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An inequality for finite permutation groups

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0. Introduction

Let (G,Ω) be a permutation group of degree n. For any subset X of G, we put

$$F(X) := \{\alpha \in \Omega \mid \forall x \in X \ \ \alpha^x = \alpha\}$$
$$f(X) := |F(X)| .$$

For $x \in G$, we use $f(x)$ instead of $f(\{x\})$.

(Definition 1) Let ℓ_i $(i=1,\cdots,r)$ be integers such that $0 \leq \ell_1 < \cdots < \ell_r < n$. We say that (G,Ω) is an $\{\ell_1,\cdots,\ell_r\}$-group, if $\{f(x) \mid x \in G,\ x \neq 1\} \subseteq \{\ell_1,\cdots,\ell_r\}$.

E. Bannai and M. Deza posed us the following conjecture ; if (G,Ω) is an $\{\ell_1,\cdots,\ell_r\}$-group of degree n, then $|G| \leq \prod_{i=1}^{r}(n-\ell_i)$. In §1 this conjecture is proved. In §§ 2,3 we consider the case $|G| = \prod(n-\ell_i)$. Finally in § 4, using the same method as in Theorem 1, we give a proof of the Burnside-Brauer Theorem.

1. Proof of the conjecture

Here we prove the conjecture mentioned above.

<u>Theorem 1</u>. [5] Let (G,Ω) be an $\{\ell_1,\cdots,\ell_r\}$-group of degree n. Then $|G|$ divides $\prod_{i=1}^{r}(n-\ell_i)$.

Proof. Let θ be the permutation character of G, and let 1_G be the principal character of G. Then it is well known that

$$\hat{\theta} := \prod_{i=1}^{r}(\theta - \ell_i 1_G)$$

is a generalized character of G. By the definition of $\hat{\theta}$, we have $\hat{\theta}(g)=0$ for all $g \in G$, $g \neq 1$. Hence, the multiplicity of 1_G in $\hat{\theta}$ is given by

$$(\hat{\theta}, 1_G) = \frac{1}{|G|}\sum_{g \in G}\hat{\theta}(g) = \frac{1}{|G|}\ \hat{\theta}(1) = \frac{1}{|G|}\prod_{i=1}^{r}(n-\ell_i).$$

Thus, we get the desired result.

<u>Corollary 2</u>. Assume the hypothesis of Theorem 1. Then we have that $|G| = \prod_{i=1}^{r}(n-\ell_i)$ if and only if $\hat{\theta}$ is the regular character of G, where $\hat{\theta}$ is defined in the proof of Theorem 1.

2. $\{\ell_1, \cdots, \ell_r\}$ -sharp groups

(Definition 2) Assume the hypothesis of Theorem 1. We say that (G, Ω) is an $\{\ell_1, \cdots, \ell_r\}$ -sharp group, if $|G| = \prod_{i=1}^{r}(n-\ell_i)$.

We remark that $\{0, 1, \cdots, r-1\}$-sharp group is sharply r-transitive (see Corollary 4). Hence our concept is a generalization of sharply transitivity. It is natural that one hopes to classify all $\{\ell_1, \cdots, \ell_r\}$ -sharp groups. But in general it seems to be difficult. So we must study special cases at first.

Now we state some examples and known results.

<u>Example 1</u>. $Z_\ell \wr Z_2$ is a $\{0, \ell\}$-sharp group of degree 2ℓ.

<u>Example 2</u>. $\{1, 3\}$-sharp groups

(1) $G=S_4$; $\Omega=\Delta\cup\Gamma$, $G^{\Delta}=S_3$, $G^{\Gamma}=S_4$.

(2) $G=PSL(2,7)$; $\Omega=\Delta\cup\Gamma$, G^{Δ} is 2-transitive of degree 7,
G^{Γ} is 2-transitive of degree 8.

Known results. For the following $L=\{\ell_1,\cdots,\ell_r\}$, L-sharp groups have been classified.

