is 5-transitive.

Since $G_{\alpha_1, \alpha_2, \beta_1, \beta_2}$ contains H_{α_1, α_2} , the only alternative to this assertion is that $G_{\alpha_1, \alpha_2, \beta_1, \beta_2}$ has two orbits of size $p-2$, which would imply that the order of $G_{\alpha_1, \alpha_2, \beta_1}$ is odd ([20, 3.13]). This is clearly impossible.

Step 4. G is 6-transitive.

By the Lemma at the beginning of this section, the $H_{\alpha_1 \alpha_2 \alpha_3}$ -orbits on $\Omega - \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}$ have length divisible by $q-1$. By a theorem of Nagao [14], β_3 is not fixed by $G_{\alpha_1 \alpha_2 \alpha_3 \beta_1 \beta_2}$. Bearing in mind that $q \nmid |G_{\alpha_1 \alpha_2 \alpha_3}|$, the possibilities for the length of the $G_{\alpha_1 \alpha_2 \alpha_3 \beta_1 \beta_2}$ -orbits on $\Omega - \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\}$

- therefore are
- $2q-1, q-1, q-1,$
 - $2q-1, 2q-2,$
 - $3q-2, q-1,$

or $4q-3$.

In the first three cases it follows from [19] that $G_{\alpha_1 \alpha_2 \beta_1 \beta_2}$ is imprimitive. Let B be the block containing α_3 . Then B is a union of $G_{\alpha_1 \alpha_2 \beta_1 \beta_2}$ -orbits, and since $|B|$ divides $4q-2$, we have $|B| = 2q-1$ (this already excludes the third case). Let $\Lambda = B \cup \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$. Then $G_{\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\}} \leq G_\Lambda$, and $G_{\alpha_1 \alpha_2 \beta_1 \beta_2} B$ is transitive on B . It follows that G_Λ is 5-transitive on Λ , so that $p \mid |G_\Lambda|$. This is impossible, since $p^2 \nmid |G|$.

Step 5. G is 7-transitive.

Since $H_{\alpha_1 \alpha_2 \alpha_3} \leq G_{\alpha_1 \alpha_2 \dots \beta_3}$, all the $G_{\alpha_1 \alpha_2 \dots \beta_2}$ -orbits on the rest of Ω have length divisible by $q-1$. Hence the possibilities are

- $q-1, q-1, q-1, q-1,$
- $2(q-1), q-1, q-1,$
- $2(q-1), 2(q-1),$
- $3(q-1), q-1,$

or $4(q-1)$.

We now use a variation of an argument of M.D. Atkinson in [2, Lemma] to exclude the first three cases. Let U be a Sylow 3-subgroup of $G_{\alpha_1 \alpha_2 \dots \beta_3}$, and let V be a Sylow 3-subgroup of $G_{\{\alpha_1, \alpha_2, \alpha_3\} \beta_1 \beta_2 \beta_3}$ containing U . Then $|V| = 3 \cdot |U|$. But V normalizes $G_{\alpha_1 \alpha_2 \dots \beta_2}$ and therefore permutes its

orbits. Now $G_{\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\}}$ acts as S_6 on $\{\alpha_1, \dots, \beta_3\}$ and there are only at most four $G_{\{\alpha_1, \alpha_2, \dots, \beta_3\}}$ -orbits on $\Omega - \{\alpha_1, \dots, \beta_3\}$. It follows that V fixes these orbits setwise. But then, since $q-1 \equiv 1 \pmod{3}$, we see that V fixes apart from $\beta_1, \beta_2, \beta_3$ also at least 1 or 2 points in each long $G_{\{\alpha_1, \alpha_2, \dots, \beta_3\}}$ -orbit. Hence V fixes at least six points, and since G is 6-transitive, V is conjugate to a subgroup of U . This is impossible, and so the first three cases cannot occur.

Consider now the fourth case. Here H is not 4-transitive, so the Lemma at the beginning of this section implies that R has order $\frac{1}{2}(q-1)$.

