数理解析研究所講究録
346巻 1979年 88-94

On Unorientable Surfaces in S^3

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§1. Z/4-quadratic spaces.

We recall the definition of Z/4-quadratic spaces. Let V be a finite dimensional vector space over $Z/2$ provided with a non-singular symmetric bilinear form $(x, y) \longmapsto x \cdot y \in Z/2$, and let φ be a function : $V \longrightarrow Z/4$ satisfying $\varphi(x + y) = \varphi(x) + \varphi(y) + 2(x \cdot y)$ for all $x, y \in V$. φ is called a Z/4-quadratic function and $X = (V, \cdot, \varphi)$ is called a Z/4-quadratic space.

<u>Definition</u>. A Z/4-quadratic space $(V, \cdot, \varphi)$ is <u>even</u>, if $\varphi(x) \equiv 0 \mod 2$ for all $x \in V$.

A Z/4-quadratic space $(V, \cdot, \varphi)$ is <u>odd</u>, if $\varphi(x) \equiv 1 \mod 2$ for some $x \in V$.

(Even Z/4-quadratic spaces are usually called Z/2-quadratic spaces.)

<u>Example</u>. Let F be a smoothly imbedded (not necessarily orientable) surface in S^3 whose boundary ∂F is homeomorphic to S^1. Then we can define a Z/4-quadratic function $\varphi: H_1(F;Z/2) \longrightarrow Z/4$ as follows:

Let C be an immersed circle in F. The normal bundle ν_C of C in S^3 has a unique trivialization $\nu_C = S^1 \times R^2$ such that the linking number of $C = S^1 \times 0$ and $S^1 \times *$ $(* \in R^2, \neq 0)$ is zero. Since the normal bundle of C in F defines a subbundle ν of ν_C, we can count the number $n(C)$ of right-handed half twists of ν, using the trivialization above. Now the required function φ is defined by

$$\varphi(C) = n(C) + 2\ \mathrm{Self}(C) \qquad \mathrm{mod}\ 4,$$

where Self(C) is the number of the self-intersection points of C on F.

Proposition 1. ([5], Lemma 5.1) $\varphi(C) \in Z/4$ depends only on the Z/2-homology class of C. The function $\varphi : H_1(F;Z/2) \longrightarrow Z/4$ is Z/4-quadratic with respect to the Z/2-intersection pairing of $H_1(F;Z/2)$.

Remark. Let X_F denote the Z/4-quadratic space $(H_1(F;Z/2), \cdot, \varphi)$ above. Then X_F is even, if F is orientable, and X_F is odd, if F is unorientable.

In [2], E. H. Brown defined a generalized Z/8 Arf invariant, called Brown's invariant, of Z/4-quadratic spaces. The Witt group W is isomorphic to Z/8 by Brown's invariant. (See [5] for the definition of the Witt group.) The definition of Brown's invariant is as follows:

Let X be a Z/4-quadratic space $(V, \cdot, \varphi)$. We set

$$\lambda(X) = \Sigma_{x \in V} \sqrt{-1}^{\,\varphi(x)} \in \mathbb{C}.$$

Then the complex number $\lambda(X)$ has the property that $\lambda(X)^8 \in R^+$, and the integer m modulo 8 is well-defined. It is called Brown's invariant and is denoted by $\beta(X) \in Z/8$.

Proposition 2.A (2.B). The isomorphism classes of even (odd) $Z/4$-quadratic spaces can be completely classified by the dimension of V over $Z/2$ and Brown's invariant $\beta(X)$.

For the proof, see [1], [2], and [5].

§2. Unorientable surfaces in S^3.

Let us consider smoothly imbedded surfaces in S^3. Two surfaces F and G are regular homotopic, if there is a continuous family $\{F_t\}_{0 \leq t \leq 1}$ of smoothly immersed surfaces in S^3 such that $F_0 = F$, $F_1 = G$. In [6], the author has classified orientable surfaces with boundary in S^3 by regular homotopy. (See also [4].) In this section we classify unorientable surfaces in S^3 whose boundaries are homeomorphic to S^1 by regular homotopy. See also [2] Example (1.28).

Theorem. Two smoothly imbedded (not necessarily orientable) surfaces F, G in S^3 whose boundaries are homeomorphic to S^1 are regular homotopic if and only if the associated $Z/4$-quadratic spaces X_F and X_G are isomorphic.

Corollary A (B). Two smoothly imbedded orientable (unorientable) surfaces F, G in S^3 whose boundaries are homeomorphic to S^1 are regular homotopic if and only if $\dim_{Z/2} H_1(F;Z/2) = \dim_{Z/2} H_1(G;Z/2)$ and $\beta(X_F) = \beta(X_G)$.

We prove Theorem for unorientable surfaces. See [6] for the proof of orientable surfaces. Let F and G be smoothly imbedded unorientable surfaces in S^3 whose boundaries are homeomorphic to S^1 such that X_F and X_G are isomorphic.

<u>Lemma</u> 1. Suppose that $\{e_1,\dots,e_r\}$ is a basis of $H_1(F;Z/2)$ satisfying the condition (*);

$$(*) \qquad e_i \cdot e_j = 0 \quad (i \neq j).$$

Then $e_1, \dots, e_r$ can be represented by mutually disjoint imbedded circles $c_1,\dots, c_r$.

<u>Remark</u>. By the non-singularity of the intersection pairing of $H_1(F;Z/2)$, the condition (*) implies $e_i \cdot e_i = 1 \in Z/2$ for all $i = 1,\dots, r$, and therefore $\varphi(e_i) = \pm 1 \in Z/4$.

