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Families of Linear Systems on Projective Manifolds

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1. By a complex space, we mean a reduced Hausdorff complex analytic space. Let X and S be complex spaces and $\pi: X \longrightarrow S$ a surjective proper holomorphic map. The triple (X,π,S) is called a <u>family of compact complex manifolds</u> if (1) every fiber $\pi^{-1}(s) = V_s$ is connected and (2) there are an open covering $\{X_i\}$ of X, open subsets U_i of $\mathbb{C}^d$, open subsets S_i of S and holomorphic isomorphisms $\eta_i: X_i \longrightarrow U_i \times S_i$ such that the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{\eta_i} & U_i \times S_i \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \text{proj}} & \\ & S_i & \end{array}$$

commutes. We write

$$\eta_i \eta_k^{-1}(z_k, s) = (g_{ik}(z_k, s), s).$$

We write $\{V_s\}_{s\in S}$ instead of (X,π,S). A <u>family of holomorphic vector bundles</u> is, by definition, a holomorphic vector bundle $\mathcal{F}$ on X. Put $F_s = \mathcal{F}|V_s$ and write $\{F_s, V_s\}_{s\in S}$ instead of $\mathcal{F}$. Let $H^{\nu}(V_s, \mathcal{O}(F_s))$ be the ν-th cohomology group of the sheaf $\mathcal{O}(F_s)$ over V_s of germs of holomorphic sections of F_s. We may assume that $\mathcal{F}$ is trivial on X_i. Let $\{f_{ik}(z_k, s)\}$ be the transition matrices of $\mathfrak{F}$. Then, for $o \in S$, we define a bilinear map $\tau : H^0(V_o, \Theta(F_o)) \times T_o S \longrightarrow H^1(V_o, \Theta(F_o))$ by

$$\tau(\xi, \partial/\partial s)_{ik}(z_i) = (\partial f_{ik}/\partial s)_{(z_k, o)} \xi_k(z_k) - (\partial \xi_i/\partial z_i)_{z_i} (\partial g_{ik}/\partial s)_{(z_k, o)},$$

where $\xi = \{\xi_i(z_i)\} \in H^0(V_o, \Theta(F_o))$, $\partial/\partial s \in T_o S$ and $z_k = g_{ki}(z_i, o)$. Put $\tau_\xi(\partial/\partial s) = \tau(\xi, \partial/\partial s)$.

Using Kuranishi's idea of the proof of the existence of complete families of complex structures (Kuranishi [7]), we get

Theorem 1. Let $\{F_s, V_s\}_{s \in S}$ be a family of holomorphic vector bundles. Then, for any point $o \in S$, there are an open neighborhood U of o in S and a vector bundle homomorphism

$$u : H^0(V_o, \Theta(F_o)) \times U \longrightarrow H^1(V_o, \Theta(F_o)) \times U$$

such that the (disjoint) union $\bigcup_{s \in U} H^0(V_s, \Theta(F_s))$ is identified with the Ker u. Moreover, we have

$$(du)_{(\xi, o)} = \begin{pmatrix} 0 & \tau_\xi \\ 0 & 1 \end{pmatrix}, \text{ for } \xi \in H^0(V_o, \Theta(F_o)).$$

Pathing up the local data, we have the following theorem which is considered as a special case of Schuster [11].

Theorem 2. Let $\{F_s, V_s\}_{s \in S}$ be as above. Then the disjoint union $\mathbb{H} = \bigcup_{s \in S} H^0(V_s, \Theta(F_s))$ admits a complex space structure so that $(\mathbb{H}, \lambda, S)$ is a linear fiber space in the sense of Grauert [1], where $\lambda : \mathbb{H} \longrightarrow S$ is the canonical projection.

For a complex vector space A and a non-negative integer r, we denote by $G^r(A)$ the Grassmann variety of all $(r+1)$-dimentional linear subspaces of A. (If $\dim A \leqq r$, then $G^r(A)$ is empty.)

