Non-Existence of Bounded Functions on the Universal Covering of $P^n$-n+2 Hyperplanes in General Position ($n>2$) (Geometric Theory of Several Complex Variables)

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Non-existence of bounded functions on the universal covering of $\mathbb{P}^{n+2}$ hyperplanes in general position $(n \geq 2)$.

by Isao Wakabayashi

Let $X$ be the complement of $n+2$ hyperplanes in general position in $\mathbb{P}^{n}$, and let $\pi: \widetilde{X} \rightarrow X$ be the universal covering of $X$. Then we have

THEOREM 1. If $n \geq 2$, there are no bounded, holomorphic and non-constant functions on $\widetilde{X}$. ([3])

We derive this theorem from the following theorem.

THEOREM 2. There are no bounded, holomorphic and non-constant functions on the "homological covering" of $\{\mathbb{P}^{1} - 3 \text{ points}\}$. ([4])

1°) RELATION BETWEEN THE UNIVERSAL COVERING $\widetilde{X}$ AND THE HOMOLOGICAL COVERING OF $\{\mathbb{P}^{1} - 3 \text{ points}\}$.

DEFINITION. The homological covering of a variety is the covering determined by the commutator group of the fundamental group of the variety.

Notation. We denote by $R$ the homological covering of $D=\mathbb{P}^{1}-\{0, \infty\}$.

Let $z_{1}, \ldots, z_{n}$ be the inhomogeneous coordinates of $\mathbb{P}^{n}$. We may suppose that the $n+2$ hyperplanes in general position which we take away from $\mathbb{P}^{n}$ are $\{z_{1}=0\}, \ldots, \{z_{n}=0\}, \{z_{1}+\cdots+z_{n-1}=0\}$ and the infinite hyperplane. Let $L(z_{1}, \ldots, z_{n})$ be the complex line consisting of the points whose $j$-th coordinate is equal to $z_{j}$, $j=1, \ldots, i-1, i+1, \ldots, n$. 

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Let us suppose that $z_1 + \ldots + z_{i-1} + z_{i+1} + \ldots + z_n - 1 \neq 0$, then by a topological observation we have

**PROPOSITION.** Every connected component of the inverse image of $L(z_1, \ldots, z_n)$ by $\pi$ is holomorphically isomorphic to the homological covering $R$ of $D$.

From this proposition and Theorem 2, the proof of Theorem 1 is immediate, because there are hence in $\tilde{X}$ sufficiently many analytic subsets admitting no bounded functions.

2°) **TRIANGULATION OF R.**

Let us pose $\Delta_1 = \{z \in \mathbb{C} \mid \text{Im } z > 0, z \neq 0, 1\}$ and $\Delta_2 = \{z \in \mathbb{C} \mid \text{Im } z \leq 0, z \neq 0, 1\}$. Let $A$ be the segment $(0, 1)$, $B$ be the segment $(1, \infty)$ and let $C$ be the segment $(\infty, 0)$. We consider $\Delta_1$ and $\Delta_2$ as triangles with 3 sides $A$, $B$ and $C$. The inverse image $\pi^{-1}(\Delta_1)$ of $\Delta_1$ (resp. $\pi^{-1}(\Delta_2)$ of $\Delta_2$) by the projection $\pi: R \to D$ consists of an infinite number of triangles which are holomorphically isomorphic to $\Delta_1$ (resp. $\Delta_2$). We number the triangles of $\pi^{-1}(\Delta_1)$ and those of $\pi^{-1}(\Delta_2)$ as follows: Let $p$ be a point in $\Delta_1$. Let $\alpha$ (resp. $\beta$) be a closed curve of origin $p$ and of end point $p$ winding the point $0$ (resp. the point $1$) in the positive sense. Then every element of $\pi_1(D, p)/[\pi_1(D, p), \pi_1(D, p)]$ is represented uniquely by a curve of the form $\alpha^i\beta^j$ ($i, j \in \mathbb{Z}$). This element determines a point $p^{i,j}$ situated over $p$, so we denote by $\Delta_1^{i,j}$ the triangle of $\pi^{-1}(\Delta_1)$ which contains $p^{i,j}$. We further denote by $\Delta_2^{i,j}$ the triangle of $\pi^{-1}(\Delta_2)$ whose side $B$ is identical with the side $B$ of $\Delta_1^{i,j}$. 
3°) TOPOLOGY OF $R$.

