数理解析研究所講究録
332巻 1978年 231-238

Nonlinear Dispersive Wave with Weak Dissipation

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Approximate one soliton solution is obtained for Korteweg-de Vries equation and nonlinear Schrödinger equation with a small dissipation term by means of modified conseıvation law.

§1. Introduction

Solution of a nonlinear evolution equation such as Kortweg-de Vries equation can be expressed by solitons of different amplitudes and by a wave train —— a tail. The contribution of the tail is usually small and is neglected. Then one obtain soliton solution by means of an inverse scattering method.

If a dissipation term is added in such an evolution equation, one has no longer stationary soliton solution. In this case, non-soliton part solution plays an important role and is not neglected. A perturbation theory of the inverse method[1] and the multi-time expansion method[2] give a non-stationary soliton but do not give any information on the property of a tail. In this report, we present approximate one soliton solution and simple property of a tail for K-dV equation and nonlinear Schrödinger equation with a small dissipation term.

§2. K-dV equation with dissipation

We consider the following two K-dV equations with dissipation :

$$u_t + uu_x + \frac{1}{2}u_{xxx} - C_1 u_{xx} = 0 \quad , \tag{1}$$

$$u_t + uu_x + \frac{1}{2}u_{xxx} + C_2 u = 0 \quad . \tag{2}$$

The first equation is K-dV-Burgers equation and the second, K-dV equation for an ion acoustic wave in plasma with ion-neutral collision[2].

It is well known that the K-dV equation has infinite number of conserved quantities. The first three ones among them are

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} u dx \quad , \\ I_2 &= \int_{-\infty}^{\infty} \frac{1}{2}u^2 dx \quad , \\ I_3 &= \int_{-\infty}^{\infty} \{ \frac{1}{3}u^3 - \frac{1}{2}(u_x)^2 \} dx \quad . \end{aligned} \tag{3}$$

Apart from these conserved quantities, we have another type of conserved one[3] ;

$$I_0 = \int_{-\infty}^{\infty} \{ xu - \frac{1}{2}tu^2 \} dx \quad .$$

Differentiating this equation with respect to the time, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu\, dx = \int_{-\infty}^{\infty} \frac{1}{2}u^2 dx \quad . \tag{4}$$

As the right hand side of eq.(4) is nothing but the second conserved quantity I_2, the left hand side of eq.(4) is conserved.

The modified conservation laws of the K-dV-Burgers equation are

$$\frac{d}{dt} I_1 = 0 \quad , \tag{5}$$

$$\frac{d}{dt} I_2 = -C_1 \int_{-\infty}^{\infty} (u_x)^2 dx \quad , \tag{6}$$

$$\frac{d}{dt} I_3 = C_1 \int_{-\infty}^{\infty} \{ u^2 u_{xx} + (u_{xx})^2 \} dx \quad , \tag{7}$$

and so on. The eq.(4) also holds, but is no longer time independent.

If the dissipation is weak, the solution of eq.(1) can be written as

$$u(x,t) = u^{s.}(x,t) + u^{n.s.}(x,t) \quad , \tag{8}$$

where $u^{s.}$ and $u^{n.s.}$ are soliton and non-soliton parts of the solution. The order of magnitude of $u^{s.}$ and $u^{n.s.}$ is assumed to be $O(1)$ and $O(\varepsilon)$, respectively, where ε is related to the dissipation coefficient C_1 and will be determined from the result. As a soliton part solution, we employ

$$u^{s.}(x,t) = a\ \mathrm{sech}^2\{(a/6)^{1/2}(x-\xi)\} \quad , \tag{9}$$

where the amplitude a and phase ξ change in time.

In seeking the solution, we make use of the following two assumptions. (a) Terms of order ε^2, ε^3 ... are neglected. That is, we do not retain terms of $(u^{n.s.})^2$, $(u^{n.s.})^3$ and so on. (b) The spatial overlapping between $u^{s.}$ and $u^{n.s.}$ is assumed to be small and is neglected; $u^s u^{n.s.} \simeq 0$.

