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On Symmetric Systems

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1. Introduction

Loos [5] has shown that a symmetric space can be defined as a manifold carrying a diffeomorphic binary operation which satisfies three algebraic and one topological conditions. Abstracting the algebraic property of a symmetric space, Nobusawa [7] introduced the concept of a symmetric system (or symmetric set). By his definition, a symmetric system is a set A carrying a binary operation $a \circ b$ which satisfies the following conditions:

1) $a \circ a = a$,

2) $(x \circ a) \circ a = x$,

3) $(x \circ y) \circ a = (x \circ a) \circ (y \circ a)$.

From the condition 2), we have that the mapping $\sigma(a) : A \longrightarrow A$ defined by $x^{\sigma(a)} = x \circ a$ is a bijection, and corresponding to the above conditions, we have

1') $a^{\sigma(a)} = a$,

2') $\sigma(a)^2 = 1$,

3') $\sigma(a) \in \mathrm{Aut}(A)$.

Furthermore we have

4') $\sigma(a^{\rho}) = \rho^{-1}\sigma(a)\rho \qquad (\forall\, \rho \in \mathrm{Aut}(A))$.

In particular, taking $\sigma(b)$ for ρ we have

5') $\sigma(a \circ b) = \sigma(b)^{-1}\sigma(a)\sigma(b)$.

Let $\qquad G(A) = \langle \sigma(a) \mid a \in A \rangle$

and $\qquad H(A) = \langle \sigma(a)\sigma(b) \mid a, b \in A \rangle$.

Then $|G(A) : H(A)| \leq 2$. The group $H(A)$ is usually called a group of displacements.

Example. Let G be a group, and define a binary operation in G by $a \circ b = ba^{-1}b$. Then $(G, \circ)$ is a symmetric system. Let D be a set of involutions in G such that $D^G (= \{g^{-1}dg \mid d \in D, g \in G\}) = D$. Then $(D, \circ)$ is a symmetric subsystem of $(G, \circ)$, and $a \circ b = b^{-1}ab$ in D.

We say that a symmetric system is embedded in a group G if A is isomorphic to some $(D, \circ)$, where D is a set of involutions in G such that $D^G = D$ and $G = \langle D \rangle$. In this case, indentifying A with D, we may regard A as a set of involutions in G satisfying

4) $A^G = A$,

5) $G = \langle A \rangle$.

We also have that under this situation

$$G(A) \simeq G/Z(G).$$

If A is embedded in G and $Z(G) = 1$, then we say that A is faithfully embedded in G. In this case $G(A) \simeq G$.

Now the mapping $\sigma : A \longrightarrow \sigma(A)$ $(a \longmapsto \sigma(a))$ is an epimorphism. We call A effective if σ is an isomorphism. If A is effective then A is faithfully embedded in $G(A)$, and conversely if A is faithfully embedded in some group G then A is effective.

2. Finite homogeneous symmetric systems.

A symmetric system A is called homogeneous if for any a and b in A there is c in A such that $a \circ c = b$. Let

$\phi_a : A \longrightarrow A$ be the mapping defined by $\phi_a(x) = a \circ x$. If A is homogeneous, then ϕ_a is surjective, and hence if A is finite and homogeneous then ϕ_a is also injective and for $a, b \in A$ there exists unique element c such that $a \circ c = b$. Thus we have that a finite homogeneous symmetric system A is effective and is embedded in G(A).

The main result in a joint paper [4] with M. Kano and N. Nobusawa is the following

Theorem 1 Suppose A is a finite symmetric system. Then A is homogeneous if and only if A is embedded in a group G such that the subgroup $H = \langle ab \mid a, b \in A \rangle$ is of odd order.

The "if" part is easily proved, but to prove the "only if" part we need a deep result of Glauberman which is called the Z*-Theorem. We may also have the theorems of Lagrange's type and Sylow's type for finite homogeneous symmetric systems by using the properties of a group of odd order which has an involutory automorphism.

After publishing our paper we have learned that Doro [1] has pointed out that the concept of finite homogeneous symmetric systems is equivalent to the concept of finite B-loops which were investigated by Glauberman [2], [3], and then our results are equivalent to some of the results obtained by Glauberman.

3. Simple symmetric systems.

Let A and B are symmetric systems. An epimorphism $f : A \longrightarrow B$ is called trivial if either f is an isomorphism or $|B| = 1$. A is called simple if any epimorphism of A to another symmetric system is trivial. Then we have the following

Theorem 2 Suppose A is a symmetric system with $|A| > 2$. Then A is simple if and only if A is embedded in a group G in which $H = \langle ab \mid a,b \in A \rangle$ is a minimal normal subgroup. If this is the case, H is either simple or a direct product of two simple groups which are isomorphic.

The "only if" part is proved essentially by Nobusawa [8] and the proof of "if" part will be given in [6].

Remark. If G(A) acts primitively on A then A is simple.

References

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