

Undecidable extensions of monadic first-order successor arithmetic
(abstract)

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It is proved by R.M. Robinson [8] that the monadic second-order successor arithmetic $SA[2x, x+1]$ with a function $2x$ is undecidable. Putnam proved in [6] that the monadic first-order arithmetic $FA^=[P; x+y]$ with the equality symbol, a single monadic predicate symbol P and a binary function $+$ is undecidable. In contrast with these undecidability results, both the monadic second-order successor arithmetic $SA[x+1]$ (Büchi [1]) and the monadic second-order arithmetic $SA[2x]$ with a function $2x$ are decidable. The latter is an easy corollary of Rabin's result concerning the decidability of the monadic second-order theory of two successor functions [7]. So it is very interesting to search for a boundary between these decidability and undecidability results. In [5], we have shown the following 'critical' result.

THEOREM 1. The monadic first-order successor arithmetic $FA[P; 2x, x+1]$ with a single monadic predicate symbol P and a function $2x$ (but without the equality symbol) is undecidable.

Then, the above two undecidability results follow immediately from this theorem. We will make a rough sketch of the proof of Theorem 1. Our idea is based on a reduction of the satisfiability problem of $FA[P ; 2x, x+1]$ to the meeting problem of finite causal ω^2 -systems introduced by the second author, which is shown to be unsolvable in [3]. Here, the meeting problem means the problem of deciding whether some cells will take a given special state eventually, when starting from the initial state of a given finite causal ω^2 -system. For a given finite causal ω^2 -system S and a given special state q_δ of S , we can construct such a formula $B_{S,\delta}$ (of $FA[P ; 2x, x+1]$) that $B_{S,\delta}$ is satisfiable in N (the set of natural numbers) if and only if no cells take the state q_δ in S . Thus, the unsolvability of the meeting problem implies that of the satisfiability problem of $FA[P ; 2x, x+1]$.

A question now arises as to the generalization of Theorem 1. Can we extend this ' critical ' result, namely Theorem 1, to another monadic first-order successor arithmetic with a unary function on N ? In the following, we will give an answer to this question. The following theorem can be obtained by using the same technique as in [5].

THEOREM 2. The monadic first-order successor arithmetic $FA^=[\{P_i\}_i ; f(x), x+1]$ with the equality, countable predicate symbols $\{P_i\}_i$ and a function $f(x)$ on N is undecidable, if the function $f(x)$ satisfies either of the following conditions I and II.

- I. 1) f is strictly monotone increasing, i.e.,
 $x < y$ implies $f(x) < f(y)$,
- 2) for some integer $r \geq 1$, $f(x) + r < f(x + r)$ for every x .
- II. 1) f is monotone increasing, i.e.,
 $x < y$ implies $f(x) \leq f(y)$,
- 2) for some integer $r \geq 2$, $f(x + r) < f(x) + r$ for every x ,
- 3) for every y , there exist at least one x such that $f(x) = y$.

For example, any one of the functions x^n ($n > 1$), $[e^x]$, $[\sqrt{x}]$, $[\log x]$, $[(q/p)x]$ ($p \neq q$ and $q \neq 0$) satisfies either of the above conditions. Moreover, it turns out that many undecidability results on monadic second-order successor arithmetics, for instance, the result on a monadic second-order successor arithmetic with a *hypermonotonic* function by Elgot and Rabin [2] and some results by Siefkes [9] and Thomas [10], follow as corollaries of our undecidability results on monadic *first-order* arithmetics. We notice here the limitation of our result. The condition I.2) is equivalent to the following condition;

$$1 + \frac{1}{r} \leq \frac{f(x+r) - f(x)}{r} .$$

Thus, if f satisfies the condition that $f'(x) > 1$ but $\lim_{x \rightarrow \infty} f'(x) = 1$ (in an approximate sense), then the condition I.2) can not be satisfied. So, for example, we don't know whether or not $FA^= [\{P_i\}_i ; x + [\sqrt{x}], x+1]$ is undecidable. Compare this with the undecidability of $SA[x + [\sqrt{x}], x+1]$ proved in [10].

The following theorem can be considered as a generalization of the decidability of $SA[2x]$.

THEOREM 3. The monadic second-order arithmetic $SA[f(x)]$ with a single function $f(x)$ is decidable, if $x < f(x)$ for any x and f is strictly monotone increasing.

Remark that the above theorem holds even if f is non-recursive. We notice also that if f is strictly monotone increasing and is not identical with x , then there exists the smallest x such that $x < f(x)$. When we have an effective procedure of finding such an x , we can delete the above condition that $x < f(x)$ for any x .

Now, we will give an outline of the proof of Theorem 3. A pair $\langle A, f \rangle$ is called an *algebra* (after Rabin [7]) if A is a non-empty set and f is a function defined on A . Two algebras $\langle A, f \rangle$ and $\langle B, g \rangle$ are *isomorphic* if there exists a bijective mapping φ from A to B such that $\varphi(f(x)) = g(\varphi(x))$ for any $x \in A$. Let F be any formula of $SA[f(x)]$. Then, define F^* to be a formula of $SA[g(x)]$ obtained from F by replacing every occurrence of f in F by g . Now, we have the following lemmas.

LEMMA 1. If two algebras $\langle N, f \rangle$ and $\langle N, g \rangle$ are isomorphic, then for any formula F of $SA[f(x)]$, F is valid if and only if F^* is valid.

LEMMA 2. If f is a strictly monotone increasing function such that $f(0) = 0$, $x < f(x)$ for any $x > 0$ and $f(x) - x$ is unbounded, then $\langle N, 2x \rangle$ and $\langle N, f \rangle$ are isomorphic.

By slightly modifying Lemma 2 and putting together Lemmas 1 and 2, we have Theorem 3.

Roughly speaking, it follows from Theorems 2 and 3 that we have a 'critical' result on almost all functions which increase more *rapidly* than x . On the other hand, we can not prove an analogous result on functions which increase more *slowly* than x . Indeed, the following theorem shows a difference between them.

THEOREM 4. The first-order arithmetic $FA^{\overline{=}}[f(x) \cdot]$ with a function $f(x)$, and hence the monadic second-order arithmetic $SA[f(x)]$, are undecidable for uncountably many functions $f(x)$ satisfying the condition II of Theorem 2.

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