The Godbillon-Vey class of codimension one foliations
without holonomy

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In this note we prove the following result.

THEOREM. Let $F$ be a codimension one $C^2$-foliation on a
compact smooth manifold $M$ and assume that $F$ is without
holonomy, namely the holonomy group of each leaf is trivial.
Then the Godbillon-Vey characteristic class of $F$ defined in
$H^3(M; \mathbb{R})$ ([3]) vanishes.

For the proof of the above result, the argument of Herman
used in [4] to prove the triviality of the Godbillon-Vey invariant
of foliations by planes of $T^3$ and also the work of Novikov [7]
and Imanishi [5] on codimension one foliations without holonomy
play very important roles.

1. Codimension one foliations without holonomy.

Let $M$ be a compact connected smooth manifold and let $F$
be a codimension one $C^2$-foliation without holonomy on $M$. We fix
a base point $x_0$, a flow $\Phi: M \times \mathbb{R} \to M$ whose orbits are transverse to leaves of $F$ and we denote $\varphi(t)$ for $\Phi(x_0, t)$
($t \in \mathbb{R}$). Following Novikov [7] (also see Imanishi [5]), we define
a homomorphism

$\chi: \pi_1(M, x_0) \to \text{Diff}^2_+(\mathbb{R})$

as follows, where $\text{Diff}^2_+(\mathbb{R})$ is the group of orientation preserving diffeomorphisms of class $C^2$ of $\mathbb{R}$. Let $\omega$ be an element
of $\pi_1(M, x_0)$ represented by a closed curve $p: (I, \hat{I}) \to (M, x_0)$

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and let \( t \) be a point of \( \mathbb{R} \). Then \( \chi'(\omega)(t) \) is defined to be a point \( t_1 \) of \( \mathbb{R} \) such that there is a leaf curve \( \ell : (I, 0, 1) \rightarrow (L, g(t_1), g(t)) \) (\( L \) is the leaf passing through \( g(t) \)) satisfying the condition: two curves \( p_+ \) and \( \ell_+ \) are homotopic, where \( p_+ \) is the product of two curves \( p \) and \( \varphi([0, t]) \) (if \( t \geq 0 \)) or \( \varphi([t, 0]) \) (if \( t < 0 \)), while \( \ell_- \) is the product of two curves \( \varphi([0, t_1]) \) (or \( \varphi([t_1, 0]) \)) and \( \ell \).

\( \chi \) is a well defined homomorphism (we define the product of two elements \( f \) and \( g \) of \( \text{Diff}^2_+(\mathbb{R}) \) to be \( f \circ g \)) and it is known that \( \text{Image}(\chi) \) is abelian (see [5] [7]). Now using the homomorphism \( \chi \), we can construct a locally trivial foliated \( \mathbb{R} \)-bundle (or the suspension foliation) \( E \) over \( M \) as follows.

Let \( \widetilde{M} \) be the universal covering space of \( M \). Then \( \pi_1(M, x_0) \) acts on \( \tilde{M} \times \mathbb{R} \) by the deck transformation on the first factor and through the homomorphism \( \chi \) on the second. This action is free and preserves the trivial foliation on \( \tilde{M} \times \mathbb{R} \) defined by \( \{ t = \text{constant} \} \). Therefore the quotient manifold \( E = \tilde{M} \times \mathbb{R} / \pi_1(M, x_0) \) has the structure of a locally trivial foliated \( \mathbb{R} \)-bundle over \( M \).

Now our first important step is the following.

**Proposition 1.** Let \( E \) be the locally trivial foliated \( \mathbb{R} \)-bundle over \( M \) defined by the homomorphism \( \chi \). Then there is a cross-section \( \sigma : M \rightarrow E \) such that \( \text{Image}(\sigma) \) is transverse to the codimension one foliation on \( E \) and the induced foliation on \( M \) is the same as the original one \( F \).

