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Solitons and Rational Solutions of Nonlinear Evolution Equations*

Junkichi SATSUMA (Kyoto University)

1. Introduction

It is well known that certain nonlinear evolution equations have N-soliton solutions. Recently the apparently quite different class of rational solutions was investigated by Airault, Mckean and Moser.[1)]

In this lecture, I would like to talk about the close relationship between the two types of solutions; the rational solutions can be recovered by taking a long wave limit of the soliton solutions obtained by Hirota's method. For the Korteweg-deVries (KdV) equation, I would demonstrate the results for the first few soliton solutions and then show how performing the above limiting procedure on the Bäcklund transformation of the KdV equation yields a recursion relation capable of generating the full class of rational solutions.

The method we employ can be readily adapted to the other nonlinear evolution equations that possess soliton solutions. In the case of the modified KdV equation, the nonsingular, rational solution presented by Ono is recovered from one-soliton solution, and an algebraic solution with strange behavior is obtained from two-soliton solution.

The most interesting example is that for the multi-dimensional problem. For the Kadomtsev-Petviashvili (two-dimensional KdV)

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*) This is a cooperative work with M.J.Ablowitz (Clarkson College of Technology).

equation, the long wave limit of a two soliton state gives a mode which decays algebraically in all directions (we refer to such a mode as a lump). A solution which describes a collision of two lumps is obtained from 4-soliton solution. The two-lump yields zero asymtotic phase shift after interaction. In general, a 2M-soliton solution gives an M-lump solution. Another example is the two-dimensional nonlinear Schrödinger (2DNLS) equation. In the case, two-dimensional envelope hole solutions are obtained.

2. Rational solution of KdV equation

The KdV equation,

$$u_t + 6uu_x + u_{xxx} = 0, \tag{2.1}$$

is transformed into

$$D_x(D_t + D_x^3) f\cdot f = 0, \tag{2.2}$$

through a variable transformation,

$$u = 2(\log f)_{xx}, \tag{2.3}$$

where we assume $u \to 0$ as $|x| \to \infty$ and use a special type of differential operators,

$$D_x^m D_t^n a\cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,t)\, b(x',t')\Big|_{x=x',\, t=t'}. \tag{2.4}$$

Hirota[2] obtained from (2.3) an N-soliton solution describing a multiple collision of solitons. The solution is given by

$$f \equiv f_N = \sum_{\underset{\sim}{\mu}=0,1} \exp\left(\sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \eta_i\mu_i\right), \tag{2.5}$$

$$\eta_i = k_i x - k_i^3 t + \eta_i^{(0)}, \tag{2.6}$$

$$\exp A_{ij} = (k_i - k_j)^2 / (k_i + k_j)^2, \tag{2.7}$$

where $\eta_i^{(0)}$ are arbitrary phase constants. The first two of (2.5) are written as

$$f_1 = 1 + e^{\eta_1}, \tag{2.8}$$

$$f_2 = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}} . \tag{2.9}$$

The one-soliton solution (2.8) is expressed in terms of u as

$$u = \frac{k_1^2}{2} \operatorname{sech}^2 \frac{1}{2}(k_1 x - k_1^3 t + \eta_1^{(0)}) . \tag{2.10}$$

We may choose the phase constant $\eta_1^{(0)}$ as $e^{\eta_1^{(0)}} = -1$ and then (2.10) is transformed into the singular soliton,

$$u = -\frac{k_1^2}{2} \operatorname{cosech}^2 \frac{1}{2}(k_1 x - k_1^3 t) . \tag{2.11}$$

If we take the limit $k_1 \to 0$ (i.e. the "long wave" limit) in (2.11) we obtain a rational solution,

$$u \to -2/x^2 \tag{2.12}$$

It is significant that by choosing the phase constants appropriately, all the f_N described above, have nontrivial, distinguished limits. In what follows it is easiest to develop the ideas for f_N.

In order to best explain the above results we return to (2.5), which includes (2.8) and (2.9) as the simple cases. Letting $\alpha_1 = e^{\eta_1^{(0)}}$, (2.8) is written as

$$f_1 = 1 + \alpha_1 e^{\xi_1} , \tag{2.13}$$

$$\xi_i = k_i (x - k_i^2 t) . \tag{2.14}$$

As $k_1 \to 0$, we have

$$f_1 = 1 + \alpha_1 (1 + \xi_1) + O(k_1^2) . \tag{2.15}$$

If we take $\alpha_1 = -1$, then

$$f_1 = -k_1\,(x + O(k_1))$$

$$\doteq \; x \equiv \theta_1 \,, \qquad (2.16)$$

where we define $f \doteq g$ iff $f = e^{ax+b}g$ with a,b independent of x. It is easily seen that (2.16) gives the same u as (2.12).

