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Solitary Waves on a Two-layer Fluid

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§1 Introduction and Summary

It has been known since the days of Stokes that there may exist two possible modes of gravity waves on a two-layer fluid;[1)] which we call the fast and slow modes according to the magnitude of their phase velocities. On the other hand, it is also well known that weakly nonlinear long gravity waves on a single fluid layer are governed approximately by the Korteweg-de Vries (shortly K-dV) equation[2)] which provides one of the prototype equations for nonlinear evolutional systems.

In this paper, we consider weakly nonlinear long gravity waves, which take place on a stably stratified two-layer fluid, and ask how the classical K-dV equation for waves on a single layer is modified by the presence of another fluid layer.

By using the reductive perturbation method;[3)] it is found that the fast mode is always governed by a K-dV equation whose coefficients depend on the thickness ratio m ($= h_0/H_0$) and the density ratio σ ($= \rho_2/\rho_1 < 1$); H_0 and h_0 being the undisturbed thickness of the lower and upper fluid layers while ρ_1 and ρ_2 the densities of the lower and upper layers, respectively. It should be remarked that the surface wave and interface wave are always in phase and the qualitative nature of the solution is similar to that for the classical gravity waves on a single layer. In particular, if we take the limit $\sigma \to 1$, we recover the classical K-dV equation for a single layer.[2] As a by-product we can easily determine the internal level elevation at any depth of the fluid layer.

On the other hand, the slow mode shows curious behaviours. It is found that the slow mode is also governed generally by another K-dV equation except for the critical thickness ratio given by $m = m^*(\sigma)$ and that the surface wave and interface wave are always 180° out of phase. At the critical thickness ratio, however, the coefficient of the nonlinear term vanishes and the K-dV equation ceases to describe the balance between the nonlinearity and the dispersion. Using a modified reductive perturbation method similar to that for the nonlinear Alfvén waves,[4] we can show that the slow mode is governed by a modified K-dV equation with cubic nonlinearity. It can also be shown that near the critical thickness ratio, the slow mode is described by an equation of a combined form of the K-dV and modified K-dV equation.

Steady solitary wave solutions are mainly considered in this paper. It is interesting to note, however, that this equation has a shock like steady solution in addition to the usual solitary wave and periodic wave train solutions. This shock like solution may be regarded as a sort of dispersive (lossless) bore or hydraulic jump (if we move with the wave velocity).

§2 Basic System of Equations

We consider the gravity waves which take place on a stably stratified two-layer fluid as shown in Fig.1. We assume that the fluids in both layers are inviscid and incompressible and that the flow field due to the wave motion remains irrotational, so that we may introduce velocity potentials ϕ and Φ for the upper and lower layers, respectively.

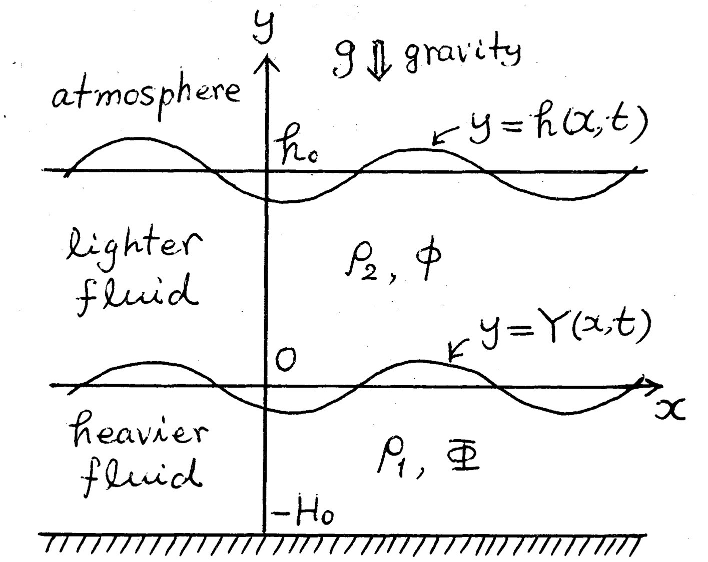

Fig. 1
Geometrical configuration of the problem

Let the elevations of the free surface and the interface be denoted by $y=h(x,t)$ and $y=Y(x,t)$, respectively, then we may obtain the following system of equations,

for the upper layer:

$$\phi_{xx}+\phi_{yy}=0 \quad \text{for } Y<y<h, \ -\infty<x<\infty, \tag{2.1}$$

$$\left.\begin{aligned} &\phi_y = h_t+\phi_x h_x \qquad (2.2)\\ &\phi_t+\tfrac{1}{2}(\phi_x^2+\phi_y^2)+(h-m)=0 \qquad (2.3)\end{aligned}\right\} \text{at } y=h(x,t),$$

for the lower layer:

$$\Phi_{xx}+\Phi_{yy}=0 \quad \text{for } -1<y<Y, -\infty<x<\infty, \tag{2.4}$$

$$\Phi_y=0 \quad \text{at} \quad y=-1, \tag{2.5}$$

and the matching conditions at the interface:

$$\left.\begin{aligned} &\phi_y = Y_t + \phi_x Y_x && (2.6)\\ &\Phi_y = Y_t + \Phi_x Y_x && (2.7)\\ &\sigma[\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + Y] \\ &= \Phi_t + \tfrac{1}{2}(\Phi_x^2 + \Phi_y^2) + Y && (2.8) \end{aligned}\right\} \text{ at } y = Y(x,t).$$

We assume here that the fluids are in the undisturbed uniform state upstream at infinity, so that we impose the following boundary conditions with respect to x :

$$x \to -\infty : h-m,\ Y,\ \phi,\ \Phi \to 0. \tag{2.9}$$

In the above system of equations (2.1) - (2.9), all the quantities are normarized with respect to the characteristic length H_0 (thickness of the lower layer) and the characteristic speed $\sqrt{gH_0}$, g being the acceleration due to gravity, and we define the thickness ratio $m = h_0/H_0$, h_0 being the thickness of the upper layer, and the density ratio $\sigma = \rho_2/\rho_1\ (<1)$, ρ_1 and ρ_2 being fluid densities of the lower and upper layers, respectively.

Before proceeding to the nonlinear problem, it is instructive to examine the linear dispersion relation. Assuming a sinusoidal wave proportional to $\exp[i(kx-\omega t)]$, k and ω being respectively the wave number and the angular frequency, and linearizing the system of equations (2.1) - (2.8), we have the linear dispersion relation between k and ω :

$$(1+\sigma \tanh k \tanh mk)\omega^4$$
$$-k(\tanh k + \tanh mk)\omega^2$$
$$+(1-\sigma)k^2 \tanh k \tanh mk = 0, \tag{2.10}$$

which was first investigated by Stokes according to a famous text book 'Hydrodynamics' written by Lamb.[1] Since the dispersion relation is biquadratic with respect to ω , there may exist two possible modes, which we call the fast and the slow modes according to the magnitude of their phase velocities. It turns out from eq.(2.10) that the both modes are dispersive except in the limit $k \to 0$. We are interested here in weakly dispersive waves and in weak nonlinearity. Therefore we consider waves for which $k^2 \ll 1$, thereby implying waves of long wavelength. In this case the phase velocities can be expanded as power series in k^2 to give

$$V_{p\pm} = \frac{\omega}{k} = V_{0\pm}(1 - D_\pm k^2 + O(k^4)), \tag{2.11}$$

which is of the same form as that for the classical waves on a single layer, where

$$V_{0\pm}^2 = \frac{1}{2}\left[(1+m) \pm \sqrt{(1+m)^2 - 4(1-\sigma)m}\,\right], \tag{2.12}$$

$$D_\pm = \frac{(1+m)[1+(3\sigma-1)m+m^2]V_{0\pm}^2 - (1-\sigma)m(1+3\sigma m+m^2)}{6[(1+m)V_{0\pm}^2 - 2(1-\sigma)m]}, \tag{2.13}$$

and the suffixed signs + and - refer to the fast and slow modes, respectively.

§3 Derivation of the Korteweg-de Vries Equation

In order to reduce a simple evolutional equation from the system of equations (2.1) - (2.9), we employ the reductive perturbation method.[3] Let us first introduce the coordinate stretching defined as

$$\xi = \varepsilon^{\frac{1}{2}}(x - V_{0\pm}t), \quad \tau = \varepsilon^{\frac{3}{2}} t, \quad y = y, \tag{3.1}$$

where the small parameter $\varepsilon\,(= O(k^2))$ measures the weakness of dispersion. Since, on the other hand, we consider weakly nonlinear waves, we expand the dependent variables around the undisturbed uniform state as power series in terms of the same parameter ε :

