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On representations of $\mathbb{R}$-elementary groups at 2

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G (finite group) の $\mathbb{R}$ (real number field) 上の表現を考えたい。ここでは，Brauer-Witt induction theorem ([4], p31) による立場に立ち，$\mathbb{R}$-elementary groups at 2 の $\mathbb{R}$ 上の表現を考える．この小文の目的は次の問題の1つの解答を得る事である。

Problem. Let G be an $\mathbb{R}$-elementary group at 2. If G has an irr. char. of Schur index 2 over $\mathbb{R}$. Then G ?

1. 2-groups

2-groups については，Witt-Roquette theorem ([2], p73) に，次の Lemma を混ぜて，ちょっとかき回してみると，問題の解答を得る事ができる．(Witt-Roquette + Lemma $\Longrightarrow$ Theorem)

Lemma. G : a finite group, H : a subgroup of G.

(1). ζ ; a linear char. of H. $\chi = \zeta^G$. Then χ is not irr. iff $\exists x \in G - H$ such that $xhx^{-1}h^{-1} \in \operatorname{Ker}\zeta$ for $\forall h \in H^x \cap H$.

(2). ζ ; a char. of H. $\chi = \zeta^G$. Then $\operatorname{Ker}\chi = \bigcap_{g \in G} (\operatorname{Ker}\zeta)^g$.

Λ を the ordinary quaternion algebra over $\mathbb{Q}$ とする.

Theorem. F : a field of characteristic 0. P ; 2-group. Then, $\exists \chi$: faith. irr. char. of P such that $m_F(\chi) = 2$ iff

(i). $P \geq {}^\exists Q > {}^\exists A > {}^\exists K$ such that $Q \rhd K$, A/K ; cyclic and $|Q : A| = 2$,

(ii). Q/K : generalized quaternion of order $2^{n+1} \geq 8$,

(iii). $x \in P - A \Rightarrow {}^\exists a \in A^x \cap A$ such that $xax^{-1}a^{-1} \notin K$.

(iv). $\bigcap_{x \in P} K^x = 1$.

(v). $F(\varepsilon_{2^n} + \varepsilon_{2^n}^{-1}) \otimes_{\mathbb{Q}} \Lambda$; a division alg., where ε_{2^n} is a primitive 2^n-th root of unity.

Remark. 条件 (v). については, Fein, Gordon and Smith [1] を参照されたい。特に $F = \mathbb{R}$ の時は, $\mathbb{R}(\varepsilon_{2^n} + \varepsilon_{2^n}^{-1}) \otimes_{\mathbb{Q}} \Lambda = \mathbb{H}$ (Hamilton's quaternion field) である.

Example. W; a generalized quaternion group.

$P \equiv W \wr \underbrace{Z_2 \wr Z_2 \wr \cdots \wr Z_2}_{r-1}$. ($P$; a Sylow 2-group of $Sp_{2^r}(8)$)

$\Rightarrow$ $\exists \chi$; faith. irr. char. of P of Schur index 2 over $\mathbb{R}$.

$M_{2^{r-1}}(\mathbb{H})$ is a simple component of $\mathbb{R}P$.

2. $\mathbb{R}$-elementary groups at 2.

この節では次の Frobenius - Schur theorem を $\mathbb{R}$-elementary group at 2 の場合に，群の言葉に書きなおす.

Frobenius - Schur theorem. ([2], P21). G : finite group.
χ : irr. char. of G. $\nu(\chi) = |G|^{-1} \sum_{g \in G} \chi(g^2)$. Then

(i) $\nu(\chi) = 1$ iff $\mathbb{R}(\chi) = \mathbb{R}$ and $m_{\mathbb{R}}(\chi) = 1$

(ii) $\nu(\chi) = -1$ iff $\mathbb{R}(\chi) = \mathbb{R}$ and $m_{\mathbb{R}}(\chi) = 2$

(iii) $\nu(\chi) = 0$ iff $\mathbb{R}(\chi) = \mathbb{C}$.

H; an $\mathbb{R}$-elementary group at 2, i.e. $H = P\langle a \rangle$ (semi-direct product), P: 2-group, $\langle a \rangle$: $2'$-group, $x \in P \Rightarrow$ $xax^{-1} = a$ or a^{-1}, とする. $A \equiv \langle a \rangle$, $Q \equiv C_P(a)$ とおく.
次を仮定する.