The same idea applies to f_2. From (2.9), we have

$$f_2 = 1 + \alpha_1 e^{\xi_1} + \alpha_2 e^{\xi_2} + \alpha_1\alpha_2 e^{\xi_1+\xi_2+A_{12}} . \qquad (2.17)$$

Choosing $\alpha_1 = -\alpha_2 = (k_1+k_2)/(k_1-k_2)$ in (2.17) and taking a limit k_1, $k_2 \to 0$ with $k_1/k_2 = O(1)$, we obtain

$$f_2 \to -\tfrac{1}{6} k_1 k_2 (k_1+k_2)\,[x^3 + 12t + O(\underset{\sim}{k})]$$

$$\doteq \; \theta_2 = x^3 + 12t \,, \qquad (2.18)$$

which gives a rational solution having three poles. Similarly, from 3-soliton solution, we have

$$f_3 \to \theta_3 = x^6 + 60x^3t - 720t^2 \qquad (2.19)$$

In principle this technique applies to any number of solitons. However the calculations are tedious, and we shall instead use the Bäcklund transformations for the KdV equation to generate a recursion relation for the polynomials.

The Bäcklund transformation of the KdV equation is given by[3)]

$$(D_x^2 - k_{N+1}^2)\, f_N \cdot f_{N+1} = 0 \,, \qquad (2.20)$$

$$(D_t + 3k_{N+1}^2 D_x + D_x^3)\, f_N \cdot f_{N+1} = 0 \,. \qquad (2.21)$$

Equations (2.20) and (2.21) yield an (N+1)-soliton solution $f_{N+1} = f_{N+1}(k_1,\cdots,k_N,k_{N+1})$, from an N-soliton solution $f_N = f_N(k_1,\cdots,k_N)$. Furthermore, from (2.20) we have a superposition formula of four soliton solutions,

$$D_x f_{N-1} \cdot f_{N+1} \propto f_N f_N' , \tag{2.22}$$

where $f_N' = f_N(k_1,\cdots,k_{N-1},k_{N+1})$ and $f_{N-1} = f_{N-1}(k_1,\cdots,k_{N-1})$. By taking a limit of $k_i \to 0$, we have $f_N \to \theta_N$ and (2.20) and (2.21) are reduced to

$$D_x^2\, \theta_N \cdot \theta_{N+1} = 0 , \tag{2.20'}$$

$$(D_t + D_x^3)\, \theta_N \cdot \theta_{N+1} = 0 , \tag{2.21'}$$

respectively. Moreover, noticing $\theta_N = x^{N(N+1)/2} + \cdots$, we obtain from (2.22),

$$D_x\, \theta_{N+1} \cdot \theta_{N-1} = (2N+1)\, \theta_N^2 . \tag{2.22'}$$

Equation (2.22') is equivalent to the recursion relation of rational solutions of KdV equation discovered by Adler and Moser[4). We can get higher order rational solutions by using (2.22') and (2.21').

3. Rational solutions of modified KdV equation

We consider the modified KdV equation,

$$v_t + 6v^2 v_x + v_{xxx} = 0 , \tag{3.1}$$

under the boundary condition $v \to v_0$ as $|x| \to \infty$. By transforming the dependent variable in (3.1) as

$$v = v_0 + i\,(\log f^*/f)_x , \tag{3.2}$$

we find that v is a solution of (3.1) if f satisfies a couple of bilinear equations,

$$(D_t + 6V_0^2 D_x + D_x^3)\, f^*\cdot f = 0 \tag{3.3}$$

$$(D_x^2 - 2iV_0 D_x)\, f^*\cdot f = 0 \tag{3.4}$$

where asterisk denotes complex conjugate.

