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<th>Title</th>
<th>Brody's Method in Value Distribution Theory (Geometric Theory of Several Complex Variables)</th>
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<tr>
<td>Author(s)</td>
<td>GREEN, Mark L.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1978), 340: 56-59</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1978-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/104262">http://hdl.handle.net/2433/104262</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Brody's Method in Value Distribution Theory

Mark L. Green

In this thesis [B], R. Brody introduced a very clever method of reparametrizing holomorphic maps from the disc to a complex manifold which allowed the use of a normal families argument. The general principle which his method yields may be embodied in the following:

**THEOREM 1.** Let $M$, $N$ be a compact complex manifolds (or more generally an analytic spaces), possibly with boundary, $H$, a differential metric on $N$. Then either

1. There exists a non-constant holomorphic map $\mathbb{C} \not\rightarrow M$ such that $|dg(0)|_H = 1$ and $|dg(z)|_H \leq 1$ for all $z \in \mathbb{C}$ or

2. Some neighborhood $M_{\epsilon}$ of $M$ in $N$ is hyperbolic; in fact, the infinitesimal Kobayashi metric is never zero on non-zero tangent vectors.

Put in this form, the theorem has the following corollaries:

**COROLLARY 1 ([B]).** Any sufficiently small deformation of a hyperbolic compact complex manifold is hyperbolic.

**Proof.** If $\mathcal{M} \not\rightarrow \Delta$ is a deformation of $M = \pi^{-1}(0)$, take $N = \pi^{-1}(|z| < \frac{1}{2})$ and apply the theorem. As any subset of a hyperbolic space is hyperbolic, we conclude $M_{\epsilon} = \pi^{-1}(t)$ is hyperbolic for $|t|$ sufficiently small.

**COROLLARY 2 ([B]).** A compact complex manifold $M$ is hyperbolic if and only if it admits no non-constant holomorphic
map \mathbb{C} \xrightarrow{g} M of order \leq 2.

Proof. Take N = M, and note that |dg(z)|_{H} \leq 1 implies g has order \leq 2.

Remark. If M is an algebraic variety containing an elliptic curve E, it is not hyperbolic and admits a non-constant holomorphic map of order 2, namely the projection \mathbb{C} \rightarrow \mathbb{C}/\Lambda = E. Thus order \leq 2 is the best statement possible.

**Corollary 3 ([G]).** If M is an analytic subspace of a complex torus T and if M contains no elliptic curves, then M is hyperbolic.

Proof. Take N = M and take H to be the Euclidean metric. If alternative (1) holds, lift g so that we have

\[
\begin{array}{ccc}
G & \xrightarrow{g} & \mathbb{C}^n \\
& & \\
\mathbb{C} & \xrightarrow{g} & \mathbb{C}^n/\Lambda = T
\end{array}
\]

G = (G_1', \ldots, G_n)

Then |G_1'|^2 + \cdots + |G_n'|^2 < 1, so by Liouville's theorem G_i' = constant for all i, thus G(z) = az + b, where a, b \in \mathbb{C}^n, i.e. g is a translate of a one-parameter subgroup of T. So \overline{g(\mathbb{C})} gives a non-trivial real subtorus contained in M, and hence a non-trivial complex subtorus and a fortiori an elliptic curve. This is forbidden by hypothesis, so alternative (1) is eliminated, leaving alternative (2).

Brody's method can be generalized to the non-compact case either by making use of Hurwitz's classical theorem on limits of nowhere zero functions, as done by Alan Howard, or by taking a complete metric with certain properties, as in [G2].
The general theorem one obtains is:

**THEOREM 2.** Let $M$ be a compact complex manifold and $D$ an analytic subset of $M$ with normal crossings. Let $X_k = \{ z \in M \mid \text{multiplicity}_D(z) = k \}$. Then either

1. $M - D$ is complete hyperbolic and hyperbolically embedded in $M$

or

2. For some $k \geq 0$, there exists a non-constant holomorphic map $\mathfrak{C} \to X_k$ of order $\leq 2$.

As corollaries we have

**COROLLARY 1.** Let $H_1, \cdots, H_{2n+1}$ be hyperplanes in general position in $\mathbb{P}^n$. Then $\mathbb{P}^n - (H_1 \cup \cdots \cup H_{2n+1})$ is complete hyperbolic and hyperbolically embedded in $\mathbb{P}^n$.

**Proof.** The irreducible components of $X_k$ are copies of $\mathbb{P}^n - (2n+1-k \text{ hyperplanes in general position in } \mathbb{P}^{n-k})$. Noting $2n+1-k \geq 2(n-k)+1$, alternative (2) is excluded by the Picard theorem for projective space minus $2n+1$ hyperplanes in general position.

**COROLLARY 2.** Let $\chi \xrightarrow{\pi} \Delta^{3g-3}$ be a local universal deformation over the unit polycylinder of dimension $3g-3$ of a stable curve $\pi^{-1}(0)$ be complete algebraic curves of genus $\geq 2$. Let $\chi_\varepsilon = \pi^{-1}(\Delta(1-\varepsilon))$ and $\chi_\varepsilon^{\text{sing}}$ be the singular fibres in $\chi_\varepsilon$. Then for any $\varepsilon > 0$, $\chi_\varepsilon - \chi_\varepsilon^{\text{sing}}$ is complete hyperbolic and hyperbolically embedded in $\chi_\varepsilon$.

**Proof.** As all of the $X_k$ project to subsets of $\Delta^{3g-3}$, any holomorphic map $\mathfrak{C} \to X_k$ goes to a fibre. But the fibres of
$X_k \xrightarrow{\pi} \pi(X_k)$ are unions of punctured curves $M - \{p_1, \ldots, p_k\}$ so that $\chi(M) + k > 0$. By the classical Picard's theorem, there are no non-constant holomorphic maps $\mathcal{C} \to M - \{p_1, \ldots, p_k\}$.

There are other results along the lines Corollary which follow by these methods. For example, one can obtain a proof of Mordell's conjecture for families of curves (Grauert, Manin) in this way.

**Corollary 3 ([G]).** Let $T$ be a complex torus, $D$ an analytic hypersurface which contains no elliptic curve. Then $T - D$ is complete hyperbolic and hyperbolically embedded in $T$.

**Proof.** This is actually a corollary of the method of proof rather than the theorem itself. We obtain that the map $\mathcal{C} \to T - D$ of alternative (2) is distance-decreasing with respect to a complete metric, and this forces $g$ to be constant. There can be no non-constant holomorphic maps $\mathcal{C} \to D$ by Corollary 3 of Theorem 1. So alternative (1) holds.

**References**