Assumption. $A \neq 1$, $P \neq Q$.

$\{1, g\}$; a set of representatives of P/Q, χ; irr. char. of H such that $\mathrm{Ker}\,\chi \cap A = 1$ ，とすると，次の事はすぐわかる。

Lemma. $\exists\mu$; irr. char. of Q, $\exists\tau$; faith linear char. of A such that $\mu^g = \overline{\mu}$, $(\mu\tau)^P = \chi$.

Lemma. (i) $\nu(\chi) = -1$ iff $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2) = -1$.

(ii). If $|P|^{-1}\sum_{h\in Q}(\mu+\mu^g)((gh)^2) = -1$, then

(1). μ^P : irr. $\Rightarrow$ $\nu(\mu)=0$, $\nu(\mu^P) = -1$.

(2). μ^P : non-irr. $\Rightarrow$ $\exists\xi, \xi'$; irr. char. of P such that $\xi \neq \xi'$ and $\mu^P = \xi + \xi'$. Further

(a). $\nu(\mu) = \nu(\xi) = \nu(\xi') = -1$

or. (b). $\nu(\mu) = 1$ and $\nu(\xi) = \nu(\xi') = 0$.

$\mathbb{R}$上 Schur index 2 の char. をもつか否かという問題は 2-group に関する議論に移る事になる。上の Lemma の(ii)のそれぞれの場合について考察する事により次の Theorem を得る.

Theorem. $\exists\chi$; irr. char. of H such that $\nu(\chi) = -1$ and $\mathrm{Ker}\,\chi \cap A = 1$ iff one of the following conditions (A), (B), (C) are satisfied.

(A). $\exists K \leq Q$ such that $K \lhd P$ and P/K is a cyclic group of order 4.

(B). $\exists K \lhd \exists T \leq Q$ which satisfy the following conditions.

(i). T/K : cyclic

(ii). $x \in Q - T \Rightarrow \exists a \in T^x \cap T$ such that $xax^{-1}a^{-1} \notin K$.

(iii). $\exists y \in P - Q$ such that $yby^{-1}b^{-1} \in K$ for $\forall b \in T^y \cap T$.

(iv). $X = \{h_1, \cdots, h_m\}$; a set of representatives of Q/T.

$\alpha = \#\{(h_i, h) \in X \times Q \mid h_i(gh)^2 h_i^{-1} \in T - K,\ h_i(gh)^4 h_i^{-1} \in K\}$, $\beta = \#\{(h_i, h) \in X \times Q \mid h_i(gh)^2 h_i^{-1} \in K\}$. Then $|Q|^{-1}(\beta - \alpha) = -1$.

(C). $\exists K \leq \exists T \leq \exists S \leq P$ which satisfy the following conditions

(i). $S \rhd K$, S/K ; a generalized quaternion group, T/K ; a cyclic subgroup of S/K of index 2.

(ii). $x \in P - T \Rightarrow \exists a \in T^x \cap T$ such that $xax^{-1}a^{-1} \notin K$.

(iii). One of the following conditions are satisfied.

(a). $K \leq T \cap Q$ and $(S \cap Q)/K$ is a cyclic subgroup of S/K of index 2.

(b). $K \leq T \cap Q$ and $(S \cap Q)/K$ is non abelien.

or. $K \nleq T \cap Q$. $X = \{h_1, \cdots, h_m\}$; a set of representatives of $Q/(T \cap Q)$. $\gamma = \#\{(h_i, h) \in X \times Q \mid h_i(gh)^2 h_i^{-1} \in T - K,\ h_i(gh)^4 h_i^{-1} \in K\}$, $\delta = \#\{(h_i, h) \in X \times Q \mid h_i(gh)^2 h_i^{-1} \in K\}$. Then $|P|^{-1}(\delta - \gamma) = -1$ and

$|T : T \cap Q| = 2$.

References.

1 B. Fein, B. Gordon and J.H. Smith, On the representation of -1 as a sum of two squares in an algebraic number field, J. of number theory 3, 310 – 315, (1971).

2. W. Feit, Characters of finite groups, Benjamin, 1967.

3. R. Gow, Real-valued characters and the Schur index, J. of Alg. 40, 258 – 270, (1976).

4. T. Yamada, The Schur subgroup of the Brauer group, Springer-Verlag, 1974.