(proof of Lemma 1) Each $Z/2$-homology class e_i can be represented by a generic immersion of S^1. Using the method illustrated in Figure 1, we may assume that the class e_i is represented by an imbedded circle c_i.

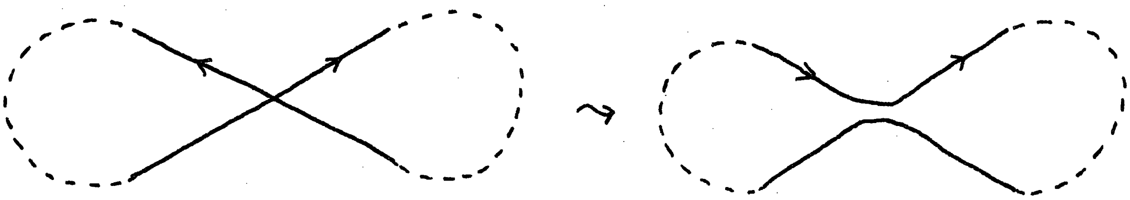

Figure 1.

Since we can prove this lemma by an induction on r, we shall prove in the case $r = 2$. Let c_1, c_2 be imbedded circles representing the elements e_1, e_2. As $e_1 \cdot e_2 = 0 \in Z/2$ $c_1 \cap c_2 = \{p_1, p_2, \ldots, p_{2k-1}, p_{2k}\}$. If $k \neq 0$, we modify the curve c_2 as the dotted line in Figure 2. This can be done, because the regular neighborhood of the circle c_1 is a Möbius band. (See Remark above.) The new curve, also denoted by c_2, has no intersection points with c_1 and represents the same $Z/2$-homology class e_2 as before, but it has some self-intersection points. Using the method illustrated in Figure 1 again, we kill these double points, and the lemma is proved.

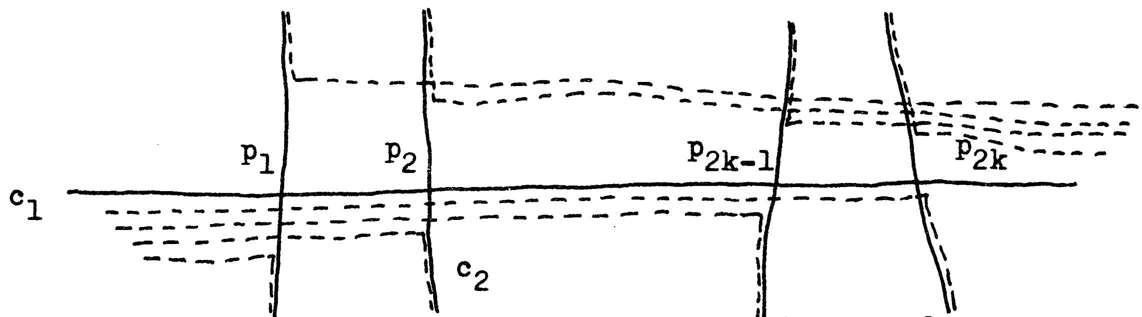

Figure 2.

From the classification of unorientable surfaces, the $Z/2$-vector space $H_1(F;Z/2)$ has a basis $\{e_1,\ldots,e_r\}$ which satisfies the condition (*). Let c_i be the imbedded circle in Lemma 1, and N_i be a regular neighborhood of c_i, for $i = 1, \ldots, r$. Let N denote the boundary-connected-sum of N_i's in F. Since the boundary ∂N_i of N_i is homeomorphic to S^1, for $i = 1,\ldots, r$, the boundary ∂N of N is also homeomorphic to S^1, and $\partial(F - \text{int } N)$ is homeomorphic to $S^1 \cup S^1$(disjoint union).

From the following Mayer-Vietoris exact sequence;

$$0 \longrightarrow \underset{\cong\, Z}{H_1(\partial N;Z)} \longrightarrow \underset{\cong\, rZ}{H_1(N;Z)} \oplus H_1(F\text{-int}N;Z) \longrightarrow \underset{\cong\, rZ}{H_1(F;Z)} \dashrightarrow 0$$

$$0 \dashrightarrow \underset{\cong\, Z}{H_0(\partial N;Z)} \longrightarrow \underset{\cong\, Z}{H_0(N;Z)} \oplus H_0(F\text{-int}N;Z) \longrightarrow \underset{\cong\, Z}{H_0(F;Z)} \longrightarrow 0$$

we obtain $H_i(F - \mathrm{int}\ N;Z) = Z$ $(i=0,1)$, and therefore $F - \mathrm{int}\ N$ is homeomorphic to $S^1 \times [0, 1]$, and

<u>Lemma</u> 2. F is regular homotopic to N.

Since the $Z/4$-quadratic space X_G is isomorphic to X_F, there is a basis $\{f_1, \ldots, f_r\}$ of $H_1(G;Z/2)$ such that

$$e_i \cdot e_j = f_i \cdot f_j \qquad (i,\ j = 1, \ldots, r)$$
$$\varphi(e_i) = \varphi(f_i) \qquad (i = 1, \ldots, r).$$

Let d_i's be mutually disjoint imbedded circles on G representing f_i's as in Lemma 1, and let M denote the boundary-connected-sum of regular neighborhoods of d_i's in G. Now from the equality $\varphi(e_i) = \varphi(f_i)$, it is easy to construct a regular homotopy between N and M ([3]), and therefore F and G are regular homotopic by Lemma 2. The converse is quite trivial and Theorem is proved.

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