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Tight spherical designs

(A preliminary report of some joint work with R. M. Damerell)

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An important generalization of the concept of t-design has been made by Delsarte [4]. Namely, he defined the concept of t-design in certain association schemes which are called Q-polynomial schemes. (See [4] for the details.)

Yet another important generalization of the concept of t-design has been made recently by Delsarte-Goethals-Seidel [5], which is the topic of the present paper.

Definition ([5]) Let Ω_d be the unit sphere in the d-dimensional Euclidean space $\mathbb{R}^d$. A subset X in Ω_d is said to be a spherical t-design (in Ω_d) if

(i) $|X| < \infty$ and

(ii) $\sum_{\xi \in X} f(\xi) = 0$ for any homogeneous harmonic polynomials f of degree 1,2,...,t . (Here, harmonic means $\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}\right) f(x_1, \cdots, x_d) = 0$.)

Spherical t-designs were studied extensively by [5]. In particular the following inequalities, which are analogous to the generalized Fisher's inequality for ordinary t-designs, have been shown.

Fisher type inequalities ([5]) If X is a spherical t-design in Ω_d, then

(i) $|X| \geqslant \binom{d+s-1}{d-1} + \binom{d+s-2}{d-1}$, if t = 2s is even, and

(ii) $|X| \geqslant 2 \cdot \binom{d+s-1}{d-1}$, if t = 2s+1 is odd.

X is said to be tight if the equality holds in either of the above inequalities.

Now, we want to classify tight spherical t-designs (in Ω_d). It is known that if d = 2 then X is a tight spherical t-design if and only if X in the set of vertices of a regular (t+1)-gon. So, without loss of generality, we may assume that $d \geqslant 3$. Delsarte-Goethals-Seidel [5] have shown that there exist no tight spherical 6-designs. The purpose of the present paper is to announce the following theorem. The details will be published in [1] and [2].

Theorem ([1], [2]) Suppose that $d \geqslant 3$.

(i) If $t = 2s \geqslant 6$, then there exist no tight spherical t-designs in Ω_d.

(ii) If $t = 2s+1 \geqslant 9$, then except for $t = 11$, $d = 24$ and $|X| = 196560$, there exist no tight spherical t-designs in Ω_d.

(The classification for $t \leqslant 5$ and $t = 7$ seems very difficult, and is still open. For $t = 11$ and $d = 24$, there actually exists a tight spherical 11-design which is constructed from the Leech lattice ([5]). However, the uniqueness problem is still open.)

Sketch of the proof of Theorem

We utilize the Lloyd type theorem that was implicitly obtained in Delsarte-Goethals-Seidel [5] and explicitly mentioned in [1, Theorem 1]. Namely, if there exists a spherical t-design in Ω_d, then all the zeros of the following polynomial $\Psi_s(x)$ of degree s are rational:

$$\Psi_s(x) = R_s(x) \qquad \text{if } t = 2s,$$
$$\Psi_s(x) = C_s(x) \qquad \text{if } t = 2s+1.$$

Here, $R_s(x) := C_s^{\frac{d}{2}}(x) + C_{s-1}^{\frac{d}{2}}(x)$, $C_s(x) := C_s^{\frac{d}{2}}(x)$

$$C_{2m}^{\nu}(x) := \frac{(-1)^m \cdot \Gamma(m+\nu)}{m!\,\Gamma(\nu)} \cdot {}_2F_1(-m, m+\nu\,;\, \tfrac{1}{2}\,;\, x^2) \quad \text{and}$$

$$C_{2m+1}^{\nu}(x) := \frac{(-1)^m\, \Gamma(m+\nu+1)}{m!\,\Gamma(\nu)}\, 2x \cdot {}_2F_1(-m, m+\nu+1\,;\, \tfrac{3}{2}\,;\, x^2).$$

(Note that ${}_2F_1$ denotes the Gauss' hypergeometric series, $C_s(x)$ is a certain Gegenbauer polynomial and $R_s(x)$ is a certain Jacobi polynomial (cf. [7].)

To begin with, we can prove that any nonzero root of the polynomial $\Psi_s(x)$ is the reciprocal of an integer. Let us denote by $\overline{\Psi}(x)$ the polynomial whose roots are the reciprocal of the nonzero roots of $\Psi_s(x)$. Our proof differ very much according as t is even or odd.

(i) (See [1]) Suppose that $t = 2s$. Then, the zeros of the polynomial $\overline{\Psi}(x)$ are almost symmetric with respect to the origin, but not exactly symmetric. In this case, we can ingeneously evaluate the range of the location of the zeros very precisely by exploiting some special properties of the associated orthogonal polynomials, and we can show that all the zeros of $\overline{\Psi}(x)$ are not integers, which leads a contradiction to the Lloyd type theorem.

(ii) (See [2]) Suppose that $t = 2s+1$. Then the zeros of $\overline{\Psi}(x)$ are exactly symmetric with respect to the origin, and so the method used in (i) is not available. However, in this case we can use the following two techniques:

(a) Let $\alpha_{-[\frac{s}{2}]}, \cdots, \alpha_{-1}, \alpha_1, \alpha_2, \cdots, \alpha_{[\frac{s}{2}]}$ be the nonzero roots of $\overline{\Psi}(x)$, (here note that $\alpha_{-i} = -\alpha_i$) .