**Proof.** We define a mapping \( \psi : \tilde{M} \rightarrow \mathbb{R} \) as follows. Let \( \tilde{q} \) be a point of \( \tilde{M} \) represented by a path \( q : (I, 0) \rightarrow (M, x_0) \). Then \( \psi(\tilde{q}) \) is defined to be a point of \( \mathbb{R} \) such that there is a leaf curve \( \ell : (I, 0, 1) \rightarrow (M, g \circ \psi(\tilde{q}), q(1)) \), so that two curves \( q \) and \( \ell_- \) are homotopic where \( \ell_- \) is the product
of two curves \( \varphi([0, \eta([\xi])), 0]) \) and \( l \). Now we define an imbedding \( \varphi: \tilde{M} \to \tilde{M} \times \mathbb{R} \) by \( \varphi(\tilde{e}) = (\tilde{e}, \varphi(\xi)) \). Then it can be checked that \( \varphi \) is equivariant with respect to the \( \pi_1(M, x_0) \)-actions. Moreover \( \varphi \) is transverse to the trivial foliation on \( \tilde{M} \times \mathbb{R} \) defined by \( \{t = \text{constant}\} \) and the induced codimension one foliation on \( \tilde{M} \) coincides with the lift to \( \tilde{M} \) of the original foliation \( F \). Therefore the induced mapping \( \sigma: M \to E \) satisfies the required conditions.

q.e.d.

**Remark 2.** In the construction above, suppose that the orbit \( \text{Image}(\varphi) \) is periodic, namely for some \( k \) the equality \( \varphi(t + k) = \varphi(t) \) holds for every \( t \in \mathbb{R} \). Then for any element \( \omega \) of \( \pi_1(M, x_0) \), \( \chi(\omega) \) is a periodic diffeomorphism of \( M \); \( \chi(\omega)(t + k) = \chi(\omega)(t) \). Thus \( \chi \) induces a homomorphism \( \chi': \pi_1(M, x_0) \to \text{Diff}^2(S^1) \) where we identify \( \mathbb{R} \mod k \mathbb{Z} \) with \( S^1 \). Imamishi \([5]\) has proved, among other things, that \( \text{Image}(\chi') \) is topologically conjugate to rotations. Now the same proof as that of Proposition 1 gives the following.

**Proposition 1'.** Let \( E' \) be the foliated \( S^1 \)-bundle over \( M \) defined by the homomorphism \( \chi' \). Then there is a cross-section \( \sigma': M \to E' \) such that \( \text{Image}(\sigma') \) is transverse to the codimension one foliation on \( E' \) and the induced foliation on \( M \) is the same as the original one \( F \).

2. The Godbillon-Vey class of foliated \( S^1 \) and \( \mathbb{R} \)-bundles.

Let \( E \) be a foliated \( S^1 \)-bundle of class \( C^2 \) over a smooth manifold \( M \) defined by a homomorphism \( \pi_1(M) \to \text{Diff}^2(S^1) \). For such object, the Godbillon-Vey class (integrated over the fibres)
is defined as an element of $H^2(\text{Diff}_+^2(S^1); \mathbb{R})$ (the 2-dimensional cohomology group with trivial coefficients $\mathbb{R}$ of $\text{Diff}_+^2(S^1)$ considered as an abstract group). According to Thurston (cf. [1] [4]), this element is represented by the following cocycle $\alpha \in C^2(\text{Diff}_+^2(S^1); \mathbb{R})$.

**DEFINITION 3.** Let $u, v$ be elements of $\text{Diff}_+^2(S^1)$. Then

$$\alpha(u, v) = \int_{S^1} \log Dv(t) \, D \log D(u)(v(t)) \, dt.$$ 

Now let $E$ be a locally trivial foliated $\mathbb{R}$-bundle over a smooth manifold $M$ defined by a homomorphism $\pi_1(M) \to \text{Diff}_+^2(\mathbb{R})$. Then similarly as above, the Godbillon-Vey class for such objects is defined as an element of $H^3(\text{Diff}_+^2(\mathbb{R}); \mathbb{R})$ as follows.

Let $f, g, h$ be elements of $\text{Diff}_+^2(\mathbb{R})$ and we set

$$A = \log Df^{-1}(t),$$
$$B = \log Dg^{-1}(f^{-1}(t)),$$
$$C = \log Dh^{-1}(g^{-1}f^{-1}(t)).$$

Let $\Delta^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 \leq 1\}$ be the 3-simplex and let $s: \Delta^3 \to \mathbb{R}$ be a function defined by

$$s(x_1, x_2, x_3) = \begin{cases} 
(x_1 + x_2 + x_3) \frac{x_2 + x_3}{x_1 + x_2 + x_3} g\left(\frac{x_3}{x_2 + x_3} h(0)\right), & x_2 + x_3 \neq 0 \\
x_1 f(0), & x_2 + x_3 = 0.
\end{cases}$$

$s$ is $C^\infty$ on the interior of $\Delta^3$, $\partial \Delta^3$, and continuous on $\Delta^3$.

