数理解析研究所講究録
340巻 1978年 69-73

Some geometric properties of compact complex manifolds uniformized by bounded domains in $\mathbb{C}^n$

B. Wong

One of the major accomplishments in the theory of Riemann surfaces is the uniformization theorem which roughly states that the universal covering space of compact Riemann surfaces of genus not less than two is complex analytically equivalent to the disc $=\{z;\ |z|<1\}$. The higher dimensional analog of this theorem is one of the basic problem in hyperbolic complex analysis. In this note we summarize one direction of this research recently moved forward by differential geometers. The starting point of us is a thoerem of H. Wu given below.

<u>Theorem</u> (Wu). Let M be a compact complex Kähler manifold of negative sectional curvature, then the universal covering $\tilde{M}$ of M is a Stein manifold.

<u>Remark</u>. Here we state Wu's theorem in a weaker form, for example "negative" can be replaced by "non-positive."

For a long time the examples of compact complex Kähler manifold of negative sectional curvature known to us were only those compact quotients of the ball. Not untill last year Mostow and Siu discovered an example which guarrantees that the studies of $\tilde{M}$ is indeed of significance from the viewpoint of uniformization theory.

<u>Theorem</u> (Mostow-Siu). There exists a compact complex Kähler surface of negative sectional curvature such that its

universal covering is not biholomorphic to a ball.

A perhaps more natural generalization of hyperbolic compact Riemann surfaces for algebraic geometers and complex analysis is the notion of negative tangent bundle in the sense of H. Grauert.

Definition. Let M be a compact complex manifold then the tangent bundle T(M) of M is said to be negative if it is a strongly pseudo-convex manifold whose only exceptional variety is the zero section.

The concept of negative tangent bundle is intimately related to the definition of holomorphic bisectional curvature introduced by I. Goldberg and S. Kobayashi.

Let M be a Kähler manifold and R its Riemannian curvature tensor. Given two complex planes σ and σ' in $T_x(M)$, we define the holomorphic bisectional curvature $H(\sigma, \sigma')$ by

$$H(\sigma, \sigma') = R(X, JX, Y, JY)$$

where J is the complex structure tensor of M. By Bianchi's identity we have

$$R(X, JX, Y, JY) = R(X, Y, X, Y) + R(X, JY, X, JY).$$

It is not difficult to show that the tangent bundle of a compact Kähler manifold of negative bisectional curvature must be negative in the sense of Grauert. The converse is almost true. From now on we may think of the condition of negative tangent bundle and negative bisectional curvature are more or less equivalent. We are now in a position to state our main probelm.

Conjecture: Let M be a compact complex surface of negative tangent bundle, then there are strong restriction on $\pi_1(M)$, $\pi_2(M)$ and $\pi_3(M)$.

For the reason we are unable to conjecture the shape of the universal covering of manifold of negative tangent bundle in the mean time, let us restrict ourselves on M by the following conditions in the sequel of this paper:

(1) $\dim_{\mathbb{C}} M = 2$

(2) the universla covering $\tilde{M}$ of M is a bounded domain in $\mathbb{C}^2$.

The only domains in $\mathbb{C}^2$ we know now which have enough discrete subgroup to form compact complex quotients are listed as follows.

(1) $B_2 = \{(Z_1, Z_2) \mid |Z_1|^2 + |Z_2|^2 < 1\}$

(2) $\Delta_2 = \{(Z_1, Z_2) \mid |Z_1| < 1, |Z_2| < 1\}$

(3) the universal covering of Kodaira surface.

B_2 and Δ_2 are symmetric domains in $\mathbb{C}^2$. The Chern numbers c_1 and c_2 of the compact complex quotients of B_2 and Δ_2 satisfy $c_1^2 = 3c_2$ and $c_1^2 = 2c_2$ respectively. They are known as the Klein-Clifford forms which have been studied by Borel, Hirzebruch and others. The following converse result was due to S.T. Yau.

<u>Theorem</u> (Yau). Let M be a compact complex two dimensional manifold uniformized by bounded domain in $\mathbb{C}^2$. If $c_1^2 = 3c_2$, then the universal covering of M is biholomorphic to B_2.

Griffiths observed that one could use the simultaneous

uniformization theorem of Bers to prove the universal covering of Kodaira surface is a bounded domain in $\mathbb{C}^2$. From our arguments of the Chern numbers described above one can conclude that this kind of domains in $\mathbb{C}^2$ are not symmetric. Actually it is a theorem of G.B. Shabat which states as follows.

<u>Theorem</u> (Shabat). Let D be the universal covering of Kodaira surface, then Aut(D) is discrete.

A global result concerning negative bisectional curvature of Paul Yang should be stated here.

<u>Theorem</u> (Yang). The polydisc Δ_n $(n > 1)$, does not admit a complete Kähler metric with its bisectional curvature bounded between two negative constants,

$$- c^2 \leq H(\sigma, \sigma') \leq - d^2 < 0$$

The immediate consequence is that if M^n $(n > 1)$ is a compact Kähler manifold with negative bisectional curvature then its universal covering cannot be a polydisc. Yang's theorem indicates the sharp distinctions between a polydisc and a ball in $\mathbb{C}^n$.

Finally we want to mention one of the new results of us in this direction.

<u>Theorem A</u>. If M is a compact complex Kähler surface of negative sectional curvature whose universal covering $\tilde{M}$ is a bounded domain in $\mathbb{C}^2$, then either

(1) $\tilde{M}$ is biholomorphic to the ball

or

(2) $\mathrm{Aut}(\tilde{M})$ is discrete.

<u>Theorem B</u>. Let M be a compact complex Kähler surface with negative bisectional curvature such that its universal covering $\tilde{M}$ is a bounded domain in $\mathbb{C}^2$, if we further assume that there exists a point $p \in \partial\tilde{M}$ such that $\exists$ neighborhood $U \ni p$ in $\mathbb{C}^2$ with $U \cap \partial M$ homeomorphic to R^3, then the same conclusion of Theorem A holds.

The general case without the assumption that $\tilde{M}$ is a bounded domain is likely to be true. A detailed proof will appear elsewhere.