Theorem 3. Let $\{F_s, V_s\}_{s \in S}$ be as above. Then the disjoint union $\mathbb{G}^r = \bigcup_{s \in S} G^r(H^0(V_s, \mathcal{O}(F_s)))$ admits a complex space structure so that the canonical projection $\mu : \mathbb{G}^r \longrightarrow S$ is a proper holomorphic map.

Sufficient conditions for the non-singularity of the spaces $\mathbb{H}$ and $\mathbb{G}^r$ are given in the next theorem, which is easily proved by using Theorem 1.

Theorem 4. Let $\{F_s, V_s\}_{s \in S}$ be as above. Let o be a non-singular point of S.

(1) For $\xi \in H^0(V_o, \mathcal{O}(F_o))$, assume that τ_ξ is surjective. Then $\mathbb{H}$ is non-singular at ξ and

$$\dim_\xi \mathbb{H} = h^0(F_o) - h^1(F_o) + \dim_o S,$$

where $h^\nu(F_o) = \dim H^\nu(V_o, \mathcal{O}(F_o))$.

(2) Let $L \in G^r(H^0(V_o, \mathcal{O}(F_o)))$ and let $\{\xi_0, \cdots, \xi_r\}$ be a basis of L. Assume that the linear map

$$\partial/\partial s \in T_o S \longmapsto (\tau_{\xi_0}(\partial/\partial s), \cdots, \tau_{\xi_r}(\partial/\partial s)) \in H^1(V_o, \mathcal{O}(F_o))^{r+1}$$

is surjective. Then $\mathbb{G}^r$ is non-singular at L and

$$\dim_L \mathbb{G}^r = (r+1)(h^0(F_o) - h^1(F_o) - r - 1) + \dim_o S.$$

2. Next, we apply the theorems in §1 to the case of line bundles on a projective manifold. Let V be a projective manifold. For a line bundle F on V, let

$$\tau : H^0(V, \mathcal{O}(F)) \times H^1(V, \mathcal{O}) \longrightarrow H^1(V, \mathcal{O}(F))$$

be the bilinear map defined by

$$\tau(\xi, h)_{ik}(z_i) = \xi_i(z_i) h_{ik}(z_i),$$

where $\xi = \{\xi_i(z_i)\} \in H^0(V, \mathcal{O}(F))$ and $h = \{h_{ik}(z_i)\} \in H^1(V, \mathcal{O})$. Put $\tau_\xi(h) = \tau(\xi, h)$.

For a cohomology class $c \in H^2(V, \mathbb{Z})$ of type $(1,1)$, we put

$$\mathrm{Pic}^c(V) = \{ F \mid F \text{ is a line bundle on } V \text{ with } c(F) = c \}.$$

($c(F)$ is the Chern class of F). Then $\mathrm{Pic}^c(V)$ is an abelian variety of dimension $q = \dim H^1(V, \mathcal{O})$, the irregularity of V, and is called the c-th <u>Picard variety of</u> V.

For $s \in \mathrm{Pic}^c(V)$, we denote by F_s the line bundle on V corresponding to s. Then $\{F_s, V\}_{s \in \mathrm{Pic}^c(V)}$ is a family of line bundles. For $r \geq 0$, we denote by $\mathfrak{G}^r_c(V)$ the complex space $\mathfrak{G}^r$ in Theorem 3 with respect to $\{F_s, V\}_{s \in \mathrm{Pic}^c(V)}$. It is regarded as the set of all linear systems g^r_c on V with "degree" c (i.e., $c([D]) = c$ for $D \in g^r_c$) and of dimension r. In particular, $\mathfrak{G}^0_c(V)$ is the set of all effective divisors D on V such that $c([D]) = c$. This is canonically isomorphic to the space introduced by Weil [13] and Kodaira [5]. The map $\mu : D \in \mathfrak{G}^0_c(V) \longmapsto [D] \in \mathrm{Pic}^c(V)$ is called the <u>Jacobi map</u>.

Definition. A linear system g_c^r on V is said to be semi-regular if there are independent $D_0, \cdots, D_r \in g_c^r$ such that the linear map

$$h \in H^1(V, \mathcal{O}) \longmapsto (\tau_{\xi_0}(h), \cdots, \tau_{\xi_r}(h)) \in H^1(V, \mathcal{O}([D]))^{r+1}$$

is surjective, where $D \in g_c^r$ and $D_\nu = (\xi_\nu)$, the zero divisor of $\xi_\nu \in H^0(V, \mathcal{O}([D]))$, $0 \leqq \nu \leqq r$.