Moreover, R has eight regular orbits $\bar{\Phi}_1$, and

$$\begin{aligned} \Delta_1 &= \{\alpha_2\} \cup \bar{\Phi}_1 \cup \bar{\Phi}_2, \\ \Delta_2 &= \{\alpha_3\} \cup \bar{\Phi}_3 \cup \bar{\Phi}_4, \\ \Delta_3 &= \{\beta_2\} \cup \bar{\Phi}_5 \cup \bar{\Phi}_6, \\ \text{and } \Delta_4 &= \{\beta_3\} \cup \bar{\Phi}_7 \cup \bar{\Phi}_8. \end{aligned}$$

Since H is not 4-transitive, the $H_{\{\alpha_1, \alpha_2, \alpha_3\}}$ -orbits on $\bar{\Gamma}_1 - \{\alpha_1, \alpha_2, \alpha_3\}$ are $\bar{\Phi}_2 \cup \bar{\Phi}_3$ and $\bar{\Phi}_1 \cup \bar{\Phi}_4$. Now the shorter $G_{\{\alpha_1, \alpha_2, \dots, \beta_3\}}$ -orbit is also an $H_{\{\alpha_1, \alpha_2, \alpha_3\}}$ -orbit, so we can assume that this is $\bar{\Phi}_2 \cup \bar{\Phi}_3$.

Let S be a complement for Q in $N_G(Q)$ which contains R . Since R has small index in S , a subgroup R_0 of small index in R is normal in S . Then R_0 fixes precisely $\alpha_2, \alpha_3, \beta_2, \beta_3$ in $\Omega - \{\alpha, \beta\}$, so that the set $\{\alpha_2, \alpha_3, \beta_2, \beta_3\}$ is S -invariant, and is permuted in precisely the same way as $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$. Let x be an element in S which interchanges α and β ; such an element exists by the Jordan lemma. Then $x \in G_{\{\alpha_1, \alpha_2, \dots, \beta_3\}}$, and so $(\bar{\Phi}_2 \cup \bar{\Phi}_3)x = \bar{\Phi}_2 \cup \bar{\Phi}_3$, so that $\{\Delta_1, \Delta_2\}$ and $\{\Delta_3, \Delta_4\}$ are x -invariant. But x involves $(\alpha \beta)$, and so it must be odd on $\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ and hence also on $\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$. It therefore involves precisely one of $(\Delta_1 \Delta_2)$ or $(\Delta_3 \Delta_4)$, and so acts differently on $\Delta_1 \cup \Delta_2$ and on $\Delta_3 \cup \Delta_4$. For instance, if x involves $(\Delta_1 \Delta_2)$ then it fixes nothing in $\Delta_1 \cup \Delta_2$ but fixes β_2 and β_3 in $\Delta_3 \cup \Delta_4$.

On the other hand, let X be the setwise stabilizer in G of $\Gamma_1 - \{\alpha\}$ and $\Gamma_2 - \{\beta\}$. Let $\pi \in \Gamma_1 - \{\alpha\}$. Then $H_{\alpha\pi}$ fixes a point $\sigma \in \Gamma_2 - \{\beta\}$, and is transitive on $\Gamma_1 - \{\alpha, \pi\}$ and on $\Gamma_2 - \{\beta, \sigma\}$. Since $H_{\alpha\pi}$ has index 2 in X_π , it follows that $X_\pi = X_\sigma$, so that X acts in the same way on $\Gamma_1 - \{\alpha\}$ and $\Gamma_2 - \{\beta\}$. This is a contradiction, since $x \in X$. The case $k = 2$ of the Theorem is now proved.

The case $k = 4$.

Step 1. G is 2-primitive.

We have already established in Section 2 that one subdegree is 3 modulo q . Since $H_\alpha \leq G_\alpha$, the subdegrees are sums of 3, $2q$, $2q$, $2q$, $2q$. Now 3 is not a subdegree by [20, 18.4], and the possibilities $4q+3$ and $6q+3$ are ruled out by [19]. Finally, any group of degree $2q+3$ whose order is divisible by q contains the alternating group by [20, 13.10], which rules out the possibility $2q+3$. Hence G is 2-transitive, and in fact 2-primitive, since the highest common divisor of 3 and 8 is 1.

