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Non-existence of bounded functions on the universal covering of \(\mathbb{P}^n - n+2\) hyperplanes in general position \((n \geq 2)\).

by Isao Wakabayashi

Let \(X\) be the complement of \(n+2\) hyperplanes in general position in \(\mathbb{P}^n\), and let \(\tilde{\pi}: \tilde{X} \to X\) be the universal covering of \(X\). Then we have

**THEOREM 1.** If \(n \geq 2\), there are no bounded, holomorphic and non-constant functions on \(\tilde{X}\). \((\text{[3]}\))

We derive this theorem from the following theorem.

**THEOREM 2.** There are no bounded, holomorphic and non-constant functions on the "homological covering" of \(\mathbb{P}^1 - 3\) points. \((\text{[4]}\))

1°) RELATION BETWEEN THE UNIVERSAL COVERING \(\tilde{X}\) AND THE HOMOLOGICAL COVERING OF \(\mathbb{P}^1 - 3\) POINTS.

**DEFINITION.** The homological covering of a variety is the covering determined by the commutator group of the fundamental group of the variety.

**Notation.** We denote by \(R\) the homological covering of \(D = \mathbb{P}^1 - \{0, 1, \infty\}\).

Let \(z_1, \ldots, z_n\) be the inhomogeneous coordinates of \(\mathbb{P}^n\). We may suppose that the \(n+2\) hyperplanes in general position which we take away from \(\mathbb{P}^n\) are \(\{z_1 = 0\}, \ldots, \{z_n = 0\}, \{z_1 + \cdots + z_n - 1 = 0\}\) and the infinite hyperplane. Let \(L(z_1, \ldots, z_n)\) be the complex line consisting of the points whose \(j\)-th coordinate is equal to \(z_j\), \(j = 1, \ldots, i-1, i+1, \ldots, n\).
Let us suppose that \( z_1 + \cdots + z_{i-1} + z_{i+1} + \cdots + z_n - 1 \neq 0 \), then by a topological observation we have

**PROPOSITION.** Every connected component of the inverse image of \( L(z_1, \ldots, z_n) \) by \( \pi \) is holomorphically isomorphic to the homological covering \( R \) of \( D \).

From this proposition and Theorem 2, the proof of Theorem 1 is immediate, because there are hence in \( \tilde{X} \) sufficiently many analytic subsets admitting no bounded functions.

2°) **TRIANGULATION OF** \( R \).

Let us pose \( \Delta_1 = \{ z \in \mathbb{C} \mid \Im z > 0, z \neq 0, 1 \} \) and \( \Delta_2 = \{ z \in \mathbb{C} \mid \Im z \leq 0, z \neq 0, 1 \} \). Let \( A \) be the segment \((0, 1)\), \( B \) be the segment \((1, \infty)\) and let \( C \) be the segment \((\infty, 0)\). We consider \( \Delta_1 \) and \( \Delta_2 \) as triangles with 3 sides \( A \), \( B \) and \( C \). The inverse image \( \pi^{-1}(\Delta_1) \) of \( \Delta_1 \) (resp. \( \pi^{-1}(\Delta_2) \) of \( \Delta_2 \)) by the projection \( \pi: R \to D \) consists of an infinite number of triangles which are holomorphically isomorphic to \( \Delta_1 \) (resp. \( \Delta_2 \)). We number the triangles of \( \pi^{-1}(\Delta_1) \) and those of \( \pi^{-1}(\Delta_2) \) as follows: Let \( p \) be a point in \( \Delta_1 \). Let \( \alpha \) (resp. \( \beta \)) be a closed curve of origin \( p \) and of end point \( p \) winding the point \( 0 \) (resp. the point 1) in the positive sense. Then every element of \( \pi_1(D, p)/[\pi_1(D, p), \pi_1(D, p)] \) is represented uniquely by a curve of the form \( \alpha^i \beta^j \) \((i, j \in \mathbb{Z})\). This element determines a point \( p^{i,j} \) situated over \( p \), so we denote by \( \Delta_1^{i,j} \) the triangle of \( \pi^{-1}(\Delta_1) \) which contains \( p^{i,j} \). We further denote by \( \Delta_2^{i,j} \) the triangle of \( \pi^{-1}(\Delta_2) \) whose side \( B \) is identical with the side \( B \) of \( \Delta_1^{i,j} \).
3°) TOPOLOGY OF $\mathbb{R}$.