Suppose that the α_i are all integers. Then $\prod_{i=1}^{[\frac{s}{2}]} \alpha_i^2$ must be $\square$ (i.e., a square of an integer), which leads the condition that

$$(*) \quad \begin{cases} \dfrac{(d+2s)(d+2s+2)\cdots(d+2s+2(s-1))}{1\cdot 3\cdot 5\cdot \cdots \cdot(2s-1)} = \square , & \text{for } s = \text{even} , \\[2ex] \dfrac{(d+2s+2)(d+2s+4)\cdots(d+4s)}{3\cdot 5\cdot \cdots \cdot(2s+1)} = \square , & \text{for } s = \text{odd} . \end{cases}$$

Now, by extending the method that was used by Erdös [5] to solve the diophantine equation

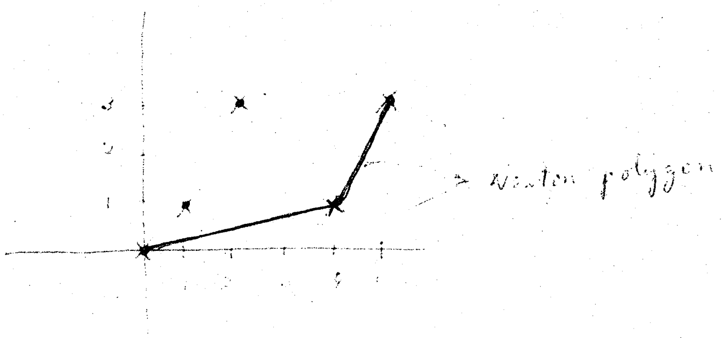
$$\binom{x}{i} = y^2 ,$$

we can show that the above diophantine equations (*) have no integral solutions for sufficiently large s (and for any $d \geq 3$). By elaborating the method of Erdös, we can actually show that it is all right if $s \geq 38$, say.

(b) Let us recall the definition and a property of Newton polygon (for a prime p) of a polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, $a_0 \neq 0$, $a_i \in \mathbb{Z}$. For the $a_i \neq 0$, let $a_i = p^{r_i}b_i$ with $(p, b_i) = 1$. We plot the points $(n-i, r_i)$ in the xy-plane. The Newton polygon (for p) of the polynomial $f(x)$ is the set of line segments which are the lower boundary of the convex hull of the points $(n-i, r_i)$ (see the figure below). Now, we have

<u>Lemma 1</u>. Suppose that $f(x) = \prod (x - \alpha_i^2)$, $\alpha_i \in \mathbb{Z}$. Then any slope of the Newton polygon must be an <u>even integer</u>.

(Example of Newton polygon. For $f(x) = x^5 + 6x^4 + 8x^2 + 10x + 8$, $p = 2$



)

Now, suppose that all the zeros of $\overline{\Psi}(x)$ are integers. Then any slope of the Newton polygon of $\overline{\Psi}(z)$ with $z = x^2$ must be an even integer. By studying very closely the Newton polygons of $\overline{\Psi}(z)$ for $p = 2$, we can show, after the <u>very complicated and tedious</u> calculations (which need improvements), that for all the remaining $s \geqslant 6$, the zeros of $\overline{\Psi}(x)$ are not all integers.

The cases $s = 4$ and 5 are eliminated separately by studying certain diophantine equations.

<u>Concluding Remark</u>

Finally, I would like to point out that the tight spherical t-designs are very important for some permutation group theoretical aspect. (cf. [3])

Let G be a transitive permutation group on a set Ω . Let $\alpha \in \Omega$ and let G_α be the stabilizer, and let $\{\alpha\}$, $\Delta_1, \Delta_2, \cdots, \Delta_r$ be the orbits of G_α . (Let $k = |\Delta_1| \leqslant |\Delta_2| \leqslant \cdots \leqslant |\Delta_r|$) .

(i) (well known) Let G be primitive, and let Δ_1 be self-paired. Then we have

$$|\Omega| \leqslant 1 + k + k(k-1) + \cdots + k(k-1)^{r-1} \qquad (\approx k^r) .$$

If the equality holds, then $|\Delta_i| = k\cdot(k-1)^{i-1}$ and the orbital graph (Ω, Δ_1) is a Moore graph, and it is shown that such a graph does not exist if $r \geqslant 3$ and $k \geqslant 3$ (Damerell-Bannai-Ito).

(ii) Let G be primitive, and let all the orbitals are self-paired. Let $\pi = 1 + \chi_1 + \cdots + \chi_s$ be the permutation character, and let

$$d = \deg.\chi_1 \le \deg.\chi_2 \le \cdots \le \deg.\chi_s .$$

Then, Cameron-Goethals-Seidel [3] have shown that

$$|\Omega| \le \binom{d+s-1}{d-1} + \binom{d+s-2}{d-1} \qquad \left(\approx \frac{d^s}{s!}\right)$$

$$= \sum_{i=1}^{s} \left(\binom{d+i-1}{d-1} - \binom{d+i-3}{d-1} \right),$$

and if the equality holds, then

$$\deg.\chi_i = \binom{d+i-1}{d-1} - \binom{d-i-3}{d-1}$$

and χ_i is the i-th symmetrized Kronecker product of χ_1, and moreover, we get a tight spherical 2s-design in Ω_d. Our theorem shows that this situation also does not hold, if $s \ge 3$ and $d \ge 3$, which solves the 'dual' extremal problem of Moore graphs.

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