$$\left.\begin{aligned} h(\xi,\tau) - m &= \varepsilon h^{(1)}(\xi,\tau) + \varepsilon^2 h^{(2)}(\xi,\tau) + \cdots, \\ Y(\xi,\tau) &= \varepsilon Y^{(1)}(\xi,\tau) + \varepsilon^2 Y^{(2)}(\xi,\tau) + \cdots, \\ \phi(\xi,y,\tau) &= \varepsilon^{\frac{1}{2}}[\phi^{(1)}(\xi,y,\tau) + \varepsilon\phi^{(2)}(\xi,y,\tau) + \cdots], \\ \Phi(\xi,y,\tau) &= \varepsilon^{\frac{1}{2}}[\Phi^{(1)}(\xi,y,\tau) + \varepsilon\Phi^{(2)}(\xi,y,\tau) + \cdots], \end{aligned}\right\} \tag{3.2}$$

thereby implying that the small parameter ε is also to be regarded as a measure of weakness of the nonlinearity. In this sense, we consider a balance between the nonlinearity and the dispersion.

Introducing (3.1) and (3.2) into eqs.(2.1) - (2.9) and carrying out the standard procedure of the reductive perturbation method[3], we obtain the following results from $O(\varepsilon^{\frac{1}{2}})$, $O(\varepsilon^{3/2})$ and $O(\varepsilon^{5/2})$:

$$\phi^{(1)} = \phi^{(1)}(\xi, \tau), \quad \Phi^{(1)} = \Phi^{(1)}(\xi, \tau), \tag{3.3}$$

$$h^{(1)} = \frac{V_{0\pm}^2}{V_{0\pm}^2 - m} Y^{(1)}, \tag{3.4}$$

$$\phi_{\xi}^{(1)} = \frac{V_{0\pm}}{V_{0\pm}^2 - m} Y^{(1)}, \tag{3.5}$$

$$\Phi_{\xi}^{(1)} = V_{0\pm} Y^{(1)}, \tag{3.6}$$

and $Y^{(1)}$ is governed by the Korteweg-de Vries equation:

$$Y_{\tau}^{(1)} + \alpha_{\pm} Y^{(1)} Y_{\xi}^{(1)} + \beta_{\pm} Y_{\xi\xi\xi}^{(1)} = 0, \tag{3.7}$$

where

$$\alpha_{\pm} = \frac{3 V_{0\pm}[(1+m\sigma) V_{0\pm}^2 - (1-\sigma) m (2-m)]}{2[\{1+(2\sigma-1)m\} V_{0\pm}^2 - (1-\sigma) m (1-m)]}, \tag{3.8}$$

$$\beta_{\pm} = V_{0\pm} D_{\pm}, \tag{3.9}$$

The first results (3.3) show, to the lowest order of perturbation, that the horizontal velocity components in both layers do not depend on the vertical coordinate and that the vertical velocity components are zero. This means, by virtue of Bernoulli's theorem, that the pressure distribution can be approximated by the hydrostatic one. This is one of the fundamental assumptions employed in the usual shallow water wave (or long wave) approximation.[5)] It should be noted that all the first order quantities can be expressed in terms of $\Upsilon^{(1)}$, more precisely, they are just proportional to the first order elevation of the interface $\Upsilon^{(1)}$, so that the first order elevation of the free surface $h^{(1)}$ and the horizontal velocity components $\phi_{\xi}^{(1)}$ and $\Phi_{\xi}^{(1)}$ are also governed by similar K-dV equations to eq.(3.7).

§4 Solitary Wave Solutions

As is well known, the steady solitary wave solution to eq.(3.7) can be expressed as

$$\Upsilon^{(1)} = a\,\mathrm{sech}^2\left[\sqrt{\frac{\alpha a}{12\beta}}\left(\xi - \frac{\alpha a}{3}\tau\right)\right], \tag{4.1}$$

where we have used the boundary condition that $\Upsilon^{(1)} \to 0$ as $\xi \to -\infty$. Since $\alpha a/(12\beta)$, which is under the square-root symbol, must be positive, the amplitude a is positive or negative according as α/β is positive or negative. Thus we may have a convex or concave solitary wave according to the sign of α/β.

