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Some problems on characters

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According to a classical theorem of Frobenius, there is a close connection between high transitivity of a permutation group H of degree n on the one hand and the behaviour of certain complex irreducible characters of the symmetric group S_n when restricted to H on the other hand. This was strengthened to some extent in [2], where it was shown for instance that (with one exception) H is 2k-transitive if and only if the character $\chi^{(n-k,k)}$ remains irreducible when restricted to H. Here $\chi^{(n-k,k)}$ is the irreducible character of S_n which is the difference between the permutation characters on k-sets and (k-1)-sets. Similar though weaker results were also obtained for other characters of S_n. Since it is known that very highly transitive groups are alternating or symmetric, it follows that there is a large set I of irreducible characters of S_n such that if $H \leq S_n$, $\chi \in I$ and if χ_H is irreducible then $A_n \leq H$. Indeed, if the famous conjecture about the non-existence of non-trivial 6-transitive groups is true, the set I consists of nearly all the irreducible characters of S_n. This leads to

Problem 1. For which pairs (G,χ), where G is a finite group and χ is an irreducible character of G, is it true that $H \leq G$ and χ_H irreducible implies that the derived group G' of G is contained in H?

Apart from the partial results for S_n mentioned above, one family of irreducible characters of the linear groups has been handled in [3], but the results should be much more general.

Let G be a finite group with a (B,N)-pair. We shall call an irreducible character χ of G a Borel character if χ is a constituent of the permutation character of G on the set of cosets of B, so that $\chi \leq 1_B^G$. We would like to obtain a solution to Problem 1 for such pairs (G,χ), and our conjecture is in

Problem 1'. Let G be a group with a (B,N)-pair, and let χ be a Borel character of G. Let $H \leqslant G$. If the character χ_H is irreducible then in general $G' \leqslant H$. Here the phrase "in general" means with few, explicitly known, exceptions.

The strategy for proving our result in [3] was as follows: Given that χ_H is irreducible, show that the permutation character π of G on some "good" G-space Ω is contained in the product $\chi\cdot\bar{\chi}$. Then

$$\langle 1,\pi\rangle_H \leqslant \langle 1,\chi\bar{\chi}\rangle_H = \langle \chi_H,\chi_H\rangle_H = 1 \ ,$$

so that H is transitive on Ω. Now if Ω is "good", then we can deduce from the transitivity of H on Ω that $G' \leqslant H$ - this is what I meant by a "good" G-space.

Hence there are two parts to the problem: one is to handle the products $\chi\cdot\bar{\chi}$, and the other is to handle the "good" spaces. It is the first part I want to concentrate on in the rest of the paper. Unfortunately, very little is known about the products of characters. They seem very difficult to handle, and the larger and more complicated the group the harder the problem seems. But let us return to the groups with a (B,N)-pair. There the situation seems more promising, in view of some results of Curtis, Iwahori and Kilmoyer [1]:

Let W be the Weyl group of G, so $W = N/B \cap N$. Then $W = \langle R \rangle$, where R is a Coxeter system for W. For $J \subseteq R$, let $W_J = \langle J \rangle$, and let $P_J = BW_JB$. Then W_J, P_J are parabolic subgroups of W, G, respectively. Finally, let $\pi_0^{(J)} = 1_{W_J}^{W}$ and $\pi^{(J)} = 1_{P_J}^{G}$. Then there is a one-to-one correspondence $\chi_0 \leftrightarrow \chi$ between the irreducible characters χ_0 of W and the irreducible Borel characters χ of G, such that

$$\langle \pi^{(J)}, \chi \rangle_G = \langle \pi_0^{(J)}, \chi_0 \rangle_W \quad \text{for all subsets } J \text{ of } R.$$

The second problem I wish to pose is whether this remarkable correspondence can be at least partially extended for products:

Problem 2. Is it true that $\langle \chi_0, \psi_0 \cdot \varphi_0 \rangle_W \neq 0$ implies $\langle \chi, \psi\varphi \rangle_G \neq 0$, or even that $\langle \chi_0, \psi_0 \varphi_0 \rangle_W \leq \langle \chi, \psi\varphi \rangle_G$?

A positive answer to Problem 2 would go a good way towards supplying an answer to Problem 1', since the characters of the Weyl groups are much easier to handle. I do not have much evidence for the conjecture in Problem 2 yet. I have worked out some of the products of characters corresponding to partitions into two parts of $GL(n,q)$ and S_n and checked the claim for these. The other indication that the conjecture may be true comes from the products of permutation characters: Let $\pi_0^{(J)}$, $\pi_0^{(L)}$ be two permutation characters of S_n; if we work out $\pi_0^{(J)} \cdot \pi_0^{(L)}$ and then replace each χ_0 by χ, we obtain a character of $GL(n,q)$ which is contained in $\pi^{(J)} \cdot \pi^{(L)}$.

References

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