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Notes on the Almost Measurability

(A Preliminary Report)

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1. Introduction. In statistical problems, we are usually concerned with two kinds of spaces, the sample space amd the parameter space, both of which are to be regarded as measurable spaces. In other words, the direct product measurable space Z of these two spaces plays the most important roles in the mathematical research of statistics. If we are of (Bayesian) opinion that the parameters are random variables, irrespectively of whether their distributions are known or not, our mainly concerned space Z is a measure space having a probability measure μ belonging to a family Σ of probability measures on Z. Regarding Z as the basi concept in mathematical research of statistics, the study of the σ-algebra and its sub-σ-algebras in Z, *e.g.*, σ-algebras in the sample space and these in the parameter space, appears to be essential in such a research.

There are many problems besides the above raised about the measurability of a given function with respect to a sub-σ-algebra; *e.g.*, sufficiency, invariance and *etc*. If a study on such problems are carried along a measure-theoretic way by using a single measure μ on a concerning measurable space Z, the result obtained by this method could not be beyond the almost measurability with respect to μ. In most statistical researches, however, such a measure μ would be taken arbitrarily from a family Σ of probability measures on Z, and the μ-almost measurabilities for all μ in Σ may be reduced to a certain extent. For instance, if Σ is the family of all probability measures on Z, whether does the μ-almost measurability for all μ in Σ imply the *pure* measurability ? This is the motivation of the present research.

2. <u>The almost measurability</u>. Let (Z,A) be a measurable space, and Σ a family of probability measures on A. Let B be a sub-σ-algebra of A. An A-measurable set A is said to be Σ-almost B-measurable, if there exists a B-measurable set B such that $\mu(A\Delta B) = 0$ for every $\mu \in \Sigma$. The family $\tilde{B}_\Sigma$ of all Σ-almost B-measurable sets is a sub-σ-algebra of A. When, in particular, the family Σ consists of a single element μ, the μ-almost-ness is used for Σ-almost-ness and $\tilde{B}_\mu$ stands for $\tilde{B}_\Sigma$. Since $\tilde{B}_\mu \supset \tilde{B}_\Sigma$ for every $\mu\in\Sigma$, it holds in general that $\bar{B}_\Sigma = \bigcap_{\mu\in\Sigma}\tilde{B}_\mu \supset \tilde{B}_\Sigma$. A sub-$\sigma$-algebra B of A is said to be Σ-completable, if $\bar{B}_\Sigma = \tilde{B}_\Sigma$.

A measurable mapping $u(z)$ of (Z,A) onto a certain measurable space (U,S) is said to possess the Bahadur property, if $S = \{S \subset U: u^{-1}S \in A\}$. (See R.R. Bahadur: Sufficiency and statistical decision functions, Ann. Math. Statist. 25 (1954), page 429.) A sub-σ-algebra B of A is said to possess the Bahadur property, if there exist a measurable space (U,S) and a measurable mapping $u(z)$ of (Z,A) onto (U,S) such that $B = u^{-1}(S)$.

3. <u>Completability Theorems</u>.

A) Let Σ_o be the family of all probability measures on (Z,A).

<u>Theorem 1</u>. A sub-σ-algebra B possessing the Bahadur property is Σ_o-completable.

<u>Proof</u>. First we shall show that for any $u \in U$ and for $A \in \bar{B}_{\Sigma_o}$,

$$u^{-1}(u)\cap A\neq\emptyset \text{ and } u^{-1}(u)-A\neq\emptyset$$

are not compatible. Assume, to the contrary, that there is a $u_o \varepsilon U$ such that $u^{-1}(u_o)\cap A\neq\emptyset$ and $u^{-1}(u_o)-A\neq\emptyset$. For these two sets we can choose a measure μ in Σ_0 such that

$$\mu^*(u^{-1}(u_o)\cap A) > 0 \text{ and } \mu^*(u^{-1}(u_o)-A) > 0,$$

and a $\mathcal{B}$-measurable set B for which $\mu(A\Delta B)=0$. Then neither $u_o \varepsilon u(B)$ nor $u_o \not\varepsilon u(B)$ holds, because the former implies $0=\mu(A\Delta B)\geqq\mu(B-A)\geqq \mu^*(u^{-1}(u_o)-A)>0$ while the latter implies $0=\mu(A\Delta B)\geqq\mu(A-B)\geqq\mu^*(u^{-1}(u_o) \cap A)>0$. This is a contradiction.

