Non-null Distribution of the Largest
Latent Root in Multivariate Analysis

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1. **Summary.** A study is made of the non-null
cumulative distribution function (cdf) of the
largest latent root which arises in testing of
equality of two covariance matrices. This
result in the null case reduces to the formula
given by Sugiyama [14]. The non-central cdf
of the largest latent root in the MANOVA
situation is also discussed.

2. **Non-null distribution of the largest latent root for
test of equality of two covariance matrices.**

Let S_1 and S_2 be independently distributed as
Wishart $W(p,n_1,\Sigma_1)$ and $W(p,n_2,\Sigma_2)$, respectively. Let
the latent roots of $S_1 S_2^{-1}$ and $\Sigma_1 \Sigma_2^{-1}$ be $f_1,\ldots,f_p$ and
$\omega_1,\ldots,\omega_p$, respectively, such that $\infty > f_1 > \ldots > f_p > 0$
and $\infty > \omega_1 \geqslant \ldots \geqslant \omega_p > 0$. Then the distribution of the
latent roots f_i depends on the latent roots ω_i and is

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(1) $g(f_1,\ldots,f_p)$

$= \text{Const } {}_1F_0[\tfrac{1}{2}(n_1+n_2);-\underset{\sim}{\Omega}^{-1},\underset{\sim}{F}]\,|\underset{\sim}{F}|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}\prod_{i<j}(f_i-f_j)$

$$[\text{cf.}[7],\text{eq.}65]$$

where $\underset{\sim}{F} = \text{diag}(f_i)$, $\underset{\sim}{\Omega} = \text{diag}(\omega_i)$,

$\text{Const}=|\underset{\sim}{\Omega}|^{-n_1/2}[\pi^{p^2/2}\Gamma_p(\tfrac{1}{2}(n_1+n_2))/\Gamma_p(\tfrac{1}{2}p)\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2)]$,

and

$${}_1F_0(\tfrac{1}{2}(n_1+n_2);-\underset{\sim}{\Omega}^{-1},\underset{\sim}{F})=\int_{O(p)} {}_1F_0(\tfrac{1}{2}(n_1+n_2);-\underset{\sim}{\Omega}^{-1}\underset{\sim}{H}\underset{\sim}{F}\underset{\sim}{H}')(d\underset{\sim}{H}).$$

Since

(2) ${}_2F_1(\alpha,\beta;\gamma;\underset{\sim}{Z})$

$= |\underset{\sim}{I}-\underset{\sim}{Z}|^{-\beta}\,{}_2F_1(\gamma-\alpha,\beta;\gamma;-\underset{\sim}{Z}(\underset{\sim}{I}-\underset{\sim}{Z})^{-1})$,

then ${}_1F_0(\tfrac{1}{2}(n_1+n_2);-\underset{\sim}{\Omega}^{-1}\underset{\sim}{H}\underset{\sim}{F}\underset{\sim}{H}')$

$= {}_2F_1(\tfrac{1}{2}(n_1+n_2),a;a;-\underset{\sim}{\Omega}^{-1}\underset{\sim}{H}\underset{\sim}{F}\underset{\sim}{H}')$ is defined for $\underset{\sim}{F} > 0$

and $\underset{\sim}{\Omega} > 0$, whenever $a > \tfrac{1}{2}(p-1)$.

Formula (1) may be rewritten as

(3) $f(f_1,\ldots,f_p) = \text{Const} \sum_{k=0}^{\infty}\sum_{\kappa}[(-1)^k(\tfrac{1}{2}(n_1+n_2))_\kappa/k!C_\kappa(\underset{\sim}{I})]$

$\cdot\int_{O(p)} |\underset{\sim}{F}|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}C_\kappa(\underset{\sim}{H}'\underset{\sim}{\Omega}^{-1}\underset{\sim}{H}\underset{\sim}{F})\prod_{i<j}(f_i-f_j)(d\underset{\sim}{H}).$

Using the result

(4) $\int_{x>\lambda_1>\ldots>\lambda_p>0}|\underset{\sim}{D}_\lambda|^{t-(p+1)/2}C_\kappa(\underset{\sim}{R}\underset{\sim}{D}_\lambda)\prod_{i<j}(\lambda_i-\lambda_j)\prod_{i=1}^{p}d\lambda_i$

$= x^{pt+k}\left(\frac{\Gamma_p(\tfrac{1}{2}p)}{\pi^{p^2/2}}\right)\frac{\Gamma_p(t,\kappa)\Gamma_p(\tfrac{1}{2}(p+1))}{\Gamma_p(t+\tfrac{1}{2}(p+1),\kappa)}\,C_\kappa(\underset{\sim}{R})$

$$[\text{cf.}[2]\text{eq.}22 \text{ and } [13]\text{eq.}2.1]$$

where $D_\lambda = \text{diag}(\lambda_i)$, $\Gamma_p(a;\kappa) = \Gamma_p(a)(a)_\kappa$,

and R is a positive definite symmetric matrix,

we obtain the cdf of the largest latent root f_1 from

(3)

$$P(f_1 < x) = [\Gamma_p(\tfrac{1}{2}(n_1+n_2))\Gamma_p(\tfrac{1}{2}(p+1))/\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))]$$