4.1 Fast Mode

Let us first consider the fast mode. Inspecting the expressions (3.8) and (3.9), it is found that $\alpha_+ > 0$ and $\beta_+ > 0$ for all possible values of m and σ, so that $a_+ > 0$. On the other hand, by virtue of (3.4), we find that both the surface wave and the interface wave are convex upward and that the amplitude of the surface wave $a_+^{(s)}$ is always larger than that of the interface wave $a_+^{(i)}$, where the subscripts s and i refer to the surface and interface waves, respectively. Thus the qualitative nature of the solution is similar to that for the classical gravity waves on a single layer. In particular, if we take the limit $\sigma \to 1$, we recover the classical K-dV equation for a single layer.[2] In fact, in terms of the first order elevation of the free surface $h^{(1)}$, we obtain

$$\left.\begin{aligned} &h_\tau^{(1)} + \frac{3V_{0+}}{2(1+m)} h^{(1)} h_\xi^{(1)} + \frac{(1+m)^2 V_{0+}}{6} h_{\xi\xi\xi}^{(1)} = 0, \\ &\text{with} \quad V_{0+} = \sqrt{1+m}. \end{aligned}\right\} \tag{4.2}$$

On the other hand, $Y^{(1)}$ gives the internal elevation at any depth, since the value of m can be chosen arbitrarily. It is thus found that the internal level elevation in a single layer is proportional to the depth, that is, the amplitude $a_+^{(i)}$ at any depth H_0 is given by

$$a_+^{(i)} = H_0 a_+^{(s)} / (H_0 + h_0), \tag{4.3}$$

where $a_+^{(s)}$ is the amplitude of the free surface wave.

4.2 Slow Mode

Let us now consider the slow mode. From the relations (3.8) and (3.9), it turns out that $\alpha_- > 0$ and $\beta_- > 0$ so that $a_- > 0$ for $m > m^*$ while $\alpha_- < 0$ and $\beta_- > 0$ so that $a_- < 0$ for $m < m^*$, where the critical thickness ratio m^*

is given by the only real root of the cubic equation:

$$m^3-(3-3\sigma-\sigma^2)m^2-(4\sigma-3)m-1=0, \qquad (4.4)$$

which is derived from the condition $\alpha_-=0$ (see eq.(3.8)*).

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*It has been found that $\beta_\pm$ are always positive for all possible values of m and σ. However, if we take into account of the effect of surface tension, $\beta_\pm$ may change signs depending upon the relative importance of the gravity and the surface tension. Such a case was dealt with by Hasimoto for waves on a single layer.[6)]

The explicit form of m^* is therefore

$$\left.\begin{aligned} m^*(\sigma) &= S_+(\sigma)+S_-(\sigma)+\tfrac{1}{3}(3-3\sigma-\sigma^2), \\ S_\pm(\sigma) &= \Big[\frac{\sigma}{54}(27+27\sigma+18\sigma^2-36\sigma^3-18\sigma^4-2\sigma^5) \\ &\qquad \pm\frac{\sigma(1+2\sigma)}{6}\sqrt{\frac{(1-\sigma)(27+5\sigma)}{3}}\ \Big]^{1/3}. \end{aligned}\right\} \qquad (4.5)$$

In Fig.2 is shown the dependence of $m^*(\sigma)$ on the density ratio σ.

It is also found, from eq.(3.4), that the free surface wave and the interface wave are always 180° out of phase, that is,

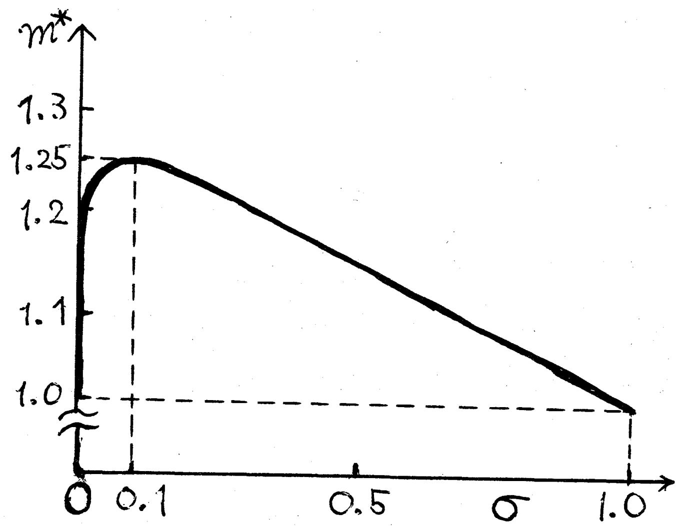

Fig. 2
dependence of m^* on σ

$$
\left.\begin{array}{llll}
a_-^{(s)} < 0 & \text{and} & a_-^{(i)} > 0 & \text{for } m > m^*, \\
a_-^{(s)} > 0 & \text{and} & a_-^{(i)} < 0 & \text{for } m < m^*,
\end{array}\right\} \quad (4.6)
$$

thereby implying that the surface wave is concave and the interface wave is convex for $m > m^*$ and vise versa for $m < m^*$. It is further found that the magnitude of the amplitudes of the surface and interface waves depends upon the thickness ratio m and the density ratio σ, that is,

$$
\left.\begin{array}{lll}
|a_-^{(s)}| > |a_-^{(i)}| & \text{for} & 0 < m < 2(1-2\sigma), \\
|a_-^{(s)}| < |a_-^{(i)}| & \text{for} & m > 2(1-2\sigma).
\end{array}\right\} \quad (4.7)
$$