Theorem 5. Let $g_c^r \in \mathbb{G}_c^r(V)$ be semi-regular. Then it is a non-singular point of $\mathbb{G}_c^r(V)$ and

$$\dim_{g_c^r} \mathbb{G}_c^r(V) = (r+1)(h^0(D)-h^1(D)-r-1) + q,$$

where $D \in g_c^r$ and $h^\nu(D) = \dim H^\nu(V, \mathcal{O}([D]))$.

Remark. If we put $r = 0$, then we get the usual semi-regularity theorem of Kodaira-Spencer [6].

The following two theorems concerning the Jacobi map are easy consequences of Theorem 1.

Theorem 6. Assume that there is $D \in \mathbb{G}_c^0(V)$ such that $h^0(D) > h^1(D)$. Then the Jacobi map $\mu : \mathbb{G}_c^0(V) \longrightarrow \mathrm{Pic}^c(V)$ is surjective and each fiber of μ has dimension at least $h^0(D)-h^1(D)-1$.

Theorem 7. For $D \in \mathbb{G}_c^0(V)$, assume that $h^0(D) \leqq h^1(D)$. Then there are an open neighborhood U of $x = \mu(D)$ in $\mathrm{Pic}^c(V)$ and a $h^0(D) \times h^1(D)$-matrix valued holomorphic function $A(y)$, $y \in U$, on U such that $\mathcal{M}(\mathbb{G}_c^0(V)) \cap U$ is the set of zeros of all $h^0(D) \times h^0(D)$-minors of $A(y)$.

Remark. If V is a compact Riemann surface, then Theorem 6 is the Jacobi inversion and Theorem 7 is known as Kempf's theorem (see Mumford[9]).

Put

$$\mathbb{F}_c^r(V) = \left\{ g_c^r \in \mathbb{G}_c^r(V) \;\middle|\; g_c^r \text{ has a fixed component or a base point} \right\}.$$

Then, we can easily show that $\mathbb{F}_c^r(V)$ is a closed complex subspace of $\mathbb{G}_c^r(V)$. Put

$$\mathbb{G}^r(V) = \bigcup_c \mathbb{G}_c^r(V), \quad \mathbb{F}^r(V) = \bigcup_c \mathbb{F}_c^r(V), \quad \text{(disjoint unions).}$$

A holomorphic map $f : V \longrightarrow \mathbb{P}^r$ is said to be non-degenerate if the image $f(V)$ is not contained in any hyperplane. Let $\mathrm{Hol}_{\text{non-deg}}(V, \mathbb{P}^r)$ be the set of all non-degenerate holomorphic maps of V into $\mathbb{P}^r$. Then, it is an open subspace of the Douady space $\mathrm{Hol}(V, \mathbb{P}^r)$. Note that $\mathrm{Aut}(\mathbb{P}^r)$ acts freely on $\mathrm{Hol}_{\text{non-deg}}(V, \mathbb{P}^r)$ by the composition of maps.

Theorem 8. The orbit space $\mathrm{Hol}_{\text{non-deg}}(V, \mathbb{P}^r)/\mathrm{Aut}(\mathbb{P}^r)$ has a complex space structure such that (1) it is biholomorphic to $\mathbb{G}^r(V) - \mathbb{F}^r(V)$ and (2) the projection

$$\mathrm{Hol}_{\text{non-deg}}(V, \mathbb{P}^r) \longrightarrow \mathrm{Hol}_{\text{non-deg}}(V, \mathbb{P}^r)/\mathrm{Aut}(\mathbb{P}^r)$$

is a principal $\mathrm{Aut}(\mathbb{P}^r)$-bundle.