A one-soliton solution of (3.3) and (3.4) is given by

$$f_1 = 1 + e^{\eta_1+\phi_1}, \tag{3.5}$$

$$\eta_i = k_i x - (6V_0^2 k_i + k_i^3)t + \eta_i^{(0)}, \tag{3.6}$$

$$e^{\phi_j} = 1 + ik_j/2V_0 . \tag{3.7}$$

Inserting (3.5) into (3.2), we have an explicit form of the one-soliton solution,

$$V = V_0 + k_1^2 / (\sqrt{4V_0^2+k_1^2}\cosh\eta_1 + 2V_0). \tag{3.8}$$

If we choose the phase constant as $e^{\eta_1^{(0)}} = -1$ and take the limit of $k_1 \to 0$ in (3.8), we obtain a rational solution,

$$V \to V_0 - 4V_0 / [4V_0^2(x-6V_0^2 t)^2 + 1]. \tag{3.9}$$

This solution was recently found by Ono[5] and is a nonsingular algebraic solution.

The two-soliton solution is given by

$$f_2 = 1 + e^{\eta_1+\phi_1} + e^{\eta_2+\phi_2} + e^{\eta_1+\eta_2+\phi_1+\phi_2+A_{12}}, \tag{3.10}$$

$$e^{A_{12}} = (k_1-k_2)^2/(k_1+k_2)^2 . \tag{3.11}$$

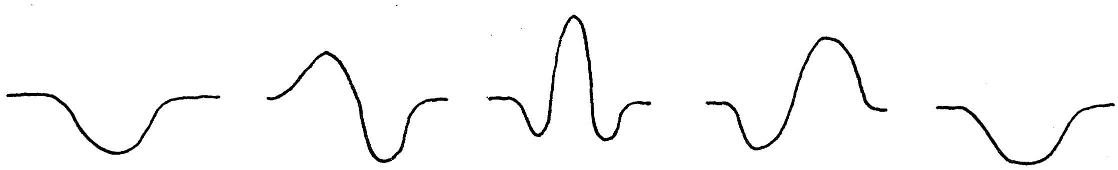
Choosing the phase constants as

$$e^{\eta_1^{(0)}} = -e^{\eta_2^{(0)}} = \frac{k_1+k_2}{k_1-k_2}\left(1+\frac{k_1k_2}{8v_0^2}\right), \qquad (3.12)$$

and taking the limit $k_i \to 0$ ($k_1/k_2 = O(1)$), we have the following rational solution;

$$v = v_0 - \frac{12v_0\left(\xi^4 + \frac{3}{2v_0^2}\xi^2 - \frac{3}{16v_0^4} - 24\xi t\right)}{4v_0^2\left(\xi^3 + 12t - \frac{3}{4v_0^2}\xi\right)^2 + 3\left(\xi^2 + \frac{1}{4v_0^2}\right)^2}, \qquad (3.13)$$

where $\xi = x - 6v_0^2 t$. This solution is not stationary and, instead, has a strange behavior. A rough sketch of time development of the solution (3.13) is shown in fig.1; It pulsates in the neighborhood of t=0 and goes to the stationary solution (3.9) as $t \to \pm\infty$.



$t=-\infty$ $\qquad$ $t=0$ $\qquad$ $t=\infty$

fig. 1

4. Multi-dimensional systems

4-1 Kadomtsev-Petviashvili(K-P) equation

The technique developed in § 2 and 3 may be extended to multi-dimensional systems. As one of the example, we consider the K-P equation,

$$(u_t + 6uu_x + u_{xxx})_x + \alpha u_{yy} = 0 \qquad (4.1)$$

where α is a constant depending on the dispersive property of the system. It was already shown that (4.1) admits an N-soliton solution describing a multiple collision of solitons each of which has a different direction of propagation.[6] Here we show that a two-dimensional lump solution which decays algebraically in all directions can be obtained by taking the long wave limit of a two soliton state.

Equation (4.1) is transformed into

$$(D_x D_t + D_x^4 + \alpha D_y^2)\, f \cdot f = 0\,, \tag{4.2}$$

through the variable transformation (2.3), where the boundary condition u → 0 as $|x| \to \infty$ is taken. The one- and two-soliton solutions of (4.2) are given by

$$f_1 = 1 + e^{\eta_1}, \tag{4.3}$$

$$f_2 = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}, \tag{4.4}$$

where

$$\eta_i = k_i[x + p_i y - (k_i^2 + \alpha p_i^2)t] + \eta_i^{(0)}, \tag{4.5}$$

$$\exp A_{12} = \{3(k_1-k_2)^2 - \alpha(p_1-p_2)^2\} / \{3(k_1+k_2)^2 - \alpha(p_1-p_2)^2\}. \tag{4.6}$$