Thus the slow mode shows entirely different behaviours from the classical waves on a single layer.

§5 Slow Mode at and near the Critical Thickness Ratio

Next step we have to proceed is to investigate the slow mode at the critical thickness ratio m^*. Since the dispersion relation (2.11) remains unchanged even at the critical ratio m^*, we adopt the same coordinate stretching as was given in eq.(3.1) with V_{0-}^*, where $V_{0-}^* = V_{0-}(m^*)$. On the other hand, in order to take 'stronger' nonlinear effect into account, we adopt the following modified expansion of the dependent variables:

$$\left.\begin{aligned} h(\xi,\tau)-m^* &= \varepsilon^{\frac{1}{2}}\tilde{h}^{(1)}(\xi,\tau)+\varepsilon\tilde{h}^{(2)}(\xi,\tau)+\cdots, \\ \Upsilon(\xi,\tau) &= \varepsilon^{\frac{1}{2}}\tilde{\Upsilon}^{(1)}(\xi,\tau)+\varepsilon\tilde{\Upsilon}^{(2)}(\xi,\tau)+\cdots, \\ \phi(\xi,y,\tau) &= \tilde{\phi}^{(0)}(\xi,y,\tau)+\varepsilon^{\frac{1}{2}}\tilde{\phi}^{(1)}(\xi,y,\tau)+\cdots, \\ \Phi(\xi,y,\tau) &= \tilde{\Phi}^{(0)}(\xi,y,\tau)+\varepsilon^{\frac{1}{2}}\tilde{\Phi}^{(1)}(\xi,y,\tau)+\cdots, \end{aligned}\right\} \quad (5.1)$$

whereby implying that the order of the nonlinearity is assumed to be not $O(\varepsilon)$ but $O(\varepsilon^{1/2})$. Similar expansion to (5.1) was first employed by Kakutani and Ono[4] for investigating nonlinear Alfvén waves in a collisionless plasma. Introducing (5.1) into eqs.(2.1) - (2.9) and carrying out tedious but straightforward manipulations, we obtain the following results; from $O(\varepsilon^0)$, $O(\varepsilon^{1/2})$, and $O(\varepsilon)$:

$$\left.\begin{aligned} \tilde{\phi}^{(0)} &= \tilde{\phi}^{(0)}(\xi,\tau), \quad \tilde{\Phi}^{(0)} = \tilde{\Phi}^{(0)}(\xi,\tau), \\ \tilde{h}^{(1)} &= \frac{V_{0-}^{*2}}{V_{0-}^{*2}-m^*}\tilde{\Upsilon}^{(1)}, \\ \tilde{\phi}^{(0)}_{\xi} &= \frac{V_{0-}^{*}}{V_{0-}^{*2}-m^*}\tilde{\Upsilon}^{(1)}, \\ \tilde{\Phi}^{(0)}_{\xi} &= V_{0-}^{*}\,\tilde{\Upsilon}^{(1)}, \end{aligned}\right\} \quad (5.2)$$

which are similar in form to those given by (3.3) - (3.6). From $O(\varepsilon^{\frac{3}{2}})$, we have

$$F(m^*,\sigma)\,\tilde{\Upsilon}^{(1)}\tilde{\Upsilon}^{(1)}_{\xi} = 0 , \quad (5.3)$$

where $F(m,\sigma)$ is a complicated function of m and σ , but it turns out that $F(m,\sigma)=0$ for $m=m^*$, so that $\tilde{\Upsilon}^{(1)}\tilde{\Upsilon}_\xi^{(1)}$ is not necessarily zero in spite of eq.(5.3). Finally, from $O(\varepsilon^2)$, we obtain the modified K-dV equation for $\tilde{\Upsilon}^{(1)}$:

$$\tilde{\Upsilon}_\tau^{(1)} + \alpha_c \tilde{\Upsilon}^{(1)2}\tilde{\Upsilon}_\xi^{(1)} + \beta_- \tilde{\Upsilon}_{\xi\xi\xi}^{(1)} = 0, \tag{5.4}$$

where

$$\alpha_c = \frac{3V_{0-}^*\left[\begin{array}{l}\{(32-64\sigma+31\sigma^2)+(-37+66\sigma-21\sigma^2-10\sigma^3)m^* \\ \quad +(14-15\sigma)m^{*2}\}V_{0-}^{*2} \\ -(1-\sigma)\{23(1-\sigma)+(-19+16\sigma+2\sigma^2)m^*+(5-6\sigma)m^{*2}\}\end{array}\right]}{4\sigma^2[(1+2\sigma m^*+m^{*2})V_{0-}^{*2}-(1-\sigma)m^*(1+m^*)]}. \tag{5.5}$$

It turns out that α_c is negative for all possible values of σ . It is found that any solitary wave solution to eq.(5.4) approaches to non-zero uniform state at infinity. More precisely, a convex solitary wave approaches to a negative uniform state, whereas a concave one to a positive uniform state. If we impose the boundary condition that $\tilde{\Upsilon}^{(1)}\to 0$ as $\xi\to-\infty$ (which is relevant to the present problem), we have no solitary waves at all, that is, the amplitude of the solitary wave with zero uniform state becomes zero! Therefore, in order to clarify the behaviour of the slow mode solitary waves across the critical thickness ratio m^* , one must examine the slow mode near the critical ratio m^*.

Assuming $\delta\equiv m-m^*=O(\varepsilon^{1/2})$, and carrying out similar calculations to those leading to eq.(5.4), we have a mixed K-dV and modified K-dV equation of the following form:

$$\tilde{Y}^{(1)}_{\tau} + \alpha^{*}\tilde{Y}^{(1)}\tilde{Y}^{(1)}_{\xi} + \alpha_c \tilde{Y}^{(1)2}\tilde{Y}^{(1)}_{\xi} + \beta_{-}\tilde{Y}^{(1)}_{\xi\xi\xi} = 0, \tag{5.6}$$

where $\alpha^{*} = \varepsilon^{-1/2}\delta(\partial\alpha_{-}/\partial m)_{m=m^{*}}$ and $\alpha^{*} \gtrless 0$ according as $\delta \gtrless 0$. The solitary wave solution to eq.(5.6) is easily obtained as

$$\left.\begin{aligned} \tilde{Y}^{(1)} &= \frac{a\,\mathrm{sech}^2\left[\sqrt{\frac{\lambda}{4\beta_{-}}}(\xi - \lambda\tau)\right]}{1 + \frac{a}{(a+2\alpha^{*}/\alpha_c)}\tanh^2\left[\sqrt{\frac{\lambda}{4\beta_{-}}}(\xi-\lambda\tau)\right]}, \\ \text{with}\quad \lambda &= \frac{a\alpha_c}{6}\left(a + \frac{2\alpha^{*}}{\alpha_c}\right), \end{aligned}\right\} \tag{5.7}$$

where $0 < a < -\alpha^{*}/\alpha_c$ for $\alpha^{*} > 0$, whereas $-\alpha^{*}/\alpha_c < a < 0$ for $\alpha^{*} < 0$. Thus the behaviour of the slow mode solitary waves across the critical ratio m^{*} can be fully understood, that is, we have convex, zero, and concave solitary waves for the lowest order elevation of the interface according as $m > m^{*}$, $m = m^{*}$, and $m < m^{*}$. It should be noted that the amplitude is of the order of ε far from the critical ratio m^{*}, but it is of the order of $\varepsilon^{1/2}$ near m^{*} and it becomes rapidly zero just at m^{*}.

In conclusion, it should be remarked that eq.(5.6) has the shock like solution:

$$\left.\begin{aligned} \tilde{Y}^{(1)} &= -\frac{\alpha^{*}}{2\alpha_c}\left[1 + \tanh\left\{\sqrt{\frac{\lambda}{4\beta_{-}}}(\xi - \lambda\tau)\right\}\right], \\ \text{with}\quad \lambda &= -\alpha^{*2}/(6\alpha_c), \end{aligned}\right\} \tag{5.8}$$

when $a = -\alpha^{*}/\alpha_c$. This may be regarded as a sort of dispersive (lossless) bore or hydraulic jump (if we move with the wave velocity). It should be compared with the usual bores or hydraulic jumps on a single layer which are accompanied with dissipation.

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