We prepare a countable number of triangles $\Delta_{1}^{i,j}$ and $\Delta_{2}^{i,j}$ ($i,j \in \mathbb{Z}$). We place them as Fig. 1.

![Fig. 1.](image)

We consider each pair of sides which are situated face to face in Fig. 1. We identify them oppositely so that the given orientations coincide. Let us denote by $\mathbb{R}'$ the surface constructed in this way. Then we can prove that the triangles $\Delta_{1}^{i,j}$ and $\Delta_{2}^{i,j}$ are patched in $\mathbb{R}$ exactly in the same way as the triangles $\Delta_{1}^{i,j}$ and $\Delta_{2}^{i,j}$ in $\mathbb{R}'$. Hence

PROPOSITION. $\mathbb{R}$ is homeomorphic to $\mathbb{R}'$.

4°) CRITERION OF A. PFLUGER.

To show Theorem 2 we use a criterion of A. Pfluger which gives a sufficient condition for a Riemann surface to belong to the class $O_{AB}$, i.e., the class of surfaces with no bounded holomorphic functions.

Let $W$ be an open Riemann surface and let $\{A_{n}^{k}\}, k=1,2,\ldots$, $k(n) < \infty$, $n=1,2,\ldots$, be a family of doubly connected domains in $W$
satisfying the following conditions:

1. \( A_n^k \) is bounded by two closed curves \( a_n^k \) and \( a'_n^k \),
2. \( A_n^k \cap A_{n'}^{k'} = \emptyset \) if \((k,n) \neq (k',n')\),
3. the complement of \( \bigcup_{k=1}^{K(n)} A_n^k \) in \( W \) has a unique compact connected component \( B_n \),
4. \( B_n \) is bounded by \( k(n) \) curves \( \{a_n^k\}_{k=1}^{K(n)} \) and contains all \( A_n^{k'} \) such that \( n' < n \).

Let us denote by \( \mu_n^k \) the harmonic module of \( A_n^k \), where the harmonic module of a domain conformally equivalent to \( \{z \in \mathbb{C} \mid r < |z| < r'\} \) is by definition \( \log r'/r \). We pose

\[ \mu_n = \min_{k} \mu_n^k, \text{ and } K(N) = \max_{n \leq N} k(n). \]

CRITERION (A. Pfluger). ([1], [2]) If

\[ \lim_{N \to \infty} \left\{ - \frac{1}{2} \log K(N) \right\} = \infty, \]

then \( W \in O_{AB} \).

5°) CONSTRUCTION OF POLYGONES \( \{P_n\} \), AND ITS BOUNDARY.

We construct on our surface \( R \) a family of polygones \( \{P_n\} \) such that

(i) each \( P_n \) consists of a finite number of triangles in 2°,
(ii) \( P_n \subset P_{n+1} \),
(iii) \( \bigcup_{n=1}^{\infty} P_n = R \).

Actually \( P_n \) is constructed as Fig.2.

We find that the boundary of \( P_n \) consists of \( 2n+1 \) connected
Figure 2.
components. For each connected component of the boundary of $P_n$, we describe a closed curve $\gamma^k_n$ ($k=-n-1, \ldots, n-1$) which runs sufficiently near to it. \{\gamma^k_n\} are given in Fig. 2.

6°) CONSTRUCTION OF DOUBLY CONNECTED DOMAINS \{\Lambda^k_n\}.

Let $r$, $r'$ be two real numbers such that $r < r'$. Let us denote by $\Delta^i,j_v(0,r,r')$ the part of $\Delta^i,j_v$ ($\nu=1,2$) lying over \{\[z \in \mathbb{C} \mid r < |z| < r'\]}. Let $\Delta^i,j_v(1,r,r')$ be the part of $\Delta^i,j_v$ lying over \{\[z \in \mathbb{C} \mid |z-1| < r'\]}. We pose $\Delta^i,j_v(\omega, r', r) = \Delta^i,j_v(0, r, r')$.