From these assumptions, we see that eqs.(6) and (7) contain only the soliton part solution. Thus we obtain from either of these equations the damping of soliton amplitude;

$$a(t) = a(0)\ \left(1+\frac{8C_1 a(0)t}{45}\right)^{-1} \quad , \tag{10}$$

where a(0) is the amplitude of soliton at t=0. Substituting eqs.(8) and (10) into eq.(5), we obtain

$$\int_{-\infty}^{\infty} u^{n.s.}\ dx = \{24a(0)\}^{1/2}\{1-\left(1+\frac{8C_1 a(0)t}{45}\right)^{-1/2}\} \geq 0 \quad . \tag{11}$$

As to the phase ξ, we have the following equation from eqs.(4) and (10);

$$\frac{d\xi}{dt} + \frac{1}{2a}\frac{da}{dt}\xi = \frac{a}{3} - \{24a(t)\}^{-1/2}\frac{d}{dt}\int_{-\infty}^{\infty} xu^{n.s.}\ dx \quad . \tag{12}$$

Although the second term on the right hand side of this equation is of order ε, we drop this term and solve eq.(12). Physically this term expresses the phase shift of soliton by emitting the tail. The solution of eq.(12) is

$$\xi = \{ \phi + \frac{45}{24C_1} - \frac{45}{24C_1}\frac{a(t)}{a(0)} \}\{ \frac{a(0)}{a(t)} \}^{1/2} \quad , \tag{13}$$

where ϕ is the position of soliton at t=0.

From eqs.(10), (11) and (13), we recognize that ε should be taken as $8C_1a(0)t/45$. Our solution holds only when $\varepsilon \ll 1$; that is, $t \ll 45/8C_1a(0)$. Expanding eq.(11) for small ε, we see the non-soliton part solution to be $O(\varepsilon)$.

The modified conservation laws for eq.(2) are given as

$$\frac{d}{dt} I_1 = - C_2 I_1 \quad , \tag{14}$$

$$\frac{d}{dt} I_2 = - 2C_2 I_2 \quad , \tag{15}$$

$$\frac{d}{dt} I_3 = - C_2 \int_{-\infty}^{\infty} \{u^3-(u_x)^2\}\, dx \quad , \tag{16}$$

......

and

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu\, dx - C_2 \int_{-\infty}^{\infty} xu\, dx = \int_{-\infty}^{\infty} \frac{1}{2} u^2 dx \quad . \tag{17}$$

The method of obtaining the solution is the same as that of K-dV-Burgers equation. The results are as follows:

$$a(t) = a(0)\exp(-4C_2t/3) \quad , \tag{18}$$

$$\int_{-\infty}^{\infty} u^{n.s.} dx = \{ 24a(0)\}^{1/2}\exp(-C_2t)\{ 1-\exp(C_2t/3)\} \leq 0 \quad , \tag{19}$$

$$\xi = \{ \phi + \frac{a(0)}{3C_2} \}\exp(-C_2t/3) - \frac{a(t)}{3C_2} \quad . \tag{20}$$

In this case, ε is equal to $\sim C_2t$. Therefor eqs.(18)~(20) hold for $t \ll 1/C_2$. The eq.(19) shows that the non-soliton part solution is proportional to ε for small ε.

Numerical solutions of eqs.(1) and (2) confirm the validity of our solutions on soliton and tail[4].

§3. Nonlinear Schrödinger equation

The following nonlinear Schrödinger equations[5,6] are considered in this section:

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u + i\varepsilon u = 0 \quad , \tag{21}$$

$$iu_t + \frac{1}{2}u_{xx} + |u|^2u + \varepsilon\, P\int_{-\infty}^{\infty}\frac{|u(x',t)|^2}{x-x'}\,dx'u = 0 \quad . \tag{22}$$

For ε=0, eq.(21) has infinite number of conserved quantities. The first three equations are

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty}|u|^2dx \quad , \\ I_2 &= \frac{1}{2i}\int_{-\infty}^{\infty}(u^*u_x-u_x{}^*u)\,dx \quad , \\ I_3 &= \int_{-\infty}^{\infty}(|u_x|^2-|u|^4)\,dx \quad . \end{aligned} \tag{23}$$

In addition to these quantity we have

$$I_0 = \int_{-\infty}^{\infty}\{x|u|^2 - \frac{t}{2i}(u^*u_x-uu_x{}^*)\}\,dx \quad .$$

Differentiation of this equation with respect to the time gives

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty}x|u|^2dx &= \frac{1}{2i}\int_{-\infty}^{\infty}(u^*u_x-uu_x{}^*)\,dx \quad . \\ &= \text{const.} \end{aligned} \tag{24}$$

In the case of eq.(21), the modified conservation laws are given as

$$\frac{d}{dt} I_1 = -2\varepsilon I_1 ,$$

$$\frac{d}{dt} I_2 = -2\varepsilon I_2 , \tag{25}$$

$$\frac{d}{dt} I_3 = -2\varepsilon \int_{-\infty}^{\infty} (|u_x|^2 - 2|u|^4)dx ,$$

and eq.(24) is modified as

$$\frac{d}{dt} \int_{-\infty}^{\infty} x|u|^2 dx = \frac{1}{2i} \int_{-\infty}^{\infty} (u_x u^* - u_x^* u)dx - 2\varepsilon \int_{-\infty}^{\infty} x|u|^2 dx . \tag{26}$$