(5)

$$\cdot |x\Omega^{-1}|^{\frac{1}{2}n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\tfrac{1}{2}(n_1+n_2))_\kappa(\tfrac{1}{2}n_1)_\kappa C_\kappa(-x\Omega^{-1})}{(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))_\kappa k!} .$$

Applying (2) to (5) we get

(6) $\quad P(f_1 < x) = [\Gamma_p(\tfrac{1}{2}(n_1+n_2))\Gamma_p(\tfrac{1}{2}(p+1))/\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))]$

$$\cdot |x\Omega^{-1}(I+x\Omega^{-1})^{-1}|^{n_1/2}$$

$$\cdot {}_2F_1(\tfrac{1}{2}(-n_2+p+1),\tfrac{1}{2}n_1;\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1);x\Omega^{-1}(I+x\Omega^{-1})^{-1}).$$

When $\Omega=I$, (6) reduces to the expression given by
Sugiyama [14]   (In [14] eq.1.7 and 1.8, $1/(1+y)$ is correct-
ed to $y/(1+y)$). We note from the generalization of the
Gauss formula ([14], eq.1.13) that $P(f_1<x) \to 1$ when $x \to \infty$.

Pillai and Sugiyana [12] derived (6) in terms of a doubly
infinite sum, and Khatri [9] expressed the probability
density function by a ${}_3F_2$ hypergeometric function.
Under the assumption that $\tfrac{1}{2}(n_2-p-1)$ is an integer,
Constantine and Venables [4] gave another polynomial
expression.

Since

$${}_2F_1(\alpha,\beta;\gamma;Z) = |I-Z|^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;Z),$$

(6) may be rewritten as a positive series by

(7) $\quad P(f_1<x)=[\Gamma_p(\tfrac{1}{2}(n_1+n_2))\Gamma_p(\tfrac{1}{2}(p+1))/\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))]$

$$\cdot |x\Omega^{-1}|^{n_1/2}|I+x\Omega^{-1}|^{-(n_1+n_2)/2}$$

$$\cdot {}_2F_1(\tfrac{1}{2}(n_1+n_2),\tfrac{1}{2}(p+1);\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1);x\Omega^{-1}(I+x\Omega^{-1})^{-1}).$$

Theorem 1. Let $\underset{\sim}{S}_1$ and $\underset{\sim}{S}_2$ be independently distributed as Wishart $W(p,n_1,\underset{\sim}{\Sigma}_1)$ and $W(p,n_2,\underset{\sim}{\Sigma}_2)$ respectively. Then the cdf of the largest latent root of $|\underset{\sim}{S}_1-f\underset{\sim}{S}_2| = 0$ is given by (6), or equivalently (7).

3. Non-central distribution of the largest root in the MANOVA case.

Let $\underset{\sim}{X}$ be a $(p\times n_1)$ matrix variate $(p\leqslant n_1)$ and $\underset{\sim}{Y}$ a $(p\times n_2)$ matrix variate $(p\leqslant n_2)$ and the columns be all independently normally distributed with covariance matrix $\underset{\sim}{\Sigma}$, $E(\underset{\sim}{X}) = \underset{\sim}{M}$ and $E(\underset{\sim}{Y}) = \underset{\sim}{O}$. Then the distribution of the latent roots f_i $(\infty>f_1>...>f_p>0)$ of $|\underset{\sim}{X}\underset{\sim}{X}'-f\underset{\sim}{Y}\underset{\sim}{Y}'|=0$ is

(8) $g(f_1,...,f_p)$

$$=\text{Const } |\underset{\sim}{I}+\underset{\sim}{F}|^{-\frac{1}{2}(n_1+n_2)} {}_1F_1(\tfrac{1}{2}(n_1+n_2);\tfrac{1}{2}n_1;\tfrac{1}{2}\underset{\sim}{\Omega},\underset{\sim}{F}(\underset{\sim}{I}+\underset{\sim}{F})^{-1})$$

$$\cdot |\underset{\sim}{F}|^{\frac{1}{2}(n_1-p-1)}\prod_{i<j}(f_i-f_j)$$

[cf.[7]eq.73 and [2]eq.41]

where ω_i are the latent roots of $|\underset{\sim}{M}\underset{\sim}{M}'-\omega\underset{\sim}{\Sigma}| = 0$,

$\underset{\sim}{\Omega} = \text{diag}(\omega_i)$, $\underset{\sim}{F} = \text{diag}(f_i)$, and

$\text{Const}= e^{-\frac{1}{2}\text{tr}\underset{\sim}{\Omega}}(\pi^{p^2/2}\Gamma_p(\tfrac{1}{2}(n_1+n_2))/\Gamma_p(\tfrac{1}{2}p)\Gamma_p(\tfrac{1}{2}n_1)\Gamma_p(\tfrac{1}{2}n_2))$.