$Aut(\mathbb{P}^r) \times Aut(V)$ acts on $Hol_{non\text{-}deg}(V, \mathbb{P}^r)$ by the composition of maps. By Holmann's theorem[2],

<u>Corollary</u>. Assume that $Aut(V)$ is compact. Then

$$M(V, \mathbb{P}^r) = Hol_{non\text{-}deg}(V, \mathbb{P}^r)/(Aut(\mathbb{P}^r) \times Aut(V))$$

admits a complex space structure such that (1) it is biholomorphic to $(\mathbb{G}^r(V) - \mathbb{F}^r(V))/Aut(V)$ and (2) if h is a $(Aut(\mathbb{P}^r) \times Aut(V))$-invariant holomorphic function on an open subset W of $Hol_{non\text{-}deg}(V, \mathbb{P}^r)$, then there is a holomorphic function $\hat{h}$ on $\alpha(W)$ such that $\hat{h}\alpha = h$, where $\alpha : Hol_{non\text{-}deg}(V, \mathbb{P}^r) \longrightarrow M(V, \mathbb{P}^r)$ is the projection.

<u>Example</u>. Let $V = \mathbb{C}/(\mathbb{Z} + \omega\mathbb{Z})$ be a complex 1-torus. Let 0 be the zero of the group V and $n0 = 0 + \cdots + 0$ be the divisor on V. Let $\Phi_{|n0|} : V \longrightarrow \mathbb{P}^{n-1}$ be the meromorphic map associated with $|n0|$. It is in fact a holomorphic imbedding for $n \geqq 3$. Put $C_n = \Phi_{|n0|}(V)$. Let $\mathcal{S}_n^r$ $(1 \leqq r \leqq n-2)$ be the open subspace of the Grassmann variety of all $(n-2-r)$-dimensional linear subspaces of $\mathbb{P}^{n-1}$ which do not intersect with C_n.

Let $t_x : y \in V \longmapsto x+y \in V$ be the translation of V by $x \in V$. Let G be the finite subgroup of $Aut(V)$ generated by t_x with $nx = 0$ (the summation in the group V) and by

(1) $s_{-1} : y \in V \longmapsto -y \in V$, if V is neither biholomorphic to $\mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$ nor to $\mathbb{C}/(\mathbb{Z} + \rho\mathbb{Z})$, $(\rho = (1+\sqrt{-3})/2)$,

(2) $s_{\sqrt{-1}} : y \in V \longmapsto \sqrt{-1}y \in V$, if $V = \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$,

(3) $s_\rho : y \in V \longmapsto \rho y \in V$, if $V = \mathbb{C}/(\mathbb{Z} + \rho\mathbb{Z})$.

Then every element of G can be extended to be a projective transformation of $\mathbb{P}^{n-1}$ mapping C_n onto itself. Thus G acts on $\mathcal{S}^r_n$.

Now, $M(V, \mathbb{P}^r)$ is, in this case, divided into connected compo~~r~~nents as follows:

$$M(V, \mathbb{P}^r) = M_{r+1}(V, \mathbb{P}^r) \cup M_{r+2}(V, \mathbb{P}^r) \cup \cdots,$$

where

$$M_{r+1}(V, \mathbb{P}^r) = \text{one point},$$
$$M_n(V, \mathbb{P}^r) \cong \mathcal{S}^r_n / G, \text{ for } n \geq r+2.$$

$M_n(V, \mathbb{P}^r)$ is considered as the moduli space of non-degenerate holomorphic maps $f : V \longrightarrow \mathbb{P}^r$ such that

$$n = (\text{ord } f)(\deg f(V)),$$

where $\text{ord } f$ is the mapping order of $f : V \longrightarrow f(V)$.

3. Next, we consider the case of compact Riemann surfaces. Let V be a compact Riemann surface of genus g. In this case, $\mathbb{G}^r_c(V)$ is written as $\mathbb{G}^r_n(V)$, where $n = \int_V c$. $\mathbb{G}^r_n(V)$ is the set of all linear systems on V of degree n and dimension r. $\mathbb{G}^0_n(V)$ is canonically isomorphic to $S^n V$, the n-th symmetric product of V. For $r \geq 1$, put as before

$$\mathbb{F}^r_n(V) = \left\{ g^r_n \in \mathbb{G}^r_n(V) \;\middle|\; g^r_n \text{ has a fixed point} \right\},$$

$$\mathrm{Hol}_{\text{non-deg}}(V, \mathbb{P}^r)_n = \left\{ f \in \mathrm{Hol}_{\text{non-deg}}(V, \mathbb{P}^r) \;\middle|\; n = (\mathrm{ord}\, f)(\deg f(V)) \right\}.$$