Solution of eq.(20) is assumed in the form:

$$u = u^{s.} + u^{n.s.} , \tag{27}$$

when $u^{s.}$ and $u^{n.s.}$ express the soliton and non-soliton parts of the solution, and is assumed to be $O(1)$ and $O(\varepsilon)$ respectively. As well as the K-dV equation in §2, we neglect the spatial overlapping between $u^{s.}$ and $u^{n.s.}$ and look for the solution of order 1 and ε. As a soliton part solution, $u^{s.}$, we employ

$$u = A \operatorname{sech} \alpha \, e^{i\beta} , \tag{28}$$

where

$$\alpha = A(x - vt - \phi) ,$$
$$\beta = vx - \frac{1}{2}(v^2 - A^2)t + \theta .$$

If $\varepsilon=0$, A, v, ϕ and θ in eq.(28) are time independent, but they change in the time for eqs.(20) and (21). We note in eqs.(25) and (26) that all modified conservation laws are expressed in terms of the quantities of $O(1)$ and $O(\varepsilon^2)$ ...., if the overlapping between $u^{s.}$ and $u^{n.s.}$ is neglected. This means that any pair among eq.(25) gives the same result in determining the velocity and amplitude of a soliton, eq.(28). The solution is

$$\frac{d}{dt} A = -2\varepsilon A \quad ,$$
$$\frac{d}{dt} v = 0 \quad . \tag{29}$$

From eq.(26), we also have

$$\frac{d}{dt} \phi = 0 \quad .$$

That is, the velocity of a soliton is constant, but the amplitude decreases as exp(-2εt).

Concerning eq.(21), the modified conservation laws are

$$\frac{d}{dt} I_1 = 0 \quad ,$$

$$\frac{d}{dt} I_2 = -\varepsilon P \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|u(x',t)|^2}{x-x'} (u^* u_x + u_x^* u) dx' dx \quad ,$$

$$\frac{d}{dt} I_3 = i \varepsilon P \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|u(x',t)|^2}{x-x'} (U_{xx}^* u - u^* u_{xx}) dx' dx \quad ,$$

and so on. For eq.(21), eq.(24) holds but is no longer time independent.

The solution of eq.(21) cam be obtained in the same way as eq.(20);

$$\frac{dA}{dt} = 0 \quad ,$$
$$\frac{dv}{dt} = \varepsilon I A^3 \quad , \tag{30}$$
$$\frac{d\phi}{dt} = -\varepsilon I A^3 t \quad ,$$

where

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\operatorname{sech}^2 x'}{x-x'} \operatorname{sech}^2 x \tanh x \, dx' dx$$
$$\simeq 1.462 \quad .$$

The amplitude of a soliton does not change, but the trajectory changes as $\frac{1}{2}\varepsilon IA^3t^2$. The soliton accelerates in time without changing the amplitude.

§4. Discussion

For the K-dV equation with dissipation, the damping of a soliton obtained in §2 agrees with the results of multi-time expansion method and of the perturbation theory of inverse method. The phase ξ, however, is different from the result of Karpman and Maslov;

$$\xi = \phi + \frac{45}{24C_1} \ln\left(1 + \frac{8C_1 a(0)t}{45} \right) \quad . \tag{31}$$

This result can be obtained when the second term in the left hand side of eq.(12) is neglected as well as the second term in the right hand side. The neglected terms is easily seen to be small. In fact if we expand eqs.(13) and (31) by ε, they agree with each other up to $O(\varepsilon^2)$ exactly.

Concerning the nonlinear Schrödinger equation with dissipation, we can not obtain time dependence of phase θ from the conservation laws. We do not know at present the conservation law which describes the phase θ. The time dependence of θ, however, can be obtained in the following way. If we substitute eq.(27) into eq.(20) or (21), we obtain, after linearizing with respect to $u^{n.s.}$,

$$iu_t^{n.s.} + u_{xx}^{n.s.} = S(u^{s.}) \quad . \tag{32}$$

The right hand side of eq.(31) represents a source term in terms of $u^{s.}$. The source term generaly has a secular term. One can remove this secularity by choosing a proper time dependence of θ.

It is easily shown that the source term $S(u^{s.})$ is of order ε. That is, the non-soliton part $u^{n.s.}$ is of order ε. Therefor, the assumption employed in §3 is justified.

References

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