Using ([2],eq.1) and ([3],eq.18), Leach [11] proves by applying the Laplace transform to $\underset{\sim}{S}$ that

(9) $|\underset{\sim}{I}-\underset{\sim}{Z}|^{-a} {}_1F_1(a;b;\underset{\sim}{S},\underset{\sim}{Z}(\underset{\sim}{Z}-\underset{\sim}{I})^{-1})$

$$= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a)_\kappa L_\kappa^{b-\frac{1}{2}(p+1)}(\underset{\sim}{S})C_\kappa(\underset{\sim}{Z})}{(b)_\kappa k! C_\kappa(\underset{\sim}{I})}$$

Now (8) may be rewritten as

$$g(f_1,\ldots,f_p)$$

(10)

$$=\text{Const}\sum_{k=0}^{\infty}\sum_{\kappa}[(-1)^k(\tfrac{1}{2}(n_1+n_2))_\kappa L_\kappa^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}(\tfrac{1}{2}\Omega)/(\tfrac{1}{2}n_1)_\kappa k!C_\kappa(\underset{\sim}{I})]$$

$$\cdot|\underset{\sim}{F}|^{\frac{1}{2}(n_1-p-1)}C_\kappa(\underset{\sim}{F})\prod_{i<j}(f_i-f_j)$$

where $L_\kappa^r(\tfrac{1}{2}\Omega)$ is the generalized Laguerre polynomials. Then, using (4) as before, we obtain the cdf of the largest latent root f_1

(11) $P(f_1<x)$

$$=e^{-\frac{1}{2}\text{tr}\underset{\sim}{\Omega}}[\Gamma_p(\tfrac{1}{2}(n_1+n_2))\Gamma_p(\tfrac{1}{2}(p+1))/\Gamma_p(\tfrac{1}{2}n_2)\Gamma_p(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))]$$

$$\cdot x^{\frac{1}{2}pn_1}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\tfrac{1}{2}(n_1+n_2))_\kappa L_\kappa^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}(\tfrac{1}{2}\Omega)C_\kappa(-x\underset{\sim}{I})}{(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))_\kappa k!C_\kappa(\underset{\sim}{I})}.$$

Since $L_\kappa^r(\underset{\sim}{O})=(r+\tfrac{1}{2}(p+1))_\kappa C_\kappa(\underset{\sim}{I})$ we obtain from (2) and (11) the same cdf given by Sugiyama [14] when $\Omega=\underset{\sim}{O}$.

Theorem 2. Let $\underset{\sim}{S}_1$ be a matrix having non-central Wishart distribution with n_1 degrees of freedom and matrix of non-centrality parameter $\underset{\sim}{\Omega}$, and $\underset{\sim}{S}_2$ be a matrix having the Wishart distribution with n_2 degrees of freedom. Then the cdf of the largest latent root of $|\underset{\sim}{S}_1-f\underset{\sim}{S}_2|=0$ is given by (11) which converges for $|x|<1$.

Let $f_1=\ell_1/n_2$. When n_2 tends to infinity we obtain from (11) the cdf given by Hayakawa [6] of the largest latent root ℓ_1 of non-central Wishart matrix

$$(12) \quad P(\ell_1 < y) = e^{-\frac{1}{2}\text{tr}\Omega}[\Gamma_p(\tfrac{1}{2}(p+1))/2^{\frac{1}{2}pn_1}\Gamma(\tfrac{1}{2}n_1+\tfrac{1}{2}(p+1))]$$

$$\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{L_\kappa^{\frac{1}{2}n_1 - \frac{1}{2}(p+1)}(\tfrac{1}{2}\Omega)}{(\tfrac{1}{2}n_1 + \tfrac{1}{2}(p+1))_\kappa}\left(-\tfrac{1}{2}y\right)^k.$$

The non-central cdf of the largest latent root in MANOVA case has been studied by Hayakawa [5], Khatri and Pillai [10], and Pillai and Sugiyama [12] who all expressed it by double infinite sums. When $\frac{1}{2}(n_2-p-1)$ is an integer, Constantine and Venables [4] derived the cdf in a polynomial expression which is simpler. We note that these results do not converge to (12) when n_2 tends to infinity. It would be useful to find a transformation such as (2) which we can apply to (10).

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