Taking $e^{\eta_i^{(0)}} = -1$ and $k_i \to 0$ (with $p_i = O(1)$, $k_1/k_2 = O(1)$), we find

$$f_1 \to \theta_1\,, \tag{4.7}$$

$$f_2 \to \theta_1\theta_2 + B_{12}\,, \tag{4.8}$$

where

$$\theta_i = x + p_i y - \alpha p_i^2 t\,, \tag{4.9}$$

$$B_{ij} = 12/\alpha(p_i - p_j)^2, \tag{4.10}$$

and we have used

$$\exp A_{12} \sim 1 + 12 k_1 k_2 / \alpha(p_1-p_2)^2. \tag{4.11}$$

Although f_1, f_2 are generally singular at some position, a nonsingular solution is obtained for f_2 by taking $\alpha = -1$ (in which case the wave system has so called positive dispersion) and $p_2 = p_1^*$. In this case we have

$$f_2 = \theta_1\theta_1^* - 12/(p_1-p_1^*)^2 \quad > 0 , \tag{4.12}$$

which gives a nonsingular rational solution,

$$u = \frac{4[-(x'+p_R y')^2 + p_I^2 y'^2 + \frac{3}{p_I^2}]}{[(x'+p_R y')^2 + p_I^2 y'^2 + \frac{3}{p_I^2}]^2} , \tag{4.13}$$

where we have used

$$p_1 = p_R + i p_I ,$$

$$x' = x - (p_R^2 + p_I^2) t ,$$

$$y' = y + 2 p_R t .$$

Hence we have a permanent lump solution decaying as $O(1/x^2, 1/y^2)$ for $|x|$, $|y| \to \infty$. This solution is essentially the same as that shown by Bordag et al.[7] We note that when N = 4 one may obtain a two-lump solution which leave zero asymptotic phase shift after interaction. In general when N = 2M this method yields formulae for an M-lump solution.

4.2. Two-dimensional nonlinear Schrödinger (2DNLS) equation

As another example of multi-dimensional system, we consider 2DNLS equation,

$$i A_t - \sigma A_{xx} + A_{yy} = A|A|^2 + 2\sigma Q A , \tag{4.14}$$

$$\sigma Q_{xx} + Q_{yy} = -(|A|^2)_{xx} , \tag{4.15}$$

$$(\sigma = \pm 1)$$

Imposing a boundary condition $|A|^2 \to \rho_0^2$ as $|x| \to \infty$ and transforming the dependent variables in (4.14) and (4.15) as

$$A = g/f \,, \tag{4.16}$$

$$Q = 2(\log f)_{xx} \,, \tag{4.17}$$

f; real,

we have a couple of bilinear equations

$$(iD_t - \sigma D_x^2 + D_y^2 - \rho_0^2)\, g \cdot f = 0 \,, \tag{4.18}$$

$$(\sigma D_x^2 + D_y^2 - \rho_0^2)\, f \cdot f = -g g^* . \tag{4.19}$$

We can derive N-soliton solution from (4.18) and (4.19). By taking the same procedure as in §4.1, we obtain from a two-soliton state the following rational solution;

$$f = \theta_1 \theta_2 + \alpha_{12} \,, \tag{4.20}$$

$$g = \rho_0 e^{i(kx+ly-\omega t)} [(\theta_1 + 2iA_1)(\theta_2 + 2iA_2) + \alpha_{12}] \,, \tag{4.21}$$

where

$$\omega = -\sigma k^2 + l^2 + \rho_0^2 \,, \tag{4.22}$$

$$\theta_i = x + P_i y - [-2\sigma k + 2l P_i - \frac{P_i^2 - \sigma}{A_i}] t \,, \tag{4.23}$$

$$A_i = \sqrt{(\sigma + P_i^2)/2\rho_0^2} \,, \tag{4.24}$$

$$\alpha_{12} = \frac{(\sigma + P_1^2)(\sigma + P_2^2)}{\rho_0^2 [\sqrt{\sigma + P_1^2}\sqrt{\sigma + P_2^2} - (\sigma + P_1 P_2)]} . \tag{4.25}$$

If we choose $p_2 = p_1^*$ and $\sigma = -1$, then we have $A_2 = A_1^*$ and $\alpha_{12} > 0$, and (4.20) and (4.21) gives an envelope lump solution

$$|A|^2 = \rho_0^2 \left(1 + 4 \frac{(A_1 \theta_1^* + A_1^* \theta_1)^2}{(\theta_1 \theta_1^* + \alpha_{12})^2}\right) .$$

The solution describing a collision of these lumps can be also obtained from the soliton solutions.

References

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