Step 2. G is 3-transitive.

Since $H_\alpha \leq G_{\alpha\beta}$, the $G_{\alpha\beta}$ -orbits have size obtained out of 1, 1, $2q$, $2q$, $2q$, $2q$. Assume first that there are two of size 1 modulo q . Since G is 2-primitive and since $p \nmid |G_{\alpha\beta}|$, the only possibility is $4q+1$, $4q+1$. But then $|G_{\alpha\beta}|$ is odd by [20, 3.13], which is impossible as $|H_{\alpha\beta}|$ is certainly even. Hence one of the orbits has size 2 modulo q . Notice that $\{\alpha, \sigma\}$ is not an orbit by [20, 18.7]. Hence the only possibilities are

$$2q+2, \quad 6q,$$

$$4q, \quad 4q+2,$$

$$\text{or } 8q+2.$$

The second is clearly impossible since $4q+2 = 2p$. In the first case, let Σ be the $G_{\alpha\beta}$ -orbit of length $2q+2$. Then $j, \delta \in \Sigma$, and since $H_x \leq G_{\alpha\beta j\delta}$, we see that $G_{\alpha\beta j\delta}$ is 2-transitive on $\Sigma - \{j, \delta\}$. Hence $G_{\alpha\beta}$ is 4-transitive on Σ , which contradicts [6].

Step 3. G is 4-transitive.

If $G_{\alpha\beta}$ has two blocks of imprimitivity then by a theorem of Grün (cf. [7, 35.5]), G_{α} has a normal subgroup N of index 2 which is rank 3 on $\Omega - \{\alpha\}$ with subdegrees 1, $4q+1$, $4q+1$. But then $|N|$ is odd by [20, 3.13], which is impossible since N must have a 2-transitive section of degree $2p$.

Suppose next that $G_{\alpha\beta}$ has $4q+1$ blocks of imprimitivity. Then the stabilizer of any 3 points fixes precisely 4 points in Ω , and we obtain a Steiner system $S(3, 4, n)$ on Ω . Let Λ be a line, say $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, and assume that λ_1 and λ_2 correspond to each other under the action of H , so that $H_{\lambda_1} = H_{\lambda_2}$. If $H_{\lambda_3} = H_{\lambda_1}$, then certainly also $H_{\lambda_4} = H_{\lambda_1}$. Therefore $H_{\lambda_3} \neq H_{\lambda_1}$ implies $H_{\lambda_4} \neq H_{\lambda_1}$, and hence $H_{\lambda_3} = H_{\lambda_4}$, since H fixes Λ , whereas H_{λ_1, λ_3} has orbits of size $p-2$ on the set of points of Ω not corresponding to λ_1, λ_3 . Thus in either case, $H_{\lambda_1} = H_{\lambda_2}$ implies $H_{\lambda_3} = H_{\lambda_4}$. Consider now the line $\Lambda = \{\alpha_1, \alpha_2, \alpha_3, \omega\}$. Then ω is not fixed by $H_{\alpha_1, \alpha_2, \alpha_3}$ by the above remarks, since R is semi-regular on the set of points in Ω not corresponding to $\alpha_1, \alpha_2, \alpha_3$ under the action of H .

Hence G is 3-primitive. Now one of the non-trivial subdegrees of G is $2cq+1$ with $1 \leq c \leq 4$. Certainly $c \neq 1$ since $2q+1 = p$, and $c = 2$ and $c = 3$ are excluded by [19]. Thus G is 4-transitive.

Step 4. G is 5-transitive.

