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An Exact Solution to "Simple Harmonic Generation"

Ryogo Hirota

Department of Applied Mathematics, Faculty of Engineering
Hiroshima University, Hiroshima

We give an exact solution to an equation describing "simple harmonic generations"

$$\begin{cases} \dfrac{\partial}{\partial\xi} q_1 = q_2 q_1^{*} , & \text{(1a)} \\[2ex] \dfrac{\partial}{\partial\eta} q^2 = -2q_1^{2} , & \text{(1b)} \end{cases}$$

where $\dfrac{\partial}{\partial\xi} = \dfrac{\partial}{\partial t} + \vec{c}_1 \cdot \vec{\nabla}$ and $\dfrac{\partial}{\partial\eta} = \dfrac{\partial}{\partial t} + \vec{c}_2 \cdot \vec{\nabla}$

($\vec{c}_1$ and $\vec{c}_2$ are constants vectors).

D. J. Kaup* showed that eq.(1) can be put into the inverse scattering form

$$\begin{cases} \dfrac{\partial}{\partial\xi} v_1 + i\xi v_1 = q_2 v_2 , \\[2ex] \dfrac{\partial}{\partial\xi} v_2 - i\xi v_2 = q_2^{*} v_1 , \end{cases} \qquad (2)$$

$$\begin{cases} \dfrac{\partial}{\partial\eta} v_1 = \dfrac{1}{i\xi} (q_1^{*} q_1 v_1 + q_2^{2} v_2) , \\[2ex] \dfrac{\partial}{\partial\eta} v_2 = -\dfrac{1}{i\xi} (q_1^{*2} v_1 + q_1^{*} q_1 v_2) , \end{cases} \qquad (3)$$

where ξ is an eigenvalue.

*) D. J. Kaup, "Simple Harmonic Generation: an Exact Method of Solutions", Studies in Appl. Math. 59 (1978) 25.

He investigated the Zakharov-Shabat eigenvalue problem eq.(2) and found that the inverse scattering solution for this problem is distinctly different from any others done before, and in a sense, it is also rather singular problem.

We shall find an exact solution to eq.(1) by using the direct method and show that the solution may satisfy the resonant condition and the third wave may be generated.

$$\text{Let} \quad q_1 = G/F \quad \text{for a real } F, \tag{4}$$

then we have, from eq.(1a),

$$q_2 = \frac{D_\xi G \cdot F}{FG^*}, \tag{5}$$

where the binary operator $D_\xi^m D_\eta^n$ is defined by

$$D_\xi^m D_\eta^n \, a \cdot b \equiv \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \xi'} \right)^m \left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \eta'} \right)^n a(\xi,\eta) b(\xi',\eta') \Big|_{\substack{\xi=\xi' \\ \eta=\eta'}} \, . \tag{6}$$

Substituting eq.(5) into eq.(1b) we find

$$D_\eta (D_\xi G \cdot F) \cdot FG^* = -2(GG^*)^2 , \tag{7}$$

whose complex conjugate becomes

$$D_\eta GF \cdot (D_\xi F \cdot G^*) = -2(GG^*)^2 . \tag{8}$$

We have the mathematical formula

$$(D_\eta D_\xi a \cdot b)cd - ab(D_\eta D_\xi c \cdot d)$$

$$= D_\eta[(D_\xi a \cdot d) \cdot cb + ad \cdot (D_\xi c \cdot b)] \ . \qquad (9)$$

Adding eq.(7) to eq.(8) and using eq.(9) we find the following equation

$$(D_\eta D_\xi G \cdot G^*)F^2 - GG^*(D_\eta D_\xi F \cdot F) = -4(GG^*)^2 \ , \qquad (10)$$

which may be decomposed into the coupled bilinear equation

$$D_\eta D_\xi G \cdot G^* = 0 \ , \qquad (11)$$

$$D_\eta D_\xi F \cdot F = 4GG^* \ . \qquad (12)$$

Using the standard procedure to solve the bilinear differential equation, we find

$$\begin{cases} F = 1 + \theta(\xi) + \chi(\eta) + (1 + \alpha)\theta(\xi)\,\chi(\eta) \ , & (13a) \\ G = [(\alpha/2)\theta_\xi(\xi)\chi_\eta(\eta)]^{1/2} \exp i\phi \ , & (13b) \end{cases}$$

where α is an arbitrary parameter and θ and χ are arbitrary functions of the indicated variables and ϕ satisfies the condition $\phi_\xi = 0$ or $\phi_\eta = 0$. The solution to eq.(1) is expressed with F

$$|q_1|^2 = (1/2)(\log F)_{\xi\eta} \ . \qquad (14)$$

In eq.(13a), α is an arbitrary parameter. Hence the solution satisfies the resonance condition for $\alpha = -1$, which implies the breakdown of the Zakharov-Shabat eigenvalue problem.