$L=\{2\}$	Iwahori [3]
$L=\{3\}$	Iwahori and Kondo [4]
$L=\{0,2\}$	Tsuzuku [6]

The following lemma is due to E. Bannai.

Lemma 3. Let G be a $\{0,\ell_2,\cdots,\ell_r\}$-sharp group on Ω. Then G is transitive on Ω, and G_α is an $\{\ell_2-1,\cdots,\ell_r-1\}$-sharp group on $\Omega-\{\alpha\}$, where α is any element of Ω.

Applying Theorem 1 to G_α, we can easily get the proof of Lemma 3.

Corollary 4. Let G be a $\{0,1,\cdots,r-1\}$-sharp group. Then G is sharply r-transitive.

The following Theorem, due to T. Ito, is an extension of Corollary 4.

Theorem 5. [2] Let G be an $\{\ell,\ell+1,\cdots,\ell+r-1\}$-sharp group on $\Omega(r\geq 2)$. Then $f(G)=\ell$ and G is sharply r-transitive on $\Omega-F(G)$.

<u>Remark</u>. It looks very likely that every $\{\ell_1, \cdots, \ell_r\}$ -sharp group has ℓ_1+1 orbits. Note that Lemma 3 is a special case where $\ell_1=0$.

3. The case r=2

Now we consider the case $r=2$ i.e. $\{\ell, \ell+s\}$ -sharp groups. In this case we can show that f(G) is considerably large and that ℓ-f(G) is bounded by a function of s. Hence the essential parameter is s alone. More precisely we have

<u>Theorem 6</u>. [2] Let G be an $\{\ell, \ell+s\}$ -sharp group.
Put $s' := \max\left\{1, \left[\frac{s-1}{2}\right]\right\}$, $m := \ell+(1-s)s'+s'^2-1$.
Then we have $f(G) \geq m$.

For s=1,2,3,4 this inequality is best possible. For $s \geq 5$ we guess that $f(G)=m$ does not occur. But I can not prove it yet.

Using Theorem 6, we can classify all $\{\ell, \ell+s\}$ -sharp groups for s=1,2,3,4. For example, the $\{\ell, \ell+2\}$ -sharp groups are the following groups ; $G=D_8$, S_4, GL(2,3), PSL(2,7). These groups are determined up to permutation isomorphism. For more details see [2]. The case $s \geq 5$ is very difficult.

4. Final remark

We give another example which can be proved by the same method as in the proof of Theorem 1. Let G be a finite group, and let θ be a faithful character of G. Let $\theta(1)=\alpha_1, \alpha_2, \cdots, \alpha_m$

be the distinct values taken by θ. We put $\hat{\theta} := \prod_{i=2}^{m}(\theta-\alpha_i)$. Since θ is faithful, we have

$$\hat{\theta} = \alpha\cdot\rho_G = \sum_{\chi\in Irr(G)} \alpha\,\chi(1)\chi, \text{ where } \alpha = \frac{1}{|G|}\hat{\theta}(1) \in \mathbb{C}.$$

Since $\hat{\theta}(1)\neq 0$, we have $\alpha\neq 0$. On the other hand $\hat{\theta}$ is a $\mathbb{C}$-linear combination of θ^j for $0\leq j<m$, as it can be seen from the definition of $\hat{\theta}$. Then every $\chi\in$Irr(G) must be a constituent of some θ^j. Thus we obtain

Theorem.(Burnside-Brauer cf. [1] p49) Let θ be a faithful character of G and suppose $\theta(g)$ takes exactly m different values for $g\in G$. Then every $\chi\in$Irr(G) is a constituent of one of the characters θ^j for $0\leq j<m$.

If some $\alpha_i=0$, then $\hat{\theta}$ is a $\mathbb{C}$-linear combination of θ^j for $0<j<m$. Thus we obtain

Corollary. Assume the hypothesis of the Theorem. Suppose that $\theta(g)=0$ for some $g\in G$. Then every $\chi\in$Irr(G) is a constituent of one of the characters θ^j for $0<j<m$.

We remark that every non-linear faithful irreducible character of G satisfies the hypothesis of the Corollary.

References

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