Thus we have seen that every A in $\bar{\mathcal{B}}_{\Sigma_0}$ fulfils $A=u^{-1}(S)$ with $S=\{u:u^{-1}(u)\cap A\neq\emptyset\}$. In order to see $S \varepsilon \mathcal{S}$, it is sufficient to show that $A=u^{-1}(u(A))$, because $u^{-1}(u(A))=A\varepsilon\mathcal{A}$ implies $S=u(A)\varepsilon\mathcal{S}=\{S\subset U: u^{-1}(S)\varepsilon\mathcal{A}\}$ (the Bahadur Property). For every z in A, we have $u^{-1}(u(z))\subset A$ by the fact shown in the first half of the proof, since $u^{-1}(u(z))\cap A\neq\emptyset$. Therefore $u^{-1}(u(A))\subset A$, which implies that $u^{-1}(u(A))=A$, since the inverse inclusion is clear.

B) Let $(Z,\mathcal{A})$ be the direct product space of two measurable spaces $(U,\mathcal{S})$ and $(V,\mathcal{R})$. The projection $u(u,v)=u$ of Z to U is a measurable mapping possessing the Bahadur property, and hence the sub-σ-algebra $\mathcal{B}$ of $\mathcal{A}$ induced by the mapping $u(z)$ $(\mathcal{B}=u^{-1}(\mathcal{S}))$ does so. Suppose that for each point v in V there corresponds a probability measure $\mu(\cdot:v)$ on the measurable space $(U,\mathcal{S})$ which is simultaneously an $\mathcal{R}$-measurable function of v on $(V,\mathcal{R})$ for any fixed S in $\mathcal{S}$. For a subset A of Z, the subset $\{u:(u,v)\varepsilon A\}$ will be denoted by A_v. Let Ξ_0 be the family of all probability measures on $(V,\mathcal{R})$, and let Σ_1 be the family of probability measures μ on $(Z,\mathcal{A})$ of form

$$\mu(A) = \int \mu(A_v : v)\xi(dv) \quad \text{for} \quad A \in \mathcal{A}$$

with $\xi \in \Xi_o$. If A is Σ_1-almost $\mathcal{B}$-measurable, there is an R-measurable set R in V such that $\mu(A_v:v) = 0$ for $v \varepsilon R$ and $\mu(V - A_v: v) = 0$ for $v \notin R$, that is, $|\mu(A_v:v) - 1/2| = 1/2$ for every $v \in V$. Thus we have

Theorem 2. The σ-algebra $\mathcal{B}$ induced by the projection $u(u,v) = u$ ia Σ_1-completable; in other words, $\bar{\mathcal{B}}_{\Sigma_1} = \tilde{\mathcal{B}}_{\Sigma_1}$.

C) Suppose that a probability measure ξ on (U,S) is given. We shall consider the case where the conditional distribution is unknown. However we do not know in general whether there exists a probability measure on $(Z,\mathcal{A})$ whose marginal probability measure of $u(z)$ coincides with the given ξ. Several authors treated this problem under restricted circumstances. For metric spaces, see V. Strassen: The existence of probability measures with given marginals, Ann. Math. Statist. 36 (1965),423-437; and for direct product cases, see D. Bierlein: Die Konstruction eines Masses $\mu|^{\mathcal{B}}(\mathcal{K}\times\mathcal{B})$ zu vorgegebenen Marginalmass $p|\mathcal{K}$ mit $\mu(B_o)=1$ für eine vorgegebene Menge B_o, Z. Wahrscheinlichkeitstheorie 1 (1962), 126-140.

Let $\Sigma(\xi)$ be the family of probability measures on (U,S) with a given marginal ξ.

Assumption. For any $\mathcal{A}$-measurable set A in Z with $\xi^*(uA) > 0$, there corresponds a probability measure $\mu \in \Sigma(\xi)$ such that its marginal of u coincides with ξ and $\mu(A) = \xi^*(uA)$.

Theorem 3. Under the Assumption, the σ-algebra $\mathcal{B}$ induced by $u(z)$ is $\Sigma(\xi)$-completable; *i.e.*, $\tilde{\mathcal{B}}_{\Sigma(\xi)} = \bar{\mathcal{B}}_{\Sigma(\xi)}$, whenever $u(z)$ has the Bahadur property.