Let $S: \Delta^3 \to \Delta^3 \times \mathbb{R}$ be defined by $S(x_1, x_2, x_3) = (x_1, x_2, x_3, s(x_1, x_2, x_3))$. Now we define a cochain $\beta \in C^3(\text{Diff}_+^2(\mathbb{R}); \mathbb{R})$ by the formula
DEFINITION 4.

\[ \beta(f, g, h) = \int_{\Delta^3} s^* \left\{ \left[ A \, dx_1 + (A + B) \, dx_2 + (A + B + C) \, dx_3 \right] \left[ A' \, dtdx_1 + (A' + B') \, dtdx_2 + (A' + B' + C') \, dtdx_3 \right] \right\}. \]

Since the derivatives \( \frac{\partial s}{\partial x_1}, \frac{\partial s}{\partial x_2}, \frac{\partial s}{\partial x_3} \) are bounded over \( \Delta^3 \), the integral exists. We can show

PROPOSITION 5. The cochain \( \beta \) is a cocycle.

Thus \( \beta \) defines an element \( [\beta] \in H^3(\text{Diff}^2_+(\mathbb{R}); \mathbb{R}) \).

A proof of Proposition 5 together with related topics will be given in [6]. This is because, for a proof of our THEOREM, the form of the cocycle \( \beta \) is not essential. We need only the fact that the Godbillon-Vey class of a locally trivial foliated \( \mathbb{R} \)-bundle can be calculated by group cohomology argument. More precisely, let \( \rho : \pi_1(T^3) = \mathbb{Z} \rightarrow \text{Diff}^2_+(\mathbb{R}) \) be a homomorphism defined by three mutually commuting diffeomorphisms \( f, g, h \) of \( \mathbb{R} \) and let \( E \) be the locally trivial foliated \( \mathbb{R} \)-bundle over \( T^3 \) defined by \( \rho \). Then the Godbillon-Vey class of this foliation on \( E \) is an element of \( H^3(E; \mathbb{R}) \) \( \otimes H^3(T^3; \mathbb{R}) \otimes \mathbb{R} \). Let us denote \( GV(f, g, h) \) for the corresponding real number. Under these situation, we have

PROPOSITION 6. Let \( f, g, h \) be mutually commuting elements of \( \text{Diff}^2_+(\mathbb{R}) \). Then \( z = (f, g, h) - (f, h, g) + (g, h, f) - (g, f, h) + (h, f, g) - (h, g, f) \) is a cycle (of the group \( \text{Diff}^2_+(\mathbb{R}) \)) and the equality

\[ GV(f, g, h) = \beta(z) \]

holds.

A proof of this Proposition will also be given in [6].
3. Foliated $S^1$ and $\mathbb{R}$-bundles over tori.

In [4], Herman has proved the following

THEOREM 7. Let $E$ be a foliated $S^1$-bundle of class $C^2$ over $T^2$. Then the Godbillon-Vey invariant of the codimension one foliation on $E$ is zero.

In this section, we prove the following results which can be considered as generalizations of Theorem 7.

THEOREM 8. Let $E$ be a foliated $S^1$-bundle of class $C^2$ over a torus $T^k$ ($k \geq 2$). Then the Godbillon-Vey class of the codimension one foliation on $E$ vanishes.

THEOREM 9. Let $E$ be a locally trivial foliated $\mathbb{R}$-bundle over a torus $T^k$ ($k \geq 3$). Then the Godbillon-Vey class of the codimension one foliation on $E$ vanishes.