In particular, we put

$$R_n(V) = \mathrm{Hol}_{\text{non-deg}}(V, \mathbb{P}^1)_n = \left\{ f \;\middle|\; f \text{ is a meromorphic function on } V \text{ of order } n \right\}$$

By Theorem 8,

$$\mathrm{Hol}_{\text{non-deg}}(V, \mathbb{P}^r)_n / \mathrm{Aut}(\mathbb{P}^r) \cong \mathbb{G}^r_n(V) - \mathbb{F}^r_n(V).$$

In particular,

$$R_n(V)/\mathrm{Aut}(\mathbb{P}^1) \cong \mathbb{G}^1_n(V) - \mathbb{F}^1_n(V).$$

It is a difficult problem to determine n with non-empty $\mathrm{Hol}_{\text{non-deg}}(V, \mathbb{P}^r)_n$ and to determine the structure of it for such n. Even for $R_n(V)$, it seems difficult. Note that $R_n(V)$ is non-empty for $n \geqq g+1$. $R_g(V)$ is non-empty unless V is hyperelliptic and g is odd. If $n \geqq g$, then $R_n(V)$ is non-singular and of dimension $2n+1-g$.

Example. Let V be a non-singular model (of the closure in $\mathbb{P}^2$) of the curve $y^3 = x^8-1$. It has the genus 7. By some calculations, we can show that (1) $R_3(V) \cong \mathrm{Aut}(\mathbb{P}^1)$, (2) $R_4(V)$ and $R_5(V)$ are empty and (3) $R_6(V)$ is of dimension 6 and singular at $f = x^2$. In fact, the tangent cone to $R_6(V)$ at f is given by $\left\{ (z_1, \cdots, z_7) \in \mathbb{C}^7 \;\middle|\; z_1 z_2 = 0 \right\}$.

We give here a simple theorem.

<u>Theorem 9</u>. Let V be a compact Riemann surface of genus g. Let m and n be positive integers such that (1) m and n are relatively prime and (2) $(m-1)(n-1) \leq g-1$. Then, at least one of $R_m(V)$ and $R_n(V)$ is empty.

<u>Corollary</u>. Let p be a prime number such that $R_p(V)$ is non-empty. Let n be a positive integer such that $(p-1)(n-1) \leq g-1$. Then

$$R_n(V) \begin{cases} = \text{empty}, & \text{if } n \not\equiv 0 \pmod p \\ \cong R_{n/p}(\mathbb{P}^1), & \text{if } n \equiv 0 \pmod p. \end{cases}$$

Now, using Serre duality, the semi-regularity condition is expressed in this case as follows.

<u>Theorem 10</u>. A linear system $g_n^r \in \mathfrak{G}_n^r(V)$ is semi-regular if and only if, for a basis $\{\xi^0, \cdots, \xi^r\}$ of the linear subspace of $H^0(V, \mathcal{O}([D]))$, $(D \in g_n^r)$, corresponding to g_n^r and for $\eta^0, \cdots, \eta^r \in H^0(V, \mathcal{O}(K_V \otimes [-D]))$, ($K_V$ = the canonical bundle of V), the equality

$$\xi^0 \eta^0 + \cdots + \xi^r \eta^r = 0 \in H^0(V, \mathcal{O}(K_V))$$

implies

$$\eta^0 = \cdots = \eta^r = 0.$$

<u>Corollary</u>. (1) Every divisor $D \in S^n V$ is semi-regular.
(2) $g_n^1 \in \mathfrak{G}_n^1(V)$ is semi-regular if and only if $h^1(2D-D_0) = 0$,

where $D \in g_n^1$ and D_o is the fixed part of g_n^1.

(3) If $h^1(D) \leqq 1$ for $D \in g_n^r$, then g_n^r is semi-regular.

