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SOME OPEN PROBLEMS IN THE STUDY OF NONCOMPACT KÄHLER MANIFOLDS

by

H. Wu

In this short note, I wish to discuss several open problems in complex analysis. Although they are specifically phrased in the language of Kähler geometry, the circle of ideas surrounding them seems to have wider ramifications. Most of these ideas were inspired by the recent papers [SY] and [GW1]. I may also point out that all the problems below, after suitable modifications, become problems concerning harmonic functions on noncompact Riemannian manifolds.

(I wish to express my sincere thanks to K. Yagi for pointing out a distressing error in the original formulation of Problem 1 below. Its present formulation is due to him.)

The first half of this discussion will be centered around two well-known theorems of H. Grauert. The first is his solution of the Levi

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problem: a complex manifold carrying a C^∞ strictly psh (abbreviation for plurisubharmonic) function is a Stein manifold. Intuitively, if more is known about the exhaustion function, then more could be said about the Stein manifold. It is worthwhile to try to convert this vague statement into real mathematics because among Stein manifolds, one should be able to distinguish between (say) bounded domains of holomorphy and $\mathbb{C}^n$. So let me begin by addressing myself to these two extreme cases. First consider a noncompact Kähler manifold M with a pole $o \in M$, i.e., $\exp: M_o \to M$ is a diffeomorphism, and let ρ denote the distance function relative to o, i.e., $\rho(x)$ = distance from x to o. Then ρ^2 is a C^∞ exhaustion function on M. Let $G = \sum_{i,j} G_{ij}\, dz^i d\bar{z}^j$ be the Kähler metric of M and let $L\rho^2 = \sum_{i,j} \frac{\partial^2 \rho^2}{\partial z^i \partial \bar{z}^j} dz^i d\bar{z}^j$ be the Levi form of ρ^2. Write $L\rho^2 \approx G$ if the eigenvalues of $L\rho^2$ with respect to G uniformly approach 1 as $\rho \to \infty$. Such is obviously the case if $M = \mathbb{C}^n$, o = the origin, and $G = \sum_i dz^i\, d\bar{z}^i$.

Problem 1. Let M be a Kähler manifold with a pole o and let G and ρ be as above. If $L\rho^2 \approx G$, is M biholomorphic to $\mathbb{C}^n$?

As background information, one has the following two theorems:

Theorem 1. Let M be a Kähler manifold with a pole and let G and ρ be as above. Then $L\rho^2 \approx G$ if for some continuous functions $K, k: [0, \infty) \to [0, \infty)$, the (sectional) curvature of M satisfies:

$$(\#)\qquad \begin{cases} -k(\rho) \leqslant \text{curvature} \leqslant K(\rho)\ , \\ \displaystyle\int_0^\infty s\ K(s)ds \leqslant 1\ , \\ \displaystyle\int_0^\infty s\ k(s)ds < \infty\ . \end{cases}$$

Theorem 2. Let M be a n-dimensional complete simply connected Kähler manifold whose (sectional) curvature satisfies the following condition:

$$(\#\#)\qquad \begin{cases} -k(\rho) \leqslant \text{curvature} \leqslant 0\ , \\ \displaystyle\int_0^\infty s\ k(s)ds < \infty\ , \\ k \text{ is monotone decreasing on } [\theta, \infty) \text{ for some } \theta > 0\ . \end{cases}$$

Then M is biholomorphic to $\mathbb{C}^n$.

Theorem 1 follows from Tneorem A together with the proofs of Lemmas 4.5 and 4.6 in [GW1] while Theorem 2 is the theorem of Siu-Yau ([SY]) with a technical improvement due to Greene-Wu([GW1], Theorem J).
If Problem 1 has an affirmative solution, then as a corollary, one would obtain a far-reaching generalization of Theorem 2, namely a Kähler manifold satisfying the assumption of Theorem 1 is biholomorphic to $\mathbb{C}^n$. I should emphasize that this generalization involves more than the superficial comparison between the two sets of curvature assumptions (#) and (##), as the following discussion will hopefully bear this out.
The proof of Theorem 2 as given in [SY] and closely followed in [GW1]

makes strong use of the fact that every point of M is a pole, and the main weight of the analysis lies in obtaining good L^2 estimates for the solution of the $\bar{\partial}$ equation on $(n, 1)$ forms. Since each point of M is a pole, sub-mean-value theorems then convert these L^2 estimates into pointwise estimates of the holomorphic n-forms so produced, and taking quotients of suitable holomorphic n-forms yields the global coordinate functions on M. The main point about the proposed generalization of Theorem 2, and hence about Problem 1, is that faced with the existence of possibly only one pole $o \in M$, one will have to dispense with this approach and will have to extract holomorphic functions of linear growth from the $\bar{\partial}$ equation on $(0, 1)$ forms <u>using only the fact that there exists one good exhaustion function</u> ρ^2. This would seem to necessitate a deeper L^2 understanding of $\bar{\partial}$, particularly as regards the influence of ρ^2 in its rôle as a weight factor on the functional solution u of $\bar{\partial}u = f$ (cf. [H]). Such an understanding would be in the spirit that Problem 1 is being posed and would ultimately be essential for a better understanding of Theorem 2 itself.