Before proving the above Theorems, let us recall the argument of Herman [4] briefly. Let $E$ be a foliated $S^1$-bundle over $T^2$ defined by commuting diffeomorphisms $u, v \in \text{Diff}_+^2(S^1)$. Then $c = (u, v) - (v, u)$ is a cycle of the group $\text{Diff}_+^2(S^1)$ and by Thurston (cf. [1] [4]), the Godbillon-Vey invariant of $E$, denoted by $Gv(u, v)$, is given by

$$Gv(u, v) = \alpha(c).$$

Herman has proved $\alpha(c) = 0$ by an elegant argument using known properties of elements of $\text{Diff}_+^2(S^1)$. Now we prove Theorems 8 and 9.

Proof of Theorem 9. Since the cohomology group $H^3(T^k; \mathbb{R})$ ($k \geq 3$) is generated by 3-dimensional cohomologies of various 3-dimensional subtori of $T^k$, we have only to prove the case $k = 3$. Thus let $f, g, h \in \text{Diff}_+^2(\mathbb{R})$ be mutually commuting diffeomorphisms and let $E$ be the locally trivial foliated
A-bundle over $T^3$ defined by them. We have to prove $GV(f, g, h) = 0$. We consider two cases.

Case 1. All of $f$, $g$, $h$ have fixed points.

In this case it can be proved that $f$, $g$, $h$ have a common fixed point. In fact this follows from the following general statement.

**Proposition 10.** Let $f_1, \ldots, f_r$ be mutually commuting homeomorphisms of $\mathbb{R}$ and assume that all of $f_i$ have fixed points. Then there is a common fixed point of $f_1, \ldots, f_r$.

**Proof.** If $f$ is an orientation reversing homeomorphism of $\mathbb{R}$, then $f$ has a unique fixed point $p$ and for any homeomorphism $g$ of $\mathbb{R}$ such that $f \circ g = g \circ f$, clearly $g(p) = p$ holds. Therefore if at least one of $f_1, \ldots, f_r$ reverses the orientation, then the assertion is clear. Hence we assume that all of $f_1, \ldots, f_r$ preserve the orientation. Now first assume that at least one of $f_1, \ldots, f_r$, say $f_1$, has a maximum (or minimum) fixed point $p$. Then since any $f_j$ ($j = 1, \ldots, r$) leaves the fixed point set of $f_1$, $F(f_1)$, invariant, we have $f_j(p) = p$. So $p$ is a common fixed point. Next assume the contrary and let $(a, b)$ be a maximal open interval contained in $\mathbb{R} - F(f_1)$, thus $a, b \in F(f_1)$. Let $(a_1, b_1)$ be the maximal open interval containing $(a, b)$ such that $(a_1, b_1)$ is contained in $\mathbb{R} - F(f_i)$ for some $i$. We claim that $a_1$ and $b_1$ are common fixed points of $f_1, \ldots, f_r$. For from the definition, either $(a_1, b_1) \subset \mathbb{R} - F(f_j)$ or $f_j$ has a fixed point on $(a_1, b_1)$. But in either case we should have $f_j(a_1) = a_1$ and $f_j(b_1) = b_1$. This completes the proof of Proposition 10.

**Remark 11.** In Proposition 10, if we assume that $f_1, \ldots, f_r$ are orientation preserving diffeomorphisms of class $C^2$, then
we can obtain a stronger statement that if \((a, b)\) is a maximal open interval contained in \(\mathbb{R} - F(f_1)\), then \(a\) and \(b\) are common fixed points of \(f_1, \ldots, f_r\) (cf. [4] Lemma 1).

Now we go back to the proof of Theorem 9, Case 1.

We have just proved that \(f, g, h\) have a common fixed point \(p\). Then this fixed point defines a cross-section \(\sigma: T^2 \to E\) such that \(\text{Image}(\sigma)\) is a compact leaf of the foliation on \(E\). Since the restriction of the Godbillon-Vey class to any leaf is trivial and since \(\text{Image}(\sigma)\) generates the 3-dimensional homology group of \(E\), we conclude that \(GV(f, g, h) = 0\).

Case 2. At least one of \(f, g, h\) has no fixed point.