Since $H_x \leq G_{\alpha\beta j\delta}$, all the $G_{\alpha\beta j\delta}$ -orbits on $\Omega - \{\alpha, \beta, j, \delta\}$ have length divisible by $2q$. If one of these does have length $2q$ then the other must be $6q$ by

[6], since G is clearly 4-primitive. Hence the possibilities are

$$2q, 6q,$$

$$4q, 4q,$$

$$\text{or } 8q.$$

The Atkinson argument used in Step 5 of the case $k = 2$ excludes the second case, while an analogous argument with respect to 4 rules out the first case: Let U be a Sylow 2-subgroup of $G_{\alpha\beta\gamma\delta}$, let V be a Sylow 2-subgroup of $G_{\{\alpha, \beta, \gamma, \delta\}}$ containing U . Then $|V| = 8 \cdot |U|$, since G is 4-transitive. Now V normalizes $G_{\alpha\beta\gamma\delta}$, and so preserves the two long $G_{\alpha\beta\gamma\delta}$ -orbits. Since each of these has size 2 modulo 4, V has an orbit of size 2 in each. Let W be the pointwise stabilizer in V of two V -orbits of size 2; then the index of W in V is at most 4. On the other hand, W fixes at least four points of Ω , and since G is 4-transitive, this means that W is conjugate to a subgroup of U . This is a contradiction.

Step 5. G is 6-transitive.

Consider the length of the $G_{\alpha\beta\gamma\delta\epsilon}$ -orbits. These are sums of 1, 1, 1, $2q-1$, $2q-1$, $2q-1$. Since $8q-1$ is divisible by 3, the Atkinson argument implies that $\alpha_2, \beta_2, \gamma_2$ are all in the same orbit Σ .

Assume first that $|\Sigma| = 3$. Then G acts on a Steiner system $S(5, 8, n)$.

The line containing α, β, γ and any two of $\alpha_2, \beta_2, \gamma_2, \delta_2$ is $\{\alpha, \beta, \gamma, \delta, \alpha_2, \beta_2, \gamma_2, \delta_2\}$.

Similarly the line containing α, β, γ and two of $\alpha_3, \beta_3, \gamma_3, \delta_3$ must be

$\{\alpha, \beta, \gamma, \delta, \alpha_3, \beta_3, \gamma_3, \delta_3\}$. Consider now the line Λ on $\alpha, \beta, \gamma, \alpha_2, \alpha_3$. Then by

the above observations Λ meets $\Omega - \{\alpha, \beta, \dots, \delta_3\}$ in precisely three points

$\omega_1, \omega_2, \omega_3$. Then $\{\omega_1, \omega_2, \omega_3\}$ must be a $G_{\{\alpha, \beta, \dots, \delta_1\}}$ -invariant subset of

$\Omega - \{\alpha, \beta, \dots, \delta_3\}$. This is contrary to our knowledge of the action of $H_{\alpha_1\alpha_2\alpha_3}$.

Let $|\Sigma| = 2q+2$. Then $G_{\alpha\beta\dots\delta_2}$ is transitive, and in fact primitive,

$\Sigma = \{\beta_2, \gamma_2, \delta_2\}$, because it contains H_{α_1, α_2} . It follows by [20, 13.2] that $G_{\alpha_1, \beta_1, \delta_2}$ is 4-transitive on Σ , which is impossible since q does not divide its order.

Suppose now finally that $|\Sigma| = 4q+1$. Then H_{α_1, α_2} has two primitive orbits Σ_1, Σ_2 on $\Sigma = \{\beta_2, \gamma_2, \delta_2\}$ of size $2q-1$. Since q and p does not divide $|G_{\alpha_1, \beta_1, \delta_2}|$ we see that $G_{\alpha_1, \beta_1, \delta_2}$ is imprimitive on Σ and $\{\beta_2, \gamma_2, \delta_2\}$ is a block of imprimitivity. Consider any other block B of size 3. Then one of $B \cap \Sigma_1, B \cap \Sigma_2$ is a non-trivial block of H_{α_1, α_2} on Σ_1 or Σ_2 , contradicting its primitivity there.

Step 6. G is 7-transitive.

The $G_{\alpha_1, \beta_1, \delta_2}$ -orbits have sizes obtained out of 1, 1, $2q-1, 2q-1, 2q-1, 2q-1$. Now the Atkinson argument with respect to 4 (cf. Step 4) shows that all the $G_{\{\alpha_1, \beta_1, \dots, \beta_5\}}$ -orbits have even length and in fact only one has length not divisible by 4. Since none is divisible by p or q , the only possibilities are 2 and $8q-4$ or $8q-2$.