In the opposite extreme of Problem 1, consider the following. Define a function $\phi : M \to [0, 1)$ on a manifold M to be a <u>bounded exhaustion function</u> iff for every $t \in [0, 1)$, $\phi^{-1}([0, t])$ is compact. For example, the function $|z|^2$ on the unit ball in $\mathbb{C}^n$ is a C^∞ strictly psh bounded exhaustion function. A less obvious fact is that every bounded domain of holomorphy with C^∞ boundary possesses a C^∞ strictly psh bounded exhaustion function ([DF1]). On the other hand, there are bounded domains of holomorphy without any psh bounded exhaustion functions (cf. [DF2]). One would like to know in general how far the existence of a C^∞ strictly psh bounded exhaustion function ϕ goes towards controlling the function theory on a manifold M. First of all, such an M must be Stein because if χ: $[0, 1) \to [0, \infty)$ is a strictly increasing, strictly convex C^∞ function such that $\chi \uparrow \infty$, then $\chi(\phi)$ is a strictly psh exhaustion function on M and Grauert's theorem applies. Now consider:

<u>Problem 2</u>. Suppose a noncompact complex manifold M possesses a C^∞ strictly psh bounded exhaustion function. Is it complete hyperbolic (in the sense of Kobayashi [K])?

Again as background information, the following theorem was proved in [GW1] (Theorems F and G).

<u>Theorem 3</u>. Let M be a complete, simply connected, Kähler manifold whose curvature satisfies:

$$(\dagger) \qquad \text{curvature} \leqslant \frac{-A}{1+\rho^2}, \quad \text{outside a compact set,}$$

where A is a positive constant and ρ is the distance function relative to a fixed $o \in M$. Then: (i) M possesses a C^∞ strictly psh bounded exhaustion function, and (ii) M is complete hyperbolic.

In [GW1], (ii) was proved by making use of (i) <u>and</u> the curvature assumption $(\dagger)$. The thinking behind Problem 2 is that it may be possible to deduce (ii) directly from (i). But beyond that the two preceding problems should be viewed as sample questions in a <u>quantitative</u> study of Grauert's theorem. More refined questions in this study can be asked once these crude ones are answered.

The second theorem of Grauert's to be discussed is a corollary to his generalized Oka principle: a holomorphic vector bundle on a Stein manifold is holomorphically trivial if it is topologically trivial ([G]). In a concrete geometric context, it is natural and sometimes important to ask for the "best possible" holomorphic trivialization. As an example, let M be a complete, simply connected, Kähler manifold whose curvature satisfies $(\dagger)$ above. It was already remarked above that M is Stein. Since M is diffeomorphic to euclidean space by the Cartan-Hadamard theorem, the canonical bundle K of M is topologically and hence holomorphically trival. In other words, M has a nowhere zero holomorphic n-form $(n = \dim_{\mathbb{C}} M)$. On the other hand, Theorem G of [GW1] implies that M possesses many L^2 holomorphic n-forms. It is then natural to ask:

<u>Problem 3</u>. If M is a complete, simply-connected Kähler manifold whose curvature satisfies $(\dagger)$, does M possess a nowhere zero L^2 holomorphic n-form?

It is plausible that starting with an arbitrary nowhere zero holomorphic n-form on such an M, one can deform it in a canonical way to one that is also in L^2. However, there are as yet no available tools for this construction. For a discussion of Problem 3 from a different point of view, see the end of §8 in [GW1].

Suppose M is a complete, simply-connected, n-dimensional Kähler manifold with nonpositive curvature. It has been known for a long time that M is a Stein manifold ([W1]; see also Proposition 1.17 in [GW1]). But M being diffeomorphic to $\mathbb{C}^n$ (Cartan-Hadamard again), its holomorphic cotangent bundle T^*M is topologically and hence holomorphically trivial. Let $\omega_1, \ldots, \omega_n$ be holomorphic 1-forms which trivialize T^*M. Now we ask:

<u>Problem 4</u>. Can $\omega_1, \ldots, \omega_n$ be chosen to be closed 1-forms?

