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A note on the Dugundji extension property

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For any topological space X, let $C^*(X)$ be the vector space of continuous, bounded, real-valued functions on X, equipped with the sup-norm. Let us say that a space X has the Dugundji extension property, if for any closed subset A of X there is a linear transformation $\eta : C^*(A) \to C^*(X)$ such that for each $f \in C^*(A)$, $\eta(f)$ extends f and $\sup\{|\eta(f)(x)| : x \in X\} \leq \sup\{|f(a)| : a \in A\}$.

It is known that every metric space has the Dugundji extension property (cf. [3], [4]) and every stratifiable space also has such property (cf. [1]). On the other hand, E. van Douwen [2] constructed a first countable cosmic space H_1 which did not have that property.

For any space Y, let τY be the topology of Y. A space X is said to be a K_1-space, if for every subspace F of X there exists a function $\kappa : \tau F \to \tau X$ such that

(1) $F \cap \kappa(U) = U$ for every $U \in \tau F$,

(2) if $U \cap V = \phi$ in τF, then $\kappa(U) \cap \kappa(V) = \phi$.

It has shown that every space with the Dugundji extension property is always a K_1-space (cf. [2]), and van Douwen [2] also proved that the above space H_1 was not a K_1-space.

The space H_1 is expressed as the union of two subspaces F and C of X such that F is stratifiable as a subspace of X and that each point of C is isolated in X. Modeling the space H_1, the following result is obtained :

Theorem 1. Let X be a topological space and A a closed, G_δ-subset of X such that A is stratifiable as the subspace of X and $X \setminus A$ is a discrete subspace of X. Then the following conditions are equivalent :

(1) X is stratifiable.

(2) X has the Dugundji extension property.

(3) X is a K_1-space.

For a space X and a closed subset A of X, let us say that (X,A) is a semi-canonical pair, if there is an open cover $\mathcal{V}$ of $X \setminus A$ such that, for each point $a \in A$ and any neighborhood U of a in X, there exists a neighborhood W of a in X provided with $\bigcup\{V \in \mathcal{V} : W \cap V \neq \phi\} \subset U$.

Relating with the semi-canonical pair, the following result is obtained and it seems to suggest that the condition being semi-canonical pair works on a half part in the extension theorem :

Theorem 2. Let X be a paracompact σ- and K_1-space, and let A be a closed subset of X such that A is stratifiable as the subspace of X. Then there exists a linear transformation $\eta : C^*(A) \longrightarrow C^*(X)$ which satisfies all conditions required in the Dugundji extension property.

References

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