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On the Schur indices of algebraic groups

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Let p be a fixed prime number and k be an algebraically closed field of characteristic p. Let G be a connected, reductive linear algebraic group defined over k and σ be a surjective endomorphism of G such that $G_\sigma = \{x \in G \mid \sigma(x) = x\}$ is finite. I want to determine the rational Schur indices of G_σ. We assume that the centre Z of G is connected and that p is not a bad prime for G.

1. Semisimple and regular characters.

Let P be a Sylow p-subgroup of G_σ. The Gelfand-Graev character Γ_G of G_σ is the character of G_σ which is induced from any linear character of P in "general position". Γ_G does not depend on the choice of such linear characters of P. That Γ_G is multiplicity-free is well-known. Each irreducible component of Γ_G will be called a regular character of G_σ. Any irreducible character of G_σ whose degree is coprime to p will be called semisimple. We note that if G is defined over a finite field $\mathbb{F}_q$ ($q = p^f$, $f \geq 1$) and σ is the corresponding Frobenius endomorphism of G, the number of the regular characters of $G_\sigma = G(\mathbb{F}_q)$ coincides with that of the rational ones and it equals $|Z_\sigma| \cdot q^l$, where l is the rank of G/Z.

Theorem 1 (R. Gow, Z. Ohmori) The character of G_σ which is induced from a linear character of P is rational. In particular, Γ_G is a rational character.

Corollary 2. The Schur indices of the regular or semisimple characters of G_σ are all equal to one.

This follows from the next lemma.

Lemma 3. Let H be a finite group and ξ be a rational character of H. Then for any irreducible complex character χ of H, the Schur index $m_{\mathbb{Q}}(\chi)$ of χ divides the intertwinning number $\langle \chi, \xi \rangle_H$.

For example, any non-linear irreducible character of $G_\sigma = GL(2, q)$ ($G = GL_2$, σ is the Frobenius) is regular and the Schur indices of G_σ are one.

2. Application to $GL(n, q)$ and $U(n, q^2)$.

Let G be a general linear group GL_n or a general unitary group U_n, both defined over $\mathbb{F}_q$ and let σ be the corresponding Frobenius endomorphism of G. Then $G_\sigma = G(\mathbb{F}_q)$ is isomorphic to the finite general linear group $GL(n, q)$ or the finite general unitary group $U(n, q^2)$, respectively. We know

Theorem 4 (R. Gow) Any Schur index of G_σ divides 2.

Let u be a unipotent element of G_σ. There is a σ-stable subgroup L of G with the following properties: (1) L is a connected, reductive group with connected centre; (2) p is a good prime for L; (3) u is a regular unipotent element of L. Then by theorem 1 and lemma 3, we have

Theorem 5. Let χ be an irreducible character of G_σ and u be a unipotent element of G_σ. Then $\chi(u)$ is a rational integer and $m_{\mathbb{Q}}(\chi)$ divides $\chi(u)$.

The value $\chi(u)$ can be calculated by Green's work on the character table of $GL(n, q)$ or Ennola's conjecture on that of $U(n, q^2)$, which has been proved for $p > n$ and $q \gg 0$ by Hotta, Kazhdan, Lusztig, Springer and Srinivasan:

Theorem 6 If $G_\sigma = U(n, q^2)$, assume that $p > n$ and $q \gg 0$ (if $n \leq 5$, this assumption can be dropped, and if $G_\sigma = GL(n, q)$, no assumption is needed). Then for any irreducible character χ of G_σ, there is a unipotent element u of G_σ such that $\chi(u)$ is equal to the p-part of the degree of χ (which is a polynomial in

q with rational integral coefficients) up to the sign.

Corollary 7 (Z. Ohmori) Let the assumption be as in theorem 6. Further assume that $p > 2$. Then all the Schur indices of G_σ are one.

Remark that since any prime is good of G, any irreducible character of G_σ of degree coprime to p has Schur index one (Cor. 2). Also note that Gow has proved that all the Schur indices of $GL(2, q)$, $GL(3, q)$ or $GL(4, q)$ are equal to one.

References

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