By Theorem 5, if $g_n^r \in \mathfrak{G}_n^r(V)$ is semi-regular, then g_n^r is a non-singular point of $\mathfrak{G}_n^r(V)$ and

$$\dim_{g_n^r} \mathfrak{G}_n^r(V) = (r+1)(n-r) - rg.$$

Remark. Severi[12] says that, for a general V linear systems g_n^r on V depends $((r+1)(n-r) - rg)$-parameters.

Next, let $g \geqq 2$ and T_g be the Teichmüller space of compact Riemann surfaces of genus g. For $t \in T_g$, we denote by V_t the compact Riemann surface corresponding to t. For $n > 0$, let $J_n(V_t)$ be the Jacobi variety of degree n, i.e., the set of all line bundles on V_t of degree n. It is well known that $(J_n, \tilde{\pi}, T_g) = \{J_n(V_t)\}_{t \in T_g}$ is a family of abelian varieties.

For $s \in J_n$, put $V_s = V_{\tilde{\pi}(s)}$. Let F_s be the line bundle of degree n on V_s corresponding to s. Then $\{F_s, V_s\}_{s \in J_n}$ is a family of line bundles.

We denote by $\mathfrak{G}_n^r$ the complex space $\mathfrak{G}^r$ in Theorem 3 with respect to the family $\{F_s, V_s\}_{s \in J_n}$. Then

$$\mathfrak{G}_n^r = \bigcup_{t \in T_g} \mathfrak{G}_n^r(V_t) \quad \text{(disjoint union)}.$$

In fact, $\mathfrak{G}_n^r(V_t)$ is a fiber of

$$\pi : \mathfrak{G}_n^r \xrightarrow{\mu} J_n \xrightarrow{\tilde{\pi}} T_g.$$

We rewrite the condition of (2) of Theorem 4 in this case as follows.

Lemma. For $\xi^\nu = \{\xi_i^\nu(z_i)\} \in H^0(V, \mathcal{O}(F))$ and $\eta^\nu = \{\eta_i^\nu(z_i)\} \in H^0(V, \mathcal{O}(K_V \otimes F^{-1}))$, $0 \leq \nu \leq r$, assume that $\sum_{\nu=0}^{r} \xi^\nu \eta^\nu = 0$. Then

$$\left\{\sum_{\nu=0}^{r} \eta_i (d\xi_i^\nu / dz_i)\right\} \in H^0(V, \mathcal{O}(K_V^{\otimes 2})),$$

where z_i is a coordinate on U_i and $\{U_i\}$ is an open covering of V.

Definition. A linear system $g_n^r \in \mathfrak{G}_n^r$ is said to be weak semi-regular if, for a basis $\{\xi^0, \cdots, \xi^r\}$ of the linear subspace of $H^0(V_o, \mathcal{O}(F_o))$, $(F_o = [D], D \in g_n^r)$, and for $\eta^0, \cdots, \eta^r \in H^0(V_o, \mathcal{O}(K_{V_o} \otimes F_o^{-1}))$, the equalities

$$\begin{cases} \sum_{\nu=0}^{r} \xi^\nu \eta^\nu = 0 \in H^0(V_o, \mathcal{O}(K_{V_o})), \\ \left\{\sum_{\nu=0}^{r} \eta_i^\nu (d\xi_i^\nu / dz_i)\right\} = 0 \in H^0(V_o, \mathcal{O}(K_{V_o}^{\otimes 2})) \end{cases}$$

imply $\eta^0 = \cdots = \eta^r = 0$.

Of course, semi-regularity implies weak semi-regularity. By Theorem 4,

Theorem 11. If $g_n^r \in \mathfrak{G}_n^r$ is weak semi-regular, then g_n^r is a non-singular point of $\mathfrak{G}_n^r$ and

$$\dim_{g_n^r} \mathfrak{G}_n^r = (r+1)(n-r) - rg + 3g - 3.$$

An interesting fact is

Theorem 12. Every element of $\mathbb{G}_n^1$ is weak semi-regular. Hence $\mathbb{G}_n^1$ is non-singular and of dimension $2n+2g-5$.

