数理解析研究所講究録
365巻 1979年 19-22

或る直交変換群について

山形大 理　内田伏一

We give an example of an orthogonal transformation group of (8k-1)-sphere with codimension two principal orbits. This action possesses just two isolated singular orbits (cf.[1],p.214; [4]). This example shows that a theorem of Hsiang and Lawson ([2],Theorem 6) should be properly modified.

Let ν_m, ν_n be the standard representation of $Sp(m)$ and $Sp(n)$ on H^m and H^n respectively. Here H^m, H^n are the right quaternionic vector spaces. Let $(H^n)^*$ denote the dual vector space of H^n. $(H^n)^*$ is a left quaternionic vector space. It is well known that $H^m \otimes_H (H^n)^*$ is a real 4mn-dimensional vector space and $\nu_m \otimes_H \nu_n^*$ is a real representation of $Sp(m) \times Sp(n)$ on $R^{4mn} = H^m \otimes_H (H^n)^*$.

This representation can be regard as follows. Let $M(m,n;H)$ denote the set of all $m \times n$ quaternionic matrices. For an $m \times n$ quaternionic matrix X, let X^* denote the transpose of the conjugate of X. Then

$$Sp(m) = \left\{ A \in M(m,m;H) : A^*A = I \text{ the unit matrix} \right\},$$

the representation space $H^m \underset{H}{\otimes} (H^n)^*$ is identified with $M(m,n;H)$, and the representation $\psi = \nu_m \underset{H}{\otimes} \nu_n^*$ can be expressed by

$$\psi((A,B))\cdot X = AXB^* \quad ; \ A \in Sp(m), B \in Sp(n), X \in M(m,n;H).$$

Put

$$\langle X,Y\rangle = \text{trace } X^*Y, \ \text{Re}\,\langle X,Y\rangle = \text{real part of } \langle X,Y\rangle$$

for $X,Y \in M(m,n;H)$. $\text{Re}\,\langle X,Y\rangle$ is an $Sp(m)\times Sp(n)$-invariant inner product of the real vector space $M(m,n;H)$. For an $m\times n$ quaternionic matrix X, let rank X be the maximum number of linearly independent column vectors of X as the right quaternionic vectors.

Example. We shall consider a real 8k-dimensional representation $\psi_k = \nu_k \underset{H}{\otimes} (\nu_2^* | Sp(1)\times Sp(1))$ of the group $Sp(k)\times Sp(1)\times Sp(1)$ on $M(k,2;H)$. Suppose $k \geqq 2$ in the following. For a $k\times 2$ quaternionic matrix X, let X_1, X_2 denote the first and the second column vector of X respectively. Then the representation ψ_k can be expressed by

$$\psi_k((A,q_1,q_2))\cdot(X_1,X_2) = (AX_1\bar{q}_1, AX_2\bar{q}_2)$$

for $A \in Sp(k)$, $q_i \in Sp(1)$, $X = (X_1,X_2) \in M(k,2;H)$. Straightforward computations show the following :

(i) Suppose that rank $X = 2$ and $\langle X_1,X_2\rangle \neq 0$ for $X = (X_1,X_2)$. Then the isotropy group at X is conjugate to

$$\left\{ \left(\left(\begin{array}{cc|c} q & 0 & \\ 0 & q & 0 \\ \hline \multicolumn{2}{c|}{0} & * \end{array}\right), q, q \right) : q \in Sp(1) \right\},$$

and the orbit through X is $(8k-3)$-dimensional, which is diffeomorphic to $Sp(k)/Sp(k-2) \times S^3$.

(ii) Suppose that rank $X = 2$ and $\langle X_1, X_2 \rangle = 0$ for $X = (X_1, X_2)$. Then the isotropy group at X is conjugate to

$$\left\{ \left(\left(\begin{array}{cc|c} q_1 & 0 & \\ 0 & q_2 & 0 \\ \hline \multicolumn{2}{c|}{0} & * \end{array}\right), q_1, q_2 \right) : q_i \in Sp(1) \right\},$$

and the orbit through X is $(8k-6)$-dimensional, which is diffeomorphic to $Sp(k)/Sp(k-2)$.

(iii) Suppose that rank $X = 1$ and $\langle X_1, X_2 \rangle \neq 0$ for $X = (X_1, X_2)$. Then the isotropy group at X is conjugate to

$$\left\{ \left(\begin{pmatrix} q & 0 \\ 0 & * \end{pmatrix}, q, q \right) : q \in Sp(1) \right\},$$

and the orbit through X is $(4k+2)$-dimensional, which is diffeomorphic to $S^{4k-1} \times S^3$.

(iv) Suppose that rank $X = 1$ and $\langle X_1, X_2 \rangle = 0$ for $X = (X_1, X_2)$. Then the isotropy group at X is conjugate to

$$\left\{ \left(\begin{pmatrix} q_1 & 0 \\ 0 & * \end{pmatrix}, q_1, q_2 \right) : q_i \in Sp(1) \right\} \quad \text{for } X_1 \neq 0$$

or

$$\left\{ \left(\begin{pmatrix} q_2 & 0 \\ 0 & * \end{pmatrix}, q_1, q_2 \right) : q_i \in Sp(1) \right\} \quad \text{for} \quad X_2 \neq 0,$$

and the orbit through X is a $(4k-1)$-sphere.

Remark. (a) The representation ϕ_k induces an action of $Sp(k) \times Sp(1) \times Sp(1)$ on a sphere S^{8k-1}. The principal orbits of this action are of codimension two, and this action possesses just two isolated singular orbits which are diffeomorphic to a $(4k-1)$-sphere. (b) The representation ϕ_k is an example of a reducible compact linear group of cohomogeneity three (in the sense of Hsiang and Lawson [2]). This example shows that a theorem of Hsiang and Lawson ([2], Theorem 6) should be properly modified.

References

[1] G.E.Bredon : Introduction to Compact Transformation Groups, Academic Press, 1972.

[2] W.Y.Hsiang and H.B.Lawson : Minimal submanifolds of low cohomogeneity, J.Diff.Geometry 5(1971),1-38.

[3] F.Uchida : An orthogonal transformation group of (8k-1)-sphere, to appear.

[4] F.Uchida and T.Watabe : A note on compact connected transformation groups on spheres with codimension two principal orbits, Sci.Rep.Niigata Univ.Ser.A,16(1979),1-14.

注. この報告は [3] の一部分を紹介したものである.