In the first case $G_{\{\alpha_1, \beta_1, \dots, \alpha_2\}}$ has blocks of imprimitivity on $\Omega = \{\alpha_1, \beta_1, \dots, \alpha_2\}$ of size 3, and $\{\beta_2, \gamma_2, \delta_2\}$ is one of these. Now the block containing α_3 must contain 3 points out of $\{\beta_3, \gamma_3, \delta_3\}$, as we see from the action of $H_{\alpha_1, \alpha_2, \alpha_3}$. But this must be also true of the blocks containing β_3, γ_3 and δ_3 , which gives a contradiction.

In the second case, $G_{\{\alpha_1, \dots, \beta_2\}}$ is transitive on $\Omega = \{\alpha_1, \dots, \beta_2\}$, and since $G_{\{\alpha_1, \dots, \beta_2\}}/G_{\alpha_1, \dots, \beta_2}$ is S_6 , we see that either G is 7-transitive or $G_{\alpha_1, \dots, \beta_2}$ has two orbits of size $4q-1$. But the latter is impossible by [20, 3.13].

Step 7. G is 8-transitive.

Consider the $G_{\alpha_1, \dots, \beta_2}$ -orbits. Note that $\{\beta_2\}$ is not an orbit by [14], and

also that q does not divide the order of $G_{\alpha, \dots, \delta_2}$. Hence the possibilities

are $4q-1, 2q-1, 2q-1,$

$4q-1, 4q-2,$

$6q-2, 2q-1,$

or $8q-3.$

The first two are excluded by the Atkinson argument with respect to 4.

In the third case it follows from [19] that G is not primitive. This is however impossible, because the blocks would have size $2q$. Hence the assertion.

Step 8. G is 9-transitive.

Here all the $G_{\alpha, \dots, \delta_2}$ -orbits have length divisible by $2q-1$. The Atkinson argument with respect to 4 implies directly that $G_{\{\alpha, \dots, \delta_2\}}$ is transitive on $\Omega - \{\alpha, \dots, \delta_2\}$. Since G is 8-transitive, this shows that G is 9-homogeneous on Ω . Since $G_{\{\alpha, \dots, \delta_2\}}/G_{\alpha, \dots, \delta_2}$ is S_8 , we see that either G is 9-transitive or $G_{\alpha, \dots, \delta_2}$ has two orbits of size $4q-2$ on $\Omega - \{\alpha, \dots, \delta_2\}$.

In the latter case it follows that if Σ is any subset of Ω of size 9 then G_Σ^Σ is A_9 (and not S_9). This implies that any involution of G fixes at most ^{two} ~~any~~ points, which is clearly impossible.

Step 9. More on the action of $N(Q)$ on \mathcal{E} .

Let $N = N_G(Q)$, and let K and L be the kernels of N on Δ_0, \mathcal{E}_0 respectively. Then $L \leq K$: Otherwise $LK > K$, so $1 \neq LK/K \triangleleft N/K$. Now $N/K = S_4$, so $LK/K \cong V_4$. But this implies that $L \cap C(Q) \neq Q$, and since Q is self-centralizing on its long orbits, this is not possible.

Let $X = N/L, Y = LC(Q)/L$. We shall write \bar{K} for K/L . Then $X \leq S_8$ and $X^{\Delta_0} = S_4$, so $X \not\leq A_8$. By [20, 15.1], any 3-element of X acts on \mathcal{E}_0 as a product of two 3-cycles, since we know that all the 3-elements of X lie