Now, we consider the projection

$$\pi : \mathbb{G}_n^r \xrightarrow{\mu} J_n \xrightarrow{\tilde{\pi}} T_g.$$

It is a proper holomorphic map. Note that

$$\pi(\mathbb{G}_n^r) = \left\{ t \in T_g \;\middle|\; \text{there is a linear system } g_n^r \text{ on } V_t \right\}.$$

A famous known fact is

Theorem (Kleimann-Laksov[4], Kempf[3]). If $(r+1)(n-r) - rg \geq 0$, then $\pi(\mathbb{G}_n^r) = T_g$.

Assertion. The theorem can be proved if one finds a compact Riemann surface V and a semi-regular linear system g_n^r on V, where $n = g+r-[g/(r+1)]$.

In fact, if g_n^r is semi-regular, then $\pi^{-1}(\pi(g_n^r)) = \mathbb{G}_n^r(V)$ and $\mathbb{G}_n^r$ are non-singular at g_n^r and

$$\begin{aligned} \operatorname{codim}_{g_n^r} \pi^{-1}(\pi(g_n^r)) &= \{(r+1)(n-r) - rg + 3g - 3\} \\ &\quad - \{(r+1)(n-r) - rg\} \\ &= 3g - 3. \end{aligned}$$

Hence, by the proper mapping theorem, $\pi : \mathbb{G}_n^r \longrightarrow T_g$ is surjective. (If $\pi : \mathbb{G}_n^r \longrightarrow T_g$ is surjective, then $\pi : \mathbb{G}_m^r \longrightarrow T_g$ is also surjective for $m > n$.)

This is actually what Meis[8] did for $r = 1$. In fact, he found such V and g_n^1 as follows :

<u>Case 1 : g is even</u>. In this case, $n = (g+2)/2$. We may assume that $g \geqq 4$. Let V be a non-singular model (of the closure in $\mathbb{P}^2$) of the curve :

$$y^n = (x-1)(x-2)(x-3)(x-4)^{n-1}(x-5)^{n-1}(x-6)^{n-1}.$$

Then V has the genus g. The pencil g_n^1 determined by the meromorphic function x satisfies $h^1(2D_\infty(x)) = 0$. Hence, by (2) of Corollary to Theorem 10, g_n^1 is semi-regular.

<u>Case 2 : g is odd.</u> In this case, $n = (g+3)/2$. Let V be a non-singular model (of the closure in $\mathbb{P}^2$) of the curve :

$$y^3 = \prod_{i=1}^{n} (x-i) / \prod_{i=n+1}^{2n-2} (x-i).$$

Then V has the genus g. The pencil g_n^1 determined by the meromorphic function y satisfies $h^1(2D_\infty(y)) = 0$, so that it is semi-regular.

<u>Theorem 13.</u> Assume that $2n \geqq g+2$. Then $\pi : \mathfrak{G}_n^1 \longrightarrow T_g$ is of maximal rank at $g_n^1 \in \mathfrak{G}_n^1$ if and only if g_n^1 is semi-regular.

Finally, put

$$T_g(n) = \left\{ t \in T_g \;\middle|\; V_t \text{ has a meromorphic function of order } n \right\}.$$

Applying Corollary to Theorem 9, we get

<u>Theorem 14.</u> Let $g \geqq 2$ and let p be a prime number such that $(p-1)^2 \leqq g-1$. Then,

(1) $T_g(p)$ is an open subspace of a closed complex subspace of T_g and is of dimension $2p+2g-5$.

(2) $T_g(p)$ is singular at t if and only if $h^0(2D_\infty(f)) > 3$, where f is a meromorphic function of order p on V_t.

<u>Corollary</u>.

(1) (Rauch [10]) If $g \geq 2$, then $T_g(2)$, the hyperelliptic locus, is a non-singular closed complex subspace of T_g of dimension $2g-1$.

(2) If $g \geqq 5$, then $T_g(3)$, the locus of trigonal compact Riemann surfaces, is non-singular and of dimension $2g+1$.

(3) If $p \geqq 5$ be a prime number such that $(p-1)(2p-3) \leqq g-1$, then $T_g(p)$ is non-singular.

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