First we claim that

\[ GV(f, g, h) = GV(g, h, f) = GV(h, f, g). \]

This follows from the definition of \(GV\). It also follows from Proposition 6. Therefore to prove our assertion \(GV(f, g, h) = 0\), we may assume that \(h\) has no fixed points. Now let us define a \(\mathbb{Z}\)-action on \(\mathbb{R}\) by \(n(t) = h^n(t) (n \in \mathbb{Z}, t \in \mathbb{R})\). Then since \(h\) has no fixed points, this action is free and the quotient manifold can be identified with \(S^1\) by an orientation preserving diffeomorphism \(k: \mathbb{R}/\{h^n\} \cong S^1\). Let \(\tilde{k}: \mathbb{R} \to \mathbb{R}\) be the lift of \(k\) such that \(\tilde{k}(0) = 0\). It is a diffeomorphism of class \(C^2\).

Now we set \(f_1 = \tilde{k}^{-1}fk, g_1 = \tilde{k}^{-1}gk, h_1 = \tilde{k}^{-1}hk\). Then \(f_1, g_1, h_1\) are mutually commuting diffeomorphisms of class \(C^2\) of \(\mathbb{R}\).

Let \(\gamma = (f, g, h) - (f, h, g) + (g, h, f) - (g, f, h) + (h, f, g) - (h, g, f)\) and \(\gamma_1 = (f_1, g_1, h_1) - (f_1, h_1, g_1) + (g_1, h_1, f_1) - (g_1, f_1, h_1) + (h_1, f_1, g_1) - (h_1, g_1, f_1)\). Then the cycle \(\gamma_1\) is conjugate to \(\tilde{k}^{-1}\gamma k\). Since inner automorphisms of a group induce the
identity on the homology groups ([2]), we have

$$\beta(g_1) = \beta(g).$$

Therefore from Proposition 6, we obtain

$$GV(f, g, h) = GV(f_1, g_1, h_1).$$

Now from the construction, \( h_1 \) is the translation of \( R \) by \( 1 \) (denoted by \( T \)) or by \(-1\) according as \( h(0) > 0 \) or \( h(0) < 0 \) respectively. By the definition of GV, clearly we have

$$GV(f_1, g_1, h_1) = - GV(f_1, g_1, h_1^{-1}).$$

Therefore we may assume that \( h_1 = T \). Since \( f_1 \) and \( g_1 \) commute with \( h_1 = T \), \( f_1 \) and \( g_1 \) are lifts of some diffeomorphisms \( f_1 \) and \( g_1 \) of \( S^1 \). Now we claim

**PROPOSITION 12.** Let \( u, v \) be mutually commuting elements of \( \text{Diff}_+^2(S^1) \) and let \( \tilde{u}, \tilde{v} \) be their arbitrary lifts to \( R \).

Then we have

$$GV(\tilde{u}, \tilde{v}, T) = GV(u, v).$$

**Proof.** We consider \( R^2 \times R = \{(x_1, x_2, t); x_1, t \in R\} \),

\( R^3 \times R = \{(x_1, x_2, x_3, t); x_1, t \in R\} \) and let

$$\lambda(x_1, x_2, t) = (x_1+1, x_2, \tilde{u}(t)), \quad \lambda_1(x_1, x_2, x_3, t) = (x_1+1, x_2, x_3, \tilde{u}(t))$$

$$\mu(x_1, x_2, t) = (x_1, x_2+1, \tilde{v}(t)), \quad \mu_1(x_1, x_2, x_3, t) = (x_1, x_2+1, x_3, \tilde{v}(t))$$

$$\nu(x_1, x_2, t) = (x_1, x_2, t+1), \quad \nu_1(x_1, x_2, x_3, t) = (x_1, x_2, x_3+1, t+1).$$

Then \( \lambda, \mu, \nu \) and \( \lambda_1, \mu_1, \nu_1 \) generate free \( \mathbb{R}^2 \)-actions on \( R^2 \times R \) and \( R^3 \times R \) respectively. These actions preserve the trivial foliations defined by \( \{t = \text{constant}\} \). The quotient manifolds \( E \) and \( E_1 \) carry the structures of foliated \( S^1 \)-bundle over \( T^2 \) defined by \( u \) and \( v \) and locally trivial foliated
$\mathbb{R}$-bundle over $T^3$ defined by $\tilde{\nu}, \tilde{\gamma}, T$ respectively. Now define a mapping $\pi: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ by $\pi(x_1, x_2, x_3, t) = (x_1, x_2, t)$. Then $\pi$ is equivariant with respect to the $\mathbb{Z}^2$-actions. Therefore it induces a mapping $\pi': E_1 \to E$. Moreover it is easy to see that the pull back of the foliation on $E$ by the submersion $\pi'$ coincides with the given foliation on $E_1$. Therefore from the naturality of the Godbillon-Vey class, we obtain