Proof. First we shall show that $\tilde{\mathcal{B}}_{\Sigma(\xi)}$ consists of the sets $A \in \mathcal{A}$ such that $\xi^*(uA \cap u(Z - A)) = 0$. To prove this, suppose that $A \in \tilde{\mathcal{B}}_{\Sigma(\xi)}$; *i.e.*, there is a $\mathcal{B}$-measurable set B such that $\mu(A \Delta B) = 0$ for every $\mu \in \Sigma(\xi)$. This implies that $\xi^*(u(A \Delta B)) = 0$ by Assumption. Since $(uA \cap u(Z - A)) - u(A - B) = uA \cap u(Z - A) \cap uB \subset u(B - A)$ and $u(A - B) - (uA \cap u(Z - A)) = uA - u(Z - A) - uB \subset u(A - B)$, we have $(uA \cap u(Z - A)) \Delta u(A - B) \subset u(A \Delta B)$. From this it follows that $\xi^*(uA \cap u(Z - A)) = \xi^*(u(A - B)) = 0$. Suppose conversely that $\xi^*(uA \cap u(Z - A)) = 0$. Then there is an S-measurable set $S \subset U$ such that $uA \cap u(Z - A) \subset S$ and $\xi(S) = 0$. Taking $E = u^{-1}(S) \cup A$, we have $u^{-1}(uE) = E$. In fact, $u^{-1}uE = u^{-1}(uA - S) \cup u^{-1}S \subset u^{-1}(uA - u(Z - A)) \cup u^{-1}S \subset A \cup u^{-1}S = E$ and the evident inclusion $E \subset u^{-1}uE$ holds true. Since $E \in \mathcal{A}$, we have $uE \in S = \{S \subset U: u^{-1}S \in \mathcal{A}\}$, and hence $E \in \mathcal{B}$. Therefore $A \in \tilde{\mathcal{B}}_{\Sigma(\xi)}$, because $\mu(A \Delta E) = \mu(E - A) = \mu(u^{-1}S - A) \leq \mu(u^{-1}S) = \xi(S) = 0$.

We shall proceed to prove that $\bar{\mathcal{B}}_{\Sigma(\xi)} \subset \tilde{\mathcal{B}}_{\Sigma(\xi)}$. Suppose that $A \in \mathcal{A} - \tilde{\mathcal{B}}_{\Sigma(\xi)}$. From the fact proved above, it follows that

$$\xi^*(uA \cap u(Z - A)) > 0.$$

Therefore we can choose an S-measurable S such that $uA \cap u(Z - A) \subset S$ and $\xi(S) = \xi^*(uA \cap u(Z - A))$. Since $u^{-1}S \cap A \in \mathcal{A}$ and $u(u^{-1}S \cap A) = S \cap uA$, it follows from Assumption that there exist two measures μ and ν such that their marginal distributions of u coincide with ξ and that $\mu(u^{-1}S \cap A) = \xi^*(S \cap uA)$ and $\nu(u^{-1}S - A) = \xi^*(S \cap u(Z - A))$. Moreover, we have $\xi^*(S \cap uA) = \xi(S)$; in fact, $uA \cap u(Z - A) = S \cap uA \cap u(Z - A) \subset S \cap uA \subset S$ and so $\xi(S) = \xi^*(uA \cap u(Z - A)) \leq \xi^*(S \cap uA) \leq \xi(S)$; that is, $\xi^*(S \cap uA) = \xi(S)$. This shows that $\mu(u^{-1}S - A) = \mu(u^{-1}S) - \mu(A \cap u^{-1}S) = \xi(S) - \xi(S) = 0$. Hence we have, for an arbitrary $\mathcal{B}$-measurable B,

$$\mu(u^{-1}S \cap A - B) = \mu(u^{-1}S - B) = \xi(S - uB).$$

Similarly, we have

$$\nu(u^{-1}S \cap B - A) = \nu(u^{-1}S \cap B) = \xi(S \cap uB).$$

Putting $\lambda = (\mu + \nu)/2$, we have

$$\lambda(u^{-1}S \cap (A \Delta B)) \geqq \mu(u^{-1}S \cap A - B) + \nu(u^{-1}S \cap B - A)$$
$$= \xi(S - uB) + \xi(S \cap uB) = \xi(S) > 0.$$

Thus we have $\lambda(A \Delta B) \geqq \lambda(u^{-1}S \cap (A \Delta B)) > 0$, which shows A can not be in $\bar{\mathcal{B}}_\lambda$.

Remark. The Bahadur property of $\mathcal{B}$ cannot be removed from Theorem 1. For example, let A be the σ-algebra of analytic sets in [0,1], and $\mathcal{B}$ the σ-algebra of all Borel sets in [0,1]. Then $\bar{\mathcal{B}}_{\Sigma_o} = A$, while $\tilde{\mathcal{B}}_{\Sigma_o} = \mathcal{B}$.