Let $\Delta^i,j_v(O, r; 1, r'; \omega, r'')$ be the complement in $\Delta^i,j_v$ of the part lying over the set \{|$z| < r\} \cup \{|z-1| < r'\} \cup \{|z| > r''\}.

We construct $\Lambda^k_n$ as the union of the following sets. We take the set of the form $\Delta^i,j_v(0, r, r')$ (resp. $\Delta^i,j_v(1, r, r')$ and $\Delta^i,j_v(\omega, r', r)$) if $\Delta^i,j_v$ is situated in the interior of $P_n$ and if $\gamma^k_n$ passes near $O$ (resp. 1 and $\omega$) in $\Delta^i,j_v$. And we take the set of the form $\Delta^i,j_v(0, r; 1, r'; \omega, r'')$ if $\gamma^k_n$ passes the triangle $\Delta^i,j_v$ which is situated on the boundary of $P_n$. For example, in $\Delta^i_{-k-1}(\text{resp. } \Delta^{i+1}_{2n+1-k})$, for $-n \leq i \leq n-1$, we take the part $\Delta^i_{-k-1}(0, r_{n+1}, r_n)$ (resp. $\Delta^{i+1}_{2n+1-k}(0, r_{n+1}, r_n)$), where $r_n = Ce^{-2\pi n}$ ($C > 0$). And in $\Delta^{n+1}_{2n+1-k}$ (resp. $\Delta^{n-k-1}_1$), we take the part $\Delta^{n+1}_{2n+1-k}(0, r_{n+1}; 1, \epsilon; \omega, 1/r_n-k+1)$ (resp. $\Delta^{n-k-1}_1(0, r_{n+1}; 1, \epsilon; \omega, 1/r_n-k-1)$) with $1 \gg \epsilon > 0$. We put $\Lambda^k_n$ the union of all these sets.
7°) HARMONIC MODULE OF $A_n^k$.

We divide $A_n^k$ into some suitable parts, so that we have rectangles as their images by the map $w = \log z$ or $w = \log (z-1)$. We see that the width of each rectangle is $2\pi$ and the total length of all these rectangles is almost equal to $24\pi n$. Here $24\pi n$ is obtained as follows: $A_n^k$ passes $12n+1$ triangles among those which are in the interior of $P_n^k$. The intersection of $A_n^k$ and each triangle of this kind is mapped by the above map to a rectangle of width $2\pi$ and of length $\pi$. So the total length of the intersection of $A_n^k$ and all triangles in the interior of $P_n$ is almost equal to $12\pi n$. On the other hand, we see that the total length of the intersection of $A_n^k$ and all the triangles which are situated on the boundary of $P_n$ is almost equal to $12\pi n$. $24\pi n$ is the sum of these quantities.

We can verify that these rectangles are patched together only slightly twisted and so $A_n^k$ is almost like an annulus of width $2\pi$ and of length $24\pi n$ when we consider its harmonic module. Consequently the harmonic module $\mu_n^k$ of $A_n^k$ is almost equal to

$$\log \frac{12n+2\pi}{12n},$$

hence $\mu_n^k = \min_k \mu_n^k$ is almost equal to $\log \frac{12n+2\pi}{12n}$.

Since $K(N) = \max k(n) = 2N+1$, and $\pi/6 = 0.52\cdots > 1/2$, we obtain

$$\lim_{N \to \infty} \left\{ \sum_{n=1}^{N} \mu_n^k - \frac{1}{2} \log K(N) \right\} \geq \lim_{N \to \infty} \left\{ \sum_{n=1}^{N} \frac{\pi}{6n} - \frac{1}{2} \sum_{n=1}^{N} \frac{\pi}{6n}^2 \right\} - \frac{1}{2} \log (2N+1)$$

$$\geq \lim_{N \to \infty} \left\{ \frac{\pi}{6} \log N - \frac{1}{2} \log (2N+1) - \frac{1}{2} \sum_{n=1}^{N} \frac{\pi}{6n}^2 \right\} = \infty.$$}

Hence by virtue of the Pfluger's criterion, our Riemann surface $R$ belongs to the class $O_{AB}$ as required.
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