$$(\pi')^*(\text{gv}(E)) = \text{gv}(E_1),$$

where $\text{gv}(E)$ (resp. $\text{gv}(E_1)$) is the Godbillon-Vey class of the foliation on $E$ (resp. $E_1$). Now since $(\pi')^*$ gives an isomorphism $H^3(E; \mathbb{R}) \cong H^3(E_1; \mathbb{R}) \cong \mathbb{R}$, we obtain

$$\text{GV}(\tilde{\nu}, \tilde{\gamma}, T_1) = \text{Gv}(u, v).$$

This completes the proof of Proposition 11.

Now by the above Proposition and the argument before it, we have

$$\text{GV}(f, g, h) = \text{Gv}(f_1', g_1').$$

But Herman's result (Theorem 7) implies

$$\text{Gv}(f_1', g_1') = 0.$$ 

Hence $\text{GV}(f, g, h) = 0$. This completes the proof of Case 2 and hence Theorem 9.

q.e.d.

Next we prove Theorem 8.

Proof of Theorem 8. Since the case $k = 2$ is just Theorem 7, we assume that $k \geq 3$ and let $E$ be a foliated $S^1$-bundle of class $C^2$ over $T^k$ defined by mutually commuting diffeomorphisms $u_1, \ldots, u_k \in \text{Diff}_+(S^1)$. Since $E$ is a trivial bundle as a differentiable $S^1$-bundle, there is a cross-section $\sigma: T^k \to E$. 

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\( \sigma \) defines an isomorphism \( E \cong T^k \times S^1 \). Now the Godbillon-Vey class of the foliation on \( E \), \( gv(E) \), lies in \( H^3(E; \mathbb{R}) \cong H^3(T^k; \mathbb{R}) \oplus H^2(T^k; \mathbb{R}) \oplus H^1(S^1; \mathbb{R}) \). However Herman's result (Theorem 7) implies that the second component of \( gv(E) \) is zero. Now let \( \tilde{E} = T^k \times R \) be the covering space of \( E = T^k \times S^1 \) corresponding to the subgroup \( \pi_1(T^k) \subset \pi_1(E) \). Then the projection \( \pi : \tilde{E} \to E \) induces a codimension one foliation on \( E \). In fact \( \tilde{E} \) has the structure of locally trivial foliated \( R \)-bundle over \( T^k \) defined by mutually commuting diffeomorphisms \( \tilde{u}_1, \ldots, \tilde{u}_k \in \text{Diff}^2_+(R) \), where \( \tilde{u}_1 \) is a suitable lift of \( u_1 \) to \( R \) defined by the cross-section \( \sigma \). Hence \( gv(\tilde{E}) = 0 \) by Theorem 9. Therefore we obtain \( \pi^*(gv(E)) = gv(\tilde{E}) = 0 \). Now since \( gv(E) \) lies in \( H^3(T^k; \mathbb{R}) \subset H^3(E; \mathbb{R}) \) as remarked before, we conclude \( gv(E) = 0 \).

q.e.d.

5. Proof of THEOREM.

Let \( M \) be a compact smooth manifold, \( F \) a codimension one foliation of class \( C^2 \) over \( M \) and assume that \( F \) is without holonomy. Then by Proposition 1, there is a locally trivial foliated \( R \)-bundle \( E \) over \( M \) defined by a homomorphism \( \chi : \pi_1(M) \to \text{Diff}^2_+(R) \) and an imbedding of \( M \) in \( E \) transverse to the codimension one foliation on \( E \) such that the induced foliation on \( M \) coincides with the original one \( F \). Moreover \( \text{Image}(\chi) \) is abelian. Therefore by Theorem 9, we conclude that \( gv(E) = 0 \). Then by the naturality of the Godbillon-Vey class, we obtain \( gv(F) = 0 \). This completes the proof of THEOREM. We could also use Proposition 1' and Theorem 8 instead of Proposition 1 and Theorem 9.

q.e.d.
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