If so, then there exist holomorphic funcitons $f_1, \ldots, f_n$ such that $\omega_i = df_i$ for all i and hence the f_i's give a holomorphic immersion of M into $\mathbb{C}^n$. Nothing resembling this has been proved yet, but it makes sense to reflect on this problem a little. For, what is being asked is in essence whether the topology, or at least the geometry, of a Stein manifold has a direct bearing on the imbedding and immersion quesitons. In topology, questions of this type are standard and are dealt with extensively, viz. simplicity of the topology of a manifold (such as parallelizability) always assures its being imbeddable or immersible in a euclidean space of very low dimension. In the theory of Stein manifolds, basically nothing is known beyond the Remmert-Bishop-Narasimhan-Forster imbedding theorem, which is valid for <u>all</u> Stein manifolds (see [F1]), and a few other scattered results (cf. [F2], [FR]). For example, if an n-dimensional Stein manifold is homeomorphic to $\mathbb{C}^n$, there is no result on the value of the smallest integer k such that M can be properly imbedding or immersed in $\mathbb{C}^k$. Problem 4 is therefore an easier question along the same direction (insofar as a curvature assumption has been added). This whole area seems to deserve a systematic study. In any case, I hope the preceding discussion has at least demonstrated that the idea of looking for "best possible" trivializations will generate many problems.

The next problem to be discussed is in a sense the "finite form" of Theorem 2. To make this vague idea a bit clearer, recall that in classical complex function theory many theorems about entire functions were later discovered to be consequences of a corresponding statement about holomorphic functions defined on the unit disc. For example, Liouville's theorem that bounded entire functions are constant is an immediate consequence of the Schwarz lemma about bounds for a holomorphic function on the unit disc. Similarly, the small Picard theorem follows from the Schottky-Landau type theorems on holomorphic functions on the unit disc which omit three values. With this in mind, it is natural to ask if Theorem 2 cannot be deduced from some statement concerning the finite geodesic balls of M. To be specific:

Problem 5. Let M be a complete, simply-connected n-dimensional Kähler manifold of nonpositive curvature. Is every finite geodesic ball biholomorphic to a bounded domain in $\mathbb{C}^n$?

Now in addition to Theorem 2 (and the related Problem 1), there is also the open question of whether a Kähler manifold M satisfying the hypothesis of Theorem 3 would be biholomorphic with a bounded domain in $\mathbb{C}^n$ ([GW2], Question 5). One is thus tempted to put forth the following speculation.

Let M be as in Problem 5, fix an $o \in M$ and let $B(r)$ be the open geodesic ball of radius r around o. If Problem 5 can be affirmatively solved, one hopes that when condition (##) is added one can string together these bounded domains $B(r)$ as $r \to \infty$ to fill up all of $\mathbb{C}^n$, and that when condition (†) is added instead, one can string together these $B(r)$'s to fill up a bounded domain. Exactly how these technical problems can be overcome will depend on the exact nature of the solution to Problem 5. However, if all these turn out to be mathematically valid, then one would have a unified framework to understand the diverse phenomena (such as Theorems 2 and 3) on simply connected Kähler manifolds of nonpositive curvature.

The last problem I wish to discuss requires some definitions. Let M be a complete, noncompact Kähler manifold and let $\rho : M \to [0, \infty)$ be the distance function relative to a fixed point $o \in M$ as usual. For each holomorphic function f on M, define $M(f, r) = \max|f|$, where the maximum is taken over the geodesic sphere $S(r) \equiv \{p \in M: \rho(p) = r\}$. As usual (cf. [L], p. 373), define the <u>order</u> $\gamma(f)$ of f by:

$$\gamma(f) = \limsup_{r \to \infty} \frac{\log M(f, r)}{\log r} .$$

Thus $0 \leqslant \gamma(f) \leqslant \infty$. From the maximum principle for holomorphic functions, it is straightforward to show that the definition is independent of the choice of $o \in M$. Fix M and let f vary over all nonconstant holomorphic functions f; the question is: what is the smallest value of $\gamma(f)$? To this end, define the <u>function class</u> $\kappa(M)$ of M by:

$$\kappa(M) = \inf \gamma(f) ,$$

where the infimum is taken over all nonconstant holomorphic functions. If $\mathbb{C}$ is equipped with the usual flat metric $dz\, d\bar{z}$, then $\kappa(\mathbb{C}) = 1$. If Δ denotes the unit disc in $\mathbb{C}$ equipped with the Poincaré metric, then $\kappa(\Delta) = 0$. In general, $\kappa(M)$ can assume any value in $[0, \infty]$; this can be seen by choosing suitable functions $h(r)$ on $[0, \infty)$ and constructing complete Hermitian metrics $\{\exp h(|z|)\} dz d\bar{z}$ on $\mathbb{C}$ (note that if $h(r)$ is convex then these metrics have nonpositive curvature and $\kappa(M) \in [0, 1]$). Now the first question concerning $\kappa(M)$ is:

Problem 6. Let M, N be complete Kähler manifolds with poles x, y respectively. Suppose the curvatures of M and N do not change sign, i.e., the curvature of M is either nonpositive or nonnegative everywhere, and the same for N. If for all $x' \in M$ and for all $y' \in N$ which satisfy $d_M(x, x') = d_N(y, y')$,

$$\text{curvature}_M(x') \leqslant \text{curvature}_N(y') ,$$

then is it true that $\kappa(M) \leqslant \kappa(N)$? When does strict inequality hold?

One should note that in Problem 6, the assumption that the curvatures of both M and N keep a sign is necessary as there are easy counter-examples otherwise. However, the need for the existence of poles for M

and N is less clear, but I will assume it to facilitate the discussion below. Now the philosophy behind this problem is that "the more positive the curvature, the faster each holomorphic function must grow." Let me present the supporting evidence for this. First of all, if the curvature of M is quite negative, say, M satisfies the hypothesis of Theorem 3 and in particular (+), then there is prevalent belief that M will carry nonconstant bounded holomorphic functions (see discussion after Problem 5). Thus most likely, $\kappa(M) = 0$ in this case. Now let the curvature of M get less negative, e.g., M satisfies the hypothesis of Theorem 2 and in particular (##); then the proofs of [SY] and [GW1] yield holomorphic functions that are "almost" of linear growth, in the following sense. Given $\varepsilon > 0$, there exists a holomorphic function f on M such that $|f| \leqslant C(1 + \rho)^{1+\varepsilon}$ for some constant C. Thus $\kappa(M) = 1$, and the function class does increase with less negative curvature. (It is actually conjectured that there are functions of linear growth in this case; see [GW1], §9.) Next, suppose M has everywhere nonnegative curvature (in addition to possessing a pole); then indeed $\kappa(M) \geqslant 1$ as expected. The latter assertion is implied by a theorem of S.T. Yau ([Y], Corollary to Theorem 4′) on the growth of harmonic functions on manifolds of nonnegative Ricci curvature. Here, we are making use of the fact that a holomorphic function on a Kähler manifold is harmonic (cf. e.g., [GW1], §1) so that Yau's theorem is applicable. Without entering into a detailed discussion, I wish to interpose a remark at this point that it is actually more natural to rephrase Problem 6 in the context of harmonic functions on Riemannian manifolds. Consider finally the case where M has everywhere positive curvature, then it is likely that $\kappa(M) > 1$. One piece of evidence for this comes from an observation due to R. Schoen. He showed with a simple argument that if a 2-dimensional Riemannian manifold M with positive curvature satisfies the addition assumptions that (i) every point of M is a pole, and (ii) along each geodesic ray $\gamma: [0, \infty) \to M$, the curvature function satisfies $\int_0^\infty s\, K(s)ds \leqslant 1$, then every nonconstant harmonic function must have faster-than-linear growth. (There are manifolds satisfying these assumptions.) The proof makes use of Theorem 1.

In any case, I hope that the foregoing array of facts and conjectures has been shown to form a pattern consistent with Problem 6. If one of the chief aims of geometric function theory is to understand the interplay between curvature and holomorphic functions, Problem 6 should be one of the questions that must be answered. Let me close with the remark that if Problem 6 has an affirmative answer. then it follows that any simply connected, complete Kähler manifold M satisfying curvature $\leqslant -c$ (c is a positive constant) must have $\kappa(M) = 0$; for one can compare M with the unit ball whose Bergman metric has been normalized to satisfy $-c/4 \geqslant$ curvature $\geqslant -c$. This is not quite as strong as saying that M possesses bounded nonconstant holomorphic functions, yet even this much is not known. It has been more than twelve years since I first posed the question of whether or not such an M possesses nonconstant bounded holomorphic functions ([W2]). I believe it does, but I am sorry to say that, after so much work in the meantime, I am still no closer to the solution.

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University of California
Berkeley, Calif. 94720.
U.S.A.