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The Commutativity of the Radicals of Group Algebras

Shigeo KOSHITANI
Department of Mathematics, University of Tsukuba

Let K be a field of characteristic $p>0$, G a finite group with a p-Sylow subgroup P such that $|P| = p^a$, G' the commutator subgroup of G and KG the group algebra of G over K. For a ring R and an integer $t>0$, denote by $J(R)$ the Jacobson radical of R, by $Z(R)$ the centre of R and by R_t the ring of all $t \times t$ matrices with entries in R.

We are interested in relations between ring-theoretical properties of KG and the structure of G. In particular, we shall consider the commutativity of $J(KG)$. We shall determine G with the property that $J(KG)$ is commutative. For an odd prime p the structure of G such that $J(KG)$ is commutative has been determined by D.A.R. Wallace [3] (cf. W. Hamernik [1]). So in this note we shall obtain a necessary and sufficient condition on G for $J(KG)$ to be commutative for any prime number p.

To begin with we shall prove the following,

<u>Theorem</u> 1. Assume that $2^2 \mid |G|$ if $p = 2$ and that $p \mid |G|$ if $p \neq 2$. If $J(KG)$ is commutative, then $N_G(P) = C_G(P)$ and this group is abelian, where $N_G(P)$ and $C_G(P)$ are the normalizer of P in G and the centralizer of P in G, respectively.

<u>Proof</u>. We may assume that K is algebraically closed. By [3; Theorem 2], G is p-nilpotent and P is abelian. Thus it is clear that $N_G(P) = C_G(P)$. Put $N = N_G(P)$ and $\widetilde{H} = O_{p'}(N)$.

Since $N = P \times \tilde{H}$, it suffices to show that $\tilde{H}$ is abelian. Let $B_1, \ldots, B_n$ be all blocks of KG. Since G is p-nilpotent, by Morita's theorems [2; Theorems 2 and 7],

$$B_i \cong KHe'_{i1} \otimes_K KP_i \otimes_K K_{t_i}, \text{ as K-algebras,}$$

where $H = O_{p'}(G)$, e'_{i1} is a centrally primitive idempotent of KH, P_i is a subgroup of G such that $|P| = |P_i|t_i$. Let $KHe'_{i1} \cong K_{h_i}$ for some $h_i > 0$. Thus $B_i \cong (KP_i)_{h_i t_i} = (KP_i)_{f_i}$, where f_i is the degree of a unique irreducible Brauer character in B_i. Hence

$$(*) \qquad J(B_i) \cong (J(KP_i))_{f_i}.$$

If $J(B_i) = 0$, then $p|f_i$. If $J(B_i) \neq 0$ and $J(B_i)^2 = 0$, then $p = 2$ and $2|f_i$ from [3; Lemma 7]. If $J(B_i)^2 \neq 0$, then it follows from (*) that $f_i = 1$, and so $h_i = t_i = 1$. These show that B_i is of defect a if and only if $f_i = 1$. By rearranging the numbers $1, \ldots, n$, we can assume that $B_1, \ldots, B_m$ are all blocks of KG with defect a. By Brauer's first main theorem, there is a bijection

$$B_i \longleftrightarrow \tilde{B}_i, \quad i = 1, \ldots, m,$$

where $\tilde{B}_1, \ldots, \tilde{B}_m$ are all blocks of KN. As for B_i we can write

$$\tilde{B}_i \cong K\tilde{H}\tilde{e}'_{i1} \otimes_K K\tilde{P}_i \otimes_K K_{\tilde{t}_i}, \text{ as K-algebras,}$$

where $\tilde{e}'_{i1}$ is a centrally primitive idempotent of $K\tilde{H}$ and $\tilde{P}_i$ is a subgroup of N such that $|P| = |\tilde{P}_i|\tilde{t}_i$. Let $K\tilde{H}\tilde{e}'_{i1} \cong K_{\tilde{h}_i}$ for some $\tilde{h}_i > 0$. Since P is normal in N, all $\tilde{B}_i$ have defect a. Thus $\tilde{t}_i = 1$ for all i since $p \nmid (\tilde{h}_i \tilde{t}_i)$. Fix any i $(1 \leq i \leq m)$. Since $t_i = 1$, e'_{i1} is a centrally primitive idempotent of KG. Similarly, $\tilde{e}'_{i1}$ is a centrally primitive idempotent of KN. Thus, e'_{i1} corresponds to $\tilde{e}'_{i1}$ through the Brauer homomorphism. On the other hand, $\dim_K(KHe'_{i1}) = 1$, and so $\dim_K(K\tilde{H}\tilde{e}'_{i1}) = 1$. This implies that all irreducible $K\tilde{H}$-modules have K-dimension one, and so $\tilde{H}$ is abelian. This completes the proof.

<u>Remark</u> 1. The converse of Theorem 1 does not hold in general.

A counter-example is as follows. Assume that $p = 2$, $G = \langle x, y \mid x^8 = y^3 = 1, x^{-1}yx = y^2 \rangle$ and $P = \langle x \rangle$. Then $J(KG)^2 \neq 0$, $N_G(P) = C_G(P)$ and this group is cyclic, but $J(KG)$ is noncommutative.

Next, we can prove the following theorem as in the proof of Theorem 1.

Theorem 2. $J(KG)$ is commutative if and only if G is one of the following two types:

(i) $2^2 \nmid |G|$ if $p = 2$, and $p \nmid |G|$ if $p \neq 2$.

(ii) G is a p-nilpotent group with an abelian p-Sylow subgroup P, $b_0 = |O_{p'}(G) : G'|$, $b_1 = \dots = b_{a-2} = 0$, and if $p \neq 2$, $b_{a-1} = 0$, where $|P| = p^a$ and b_k is the number of p-regular conjugate classes K_j of G such that $p^k \mid |K_j|$ and $p^{k+1} \nmid |K_j|$ for $k = 0, \dots, a$.

Remark 2. D.A.R. Wallace [3; Theorem 1] showed that for $p \neq 2$ $J(KG)$ is commutative if and only if $J(KG) \subsetneqq Z(KG)$. But for $p = 2$ this does not hold in general. Indeed, assume that $p = 2$ and $G = \langle x, y \mid x^4 = y^3 = 1, x^{-1}yx = y^2 \rangle$. Then $J(KG)^2 \neq 0$ and $J(KG)$ is commutative, but $J(KG) \not\subseteq Z(KG)$.

References

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