We prepare a countable number of triangles $\Delta^i,j_1$ and $\Delta^i,j_2$ $(i,j \in \mathbb{Z})$. We place them as Fig. 1.

![Fig. 1.](image)

We consider each pair of sides which are situated face to face in Fig. 1. We identify them oppositely so that the given orientations coincide. Let us denote by $R'$ the surface constructed in this way. Then we can prove that the triangles $\Delta^i,j_1$ and $\Delta^i,j_2$ are patched in $R$ exactly in the same way as the triangles $\Delta^i,j_1$ and $\Delta^i,j_2$ in $R'$. Hence

PROPOSITION. $R$ is homeomorphic to $R'$.

4°) CRITERION OF A. PFLUGER.

To show Theorem 2 we use a criterion of A. Pfluger which gives a sufficient condition for a Riemann surface to belong to the class $O_{AB}$, i.e., the class of surfaces with no bounded holomorphic functions.

Let $W$ be an open Riemann surface and let $\{A^k_n\}$, $k=1,2,\ldots$, $k(n)<\infty$, $n=1,2,\ldots$, be a family of doubly connected domains in $W$
satisfying the following conditions:

1. \( A_n^k \) is bounded by two closed curves \( a_n^k \) and \( a_n' \),

2. \( A_n^k \cap A_n'^n = \emptyset \) if \( (k,n) \neq (k',n') \),

3. the complement of \( \bigcup_{k=1}^{k(n)} A_n^k \) in \( W \) has a unique compact connected component \( B_n \),

4. \( B_n \) is bounded by \( k(n) \) curves \( \{a_n^k\}_{n=1}^{k(n)} \) and contains all \( A_n^k \) such that \( n' < n \).

Let us denote by \( \mu_n^k \) the harmonic module of \( A_n^k \), where the **harmonic module** of a domain conformally equivalent to \( \{z \in \mathbb{C} \mid r < |z| < r'\} \) is by definition \( \log r'/r \). We pose

\[
\mu_n = \min_{k} \mu_n^k, \quad K(N) = \max_{n \leq N} k(n).
\]

**CRITERION (A. Pfluger).** ([1], [2]) If

\[
\lim_{N \to \infty} \left\{ \sum_{n=1}^{N} \mu_n^k \frac{1}{c} \log K(N) \right\} = \infty,
\]

then \( W \in O_{AB} \).

5°) CONSTRUCTION OF POLYGONES \( \{P_n\} \), AND ITS BOUNDARY.

We construct on our surface \( R \) a family of polygones \( \{P_n\} \) such that

(i) each \( P_n \) consists of a finite number of triangles in 2°,

(ii) \( P_n \subset P_{n+1} \),

(iii) \( \bigcup_{n=1}^{\infty} P_n = R \).

Actually \( P_n \) is constructed as Fig.2.

We find that the boundary of \( P_n \) consists of \( 2n+1 \) connected
Figure 2.
components. For each connected component of the boundary of \( P_n \),
we describe a closed curve \( \gamma^k_n \) \((k=-n-1, \cdots, n-1)\) which runs sufficiently near to it.
\( \{\gamma^k_n\} \) are given in Fig. 2.

6°) CONSTRUCTION OF DOUBLY CONNECTED DOMAINS \( \{\Lambda^k_n\} \).

Let \( r, r' \) be two real numbers such that \( r < r' \). Let us denote
by \( \Delta^{i,j}_v (0,r,r') \) the part of \( \Delta^{i,j}_v \) \((\Psi=1,2)\) lying over \( \{z \in \mathbb{C} \mid r < |z| < r'\} \). Let \( \Delta^{i,j}_v (1,r,r') \) be the part of \( \Delta^{i,j}_v \)
lying over \( \{z \in \mathbb{C} \mid r < |z-1| < r'\} \). We pose \( \Delta^{i,j}_v (\omega, r', r) = \Delta^{i,j}_v (0, r, r') \).
Let \( \Delta^{i,j}_v (0, r; 1, r'; \omega, r'') \) be the complement in \( \Delta^{i,j}_v \) of the part
lying over the set \( \{|z| < r\} \cup \{|z-1| < r'\} \cup \{|z| > r''\} \).