in Y . Now the normalizer in S_8 of such a 3-element is $Z_2 \times (Z_3 \text{ wr } Z_2) \cdot Z_2$, which has an elementary abelian Sylow 2-subgroup. Hence $X/Y \leq Z_2$ by the Frattini argument. If ~~a 2-element of Y~~ ^{an involution of $C(\Omega)$} fixed a point in Ω_0 then, being even, it would have degree at most $6q+2$; this is not possible by a theorem of Luther [11]. Hence the 2-elements in Y are semiregular on Ω_0 . Therefore $|X|$ is 24 or 48. Furthermore, if $|X| = 24$ then $X = S_4$. If X has two orbits of size 4 on Ω_0 then the permutations odd on Ω_0 are even on Ω and hence are odd on Ω , and the same is true if X is transitive on Ω_0 . If X has orbits of size 2 and 6 then the involutions of Y cannot all be semi-regular. Hence $|X| = 48$ and \bar{K} is Z_2 acting semi-regularly on Ω_0 . Finally, note that we may assume that K normalizes R . Then $\{\alpha_2, \beta_2, \dots, \delta_3\}$ is K -invariant.

Step 10. Let $D = G_{\{\alpha_2, \beta_2, \dots, \delta_3\}}$. Then D is 4-transitive on $\Omega - \{\alpha_2, \beta_2, \dots, \delta_3\}$. For, from the analysis in Step 9 it follows that $D_{\{\alpha_2, \beta_2, \dots, \delta_3\}}$ is transitive on $\Omega - \{\alpha_2, \beta_2, \dots, \delta_3\}$. Moreover, the lengths of the D_α -orbits on $\Omega - \{\alpha_2, \beta_2, \dots, \delta_3, \alpha\}$ are obtained out of 3, $2(q-1)$, $6(q-1)$, and in fact all the $D_{\alpha_2 \beta_2 \delta_3}$ -orbits on $\Omega - \{\alpha_2, \beta_2, \dots, \delta_3\}$ have length divisible by $2(q-1)$, as we see from the action of K .

Since G is 9-transitive on Ω , the Atkinson argument with respect to 9 (cf. Step 4) shows that there are at most two D_α -orbits, because $6(q-1) \equiv 6 \pmod{9}$. So the possibilities are

$$\begin{aligned} &3, \quad 8(q-1), \\ &2q+1, \quad 6(q-1), \\ &2(q-1), \quad 6q-3, \end{aligned}$$

or $8q-5$.

The first case is impossible by a theorem of Bannai [3, Theorem 2]. In the second case D is primitive, contrary to [19]. In the third case, D is

imprimitive by [19], so the blocks must have size $2q-1$. Now $6q-3$ is odd, so D_{α_3} is still transitive on the D_α -orbit of length $6q-3$. It now follows from [1, Lemma 2] that $G_{\{\alpha_2, \beta_2, \dots, \beta_3\}}$ acts as a 2-transitive group on a Steiner system $S(2, 2q, 8q-3)$, which is impossible since q does not divide its order. Hence D is 2-transitive.

Now the D_{α_3} -orbits have length obtained out of $2, 2(q-1), 2(q-1), 2(q-1), 2(q-1)$. Then the Atkinson argument with respect to 3 shows that D_{α_3} is transitive. Similarly, the only possibilities for the length of the $D_{\alpha_3\beta_3}$ -orbits are $2q-1, 6(q-1)$,
or $8q-7$.

In the first case though D_{α_3} is primitive and the suborbit of size $2q-1$ is 2-transitive, and a contradiction now comes from [5, II, Theorem 3]. Hence D is 4-transitive on $\Omega - \{\alpha_2, \beta_2, \dots, \beta_3\}$.

Conclusion. We know that the orbits of $G_{\alpha_2\beta_2\dots\beta_3\alpha_3\beta_3}$ on $\Omega - \{\alpha_2, \dots, \beta_3, \alpha_3, \beta_3, \beta_3\}$ have length combined out of 1 and eight times $q-1$. But $G_{\alpha_2\beta_2\dots\beta_3\alpha_3\beta_3}$ is normal in $D_{\alpha_3\beta_3}$, and $D_{\alpha_3\beta_3}$ is transitive on $\Omega - \{\alpha_2, \beta_2, \dots, \beta_3, \alpha_3, \beta_3, \beta_3\}$. Hence G is 12-transitive on Ω . But G contains a 3-element in $C(Q)$ fixing $2q+1$ points of Ω . Hence by a result of W.A. Manning [12, p.596], G is alternating or symmetric. This concludes the proof of the theorem.

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