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ON A PROBLEM IN DIOPHANTINE APPROXIMATION

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We are concerned in this article with a property of badly approximable real numbers. Professor W. M. Schmidt has made among many other things a number of interesting and important contributions in the study of such numbers.

An n-tuple $(\alpha_1, \dots, \alpha_n)$ of real numbers is, by definition, baddly approximable if

$$(*) \qquad |x|^{1/n} \cdot \max(\|\alpha_1 x\|, \dots, \|\alpha_n x\|) \geqq \gamma$$

for some constant $\gamma > 0$, whenever $x \neq 0$ is an integer. Here, $\| t \|$ denotes as usual the distance from the real number t to the nearest integer, so that we always have $0 < \| t \| \leqq 1/2$.

We shall prove the following

THEOREM. Let $(\alpha_1, \dots, \alpha_n)$ be a badly approximable n-tuple of real numbers, i.e. an n-tuple satisfying (*). Let $\beta_1, \dots, \beta_n$ be n arbitrary real numbers, and $X \geqq 2$ an arbitrary integer. Put

$$D = \left[\frac{(n+1)n^{n-1} X^{4n+2}}{2\gamma} \right] + 1.$$

Then, for any integer $d \gtreqless 0$ there is an integer x in the interval

$$d < x \leqq d + D$$

such that

$$\|\alpha_i x - \beta_i\| < \frac{1}{X} \qquad (i = 1, 2, \ldots, n).$$

By a transference theorem (cf. e.g. [2; Chap. V, Corollary to Theorem II]), an n-tuple $(\alpha_1, \ldots, \alpha_n)$ is badly approximable if, and only if for every integral n-vector $\underline{\underline{x}} \neq \underline{\underline{0}}$

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geqq \frac{\gamma'}{\overline{|\underline{\underline{x}}|}^n}$$

for some constant $\gamma' > 0$. Here

$$\overline{|\underline{\underline{x}}|} = \max(|x_1|, \ldots, |x_n|)$$

if $\underline{\underline{x}} = (x_1, \ldots, x_n)$. We may take $\gamma' = n^{n+1}\gamma$ in case the n-tuple $(\alpha_1, \ldots, \alpha_n)$ satisfies (*). Thus, in particular, if $(\alpha_1, \ldots, \alpha_n)$ is badly approximable, then $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over the rationals.

Now, following H. Bohr and B. Jessen [1], we define

$$F(t) = 1 + e^{2\pi i(\alpha_1 t - \beta_1)} + \ldots + e^{2\pi i(\alpha_n t - \beta_n)}$$

and, with Fejér's kernel

$$K_N(t) = \sum_{\nu=-N}^{N} \left(1 - \frac{|\nu|}{N}\right) e^{2\pi i \nu t} = \frac{1}{N}\left(\frac{\sin \pi N t}{\sin \pi t}\right)^2,$$

$$\mathbb{K}_N(t) = K_N(\alpha_1 t - \beta_1) \cdots K_N(\alpha_n t - \beta_n)$$

$$= 1 + \left(1 - \frac{1}{N}\right)(e^{-2\pi i(\alpha_1 t - \beta_1)} + \dots$$

$$+ e^{-2\pi i(\alpha_n t - \beta_n)}) + R(t);$$

here, $R(t)$ is a trigonometric polynomial whose exponents, divided by 2π, are all different from the numbers $0, -\alpha_1, \dots, -\alpha_n$.

We have

$$F(t)\,\mathbb{K}_N(t) = 1 + \left(1 - \frac{1}{N}\right) n + S(t),$$

where $S(t)$ is a trigonometric polynomial whose exponents are all different from zero modulo 2π. Hence

$$\frac{1}{D} \sum_{d < x \leqq d + D} F(x)\,\mathbb{K}_N(x) = 1 + \left(1 - \frac{1}{N}\right) n$$

$$+ \frac{1}{D} \sum_{d < x \leqq d + D} S(x),$$

where we find easily

$$\left| \frac{1}{D} \sum_{d < x \leqq d + D} S(x) \right| \leqq \frac{N^n}{D} \frac{N^n}{2\gamma'} .$$

(Note that the sum of the coefficients of $K_N(t)$ equals N.)

By the positivity of the kernel $\mathbb{K}_N(t)$ we have, since

$$\frac{1}{D} \sum_{d < x \leqq d + D} \mathbb{K}_N(x) \leqq 1 + \frac{N^n}{D} \frac{N^n}{2\gamma'},$$

$$\max_{d < x \leqq d + D} |F(x)| \geqq 1 + \left(1 - \frac{1}{N}\right) n - \frac{N^{2n}}{2\gamma' D} .$$

$$\cdot\left(1 + \frac{N^{2n}}{2\gamma' D}\right)^{-1} .$$

Taking

$$N = n X^2 ,$$

$$D = \left[\frac{(n+1) n^{2n} X^{4n+2}}{2\gamma'}\right] + 1,$$

we get

$$\max_{d < x \leqq d + D} |F(x)| \geqq n + 1 - \frac{3}{X^2} .$$

Let α, β be any one of the pairs α_i, β_i $(1 \leqq i \leqq n)$. Then, since

$$|F(x)| \leqq n - 1 + |1 + e^{2\pi i(\alpha x - \beta)}| ,$$

we have

$$|1 + e^{2\pi i(\alpha x - \beta)}| \geqq 2 - \frac{3}{X^2} .$$

Noticing that $|1 + e^{2\pi i t}| = 2|\cos \pi t|$ and $|\sin \pi t| \geqq 2\|t\|$, we deduce from the above inequalities for $|F(x)|$ that

$$\|\alpha x - \beta\| \leqq \frac{\sqrt{3}}{2} \frac{1}{X} < \frac{1}{X}$$

for some integer x, independent of the particular α, β, in the interval $d < x \leqq d + D$. This completes the proof of our theorem.

It should be observed that our method can be applied to any n-tuple of real numbers $\alpha_1, \ldots, \alpha_n$ such that 1 and the α_i's are linearly independent over the field of rational numbers, obtaining a result similar to the theorem above with a suitably defined D in terms of M_n, where

$$\frac{1}{M_n} = \min_{0 < \lceil \underline{x} \rceil \leq n} \| \alpha_1 x_1 + \ldots + \alpha_n x_n \| .$$

We thus have a sort of quantitative formulation of the (small) approximation theorem of Kronecker's.

References

[1] H. Bohr and B. Jessen: One more proof of Kronecker's theorem. J. London Math. Soc., 7(1932), 274-275.

[2] J. W. S. Cassels: An Introduction to Diophantine Approximation. Cambridge Tracts in Math. & Math. Phys. No. 45. Cambridge Univ. Press, 1957.