We construct \( \Lambda^k_n \) as the union of the following sets. We take the set of the form \( \Delta^{i,j}_v (0, r, r') \) \((\text{resp. } \Delta^{i,j}_v (1, r, r'))\) if \( \Delta^{i,j}_v \)
is situated in the interior of \( P_n \) and if \( \gamma^k \) passes near \( 0 \) \((\text{resp. } 1 \text{ and } \infty)\) in \( \Delta^{i,j}_v \). And we take the set of
the form \( \Delta^{i,j}_v (0, r; 1, r'; \omega, r'') \) if \( \gamma^k \) passes the triangle \( \Delta^{i,j}_v \)
which is situated on the boundary of \( P_n \). For example, in \( \Delta^{i,-k-1}_1 \) \((\text{resp. } \Delta^{i+1,-k}_2)\) for \( -n \leq i \leq n-1 \), we take the part
\( \Delta^{i,-k-1}_1 (0, r_{n+1}, r_n) \) \((\text{resp. } \Delta^{i+1,-k}_2 (0, r_{n+1}, r_n))\), where \( r_n = C e^{-2\pi n} \)
\((C > 0)\). And in \( \Delta^{n+1,-k}_2 \) \((\text{resp. } \Delta^{n,-k-1}_1)\), we take the part
\( \Delta^{n+1,-k}_2 (0, r_{n+1}; 1, \varepsilon; \omega, 1/r_{n-k+1}) \) \((\text{resp. } \Delta^{n,-k-1}_1 (0, r_{n+1}; 1, \varepsilon; \omega, 1/r_{n-k-1}))\)
with \( 1 \gg \varepsilon > 0 \). We put \( \Lambda^k_n \) the union of all these sets.
7°) HARMONIC MODULE OF $A_n^k$.

We divide $A_n^k$ into some suitable parts, so that we have rectangles as their images by the map $w = \log z$ or $w = \log (z-1)$. We see that the width of each rectangle is $2\pi$ and the total length of all these rectangles is almost equal to $24\pi n$. Here $24\pi n$ is obtained as follows: $A_n^k$ passes $12n+1$ triangles among those which are in the interior of $P_n$. The intersection of $A_n^k$ and each triangle of this kind is mapped by the above map to a rectangle of width $2\pi$ and of length $\pi$. So the total length of the intersection of $A_n^k$ and all triangles in the interior of $P_n$ is almost equal to $12\pi n$. On the other hand, we see that the total length of the intersection of $A_n^k$ and all the triangles which are situated on the boundary of $P_n$ is almost equal to $12\pi n$. $24\pi n$ is the sum of these quantities.

We can verify that these rectangles are patched together only slightly twisted and so $\mu_n^k$ is almost like an annulus of width $2\pi$ and of length $24\pi n$ when we consider its harmonic module. Consequently the harmonic module $\mu_n^k$ of $A_n^k$ is almost equal to

$$\log \frac{12n+2\pi}{12n},$$

hence $\mu_n^k = \min_k \mu_n^k$ is almost equal to $\log \frac{12n+2\pi}{12n}$.

Since $K(N) = \max k(n) = 2N+1$, and $\pi/6 = 0.52\cdots > 1/2$, we obtain

$$\lim_{N \to \infty} \left\{ \sum_{n=1}^{N} \frac{n}{6n} - \frac{1}{2} \log \frac{N}{6n} \right\} + \lim_{N \to \infty} \left\{ \frac{N}{6n} \sum_{n=1}^{N} \left( \frac{1}{6n} \right)^2 - \frac{1}{2} \log (2N+1) \right\}$$

$$\geq \lim_{N \to \infty} \left\{ \frac{N}{6} \log N - \frac{1}{2} \log (2N+1) - \frac{1}{2} \sum_{n=1}^{N} \left( \frac{1}{6n} \right)^2 \right\} = \infty.$$  

Hence by virtue of the Pfluger's criterion, our Riemann surface $R$ belongs to the class $O_{AB}$ as required.
BIBLIOGRAPHIE


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