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On a Decomposability of Homogeneous Linear System Representations of a Locally Compact Group

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On a decomposability of homogeneous linear system representations of a locally compact group

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§1. Linear system representations

A pair $H = \langle H_1, H_2 \rangle$ of complex linear spaces $H_1$, $H_2$ is called a linear system if a duality $\langle \xi, \eta \rangle$ is defined between $H_1$ and $H_2$. Namely, $\langle \xi, \eta \rangle$ is a complex bilinear form on $H_1 \times H_2$ with the property $\langle \xi, H_2 \rangle = 0$ only if $\xi = 0$ and $\langle H_1, \eta \rangle = 0$ only if $\eta = 0$. In this paper we consider $H_1$, $H_2$ as locally convex Hausdorff topological vector spaces with $\sigma(H)$-topology, that is, the topology generated by all functionals $\xi \mapsto \langle \xi, \eta \rangle$ on $H_1$, and by all functionals $\eta \mapsto \langle \xi, \eta \rangle$ on $H_2$ respectively.

Let $X$ be a topological group or a topological algebra over the complex number field $\mathbb{C}$. A linear system representation (LSR, for short) of $X$ means a pair $T = \langle T_1, T_2 \rangle$ of a representation $T_1$ of $X$ on $H_1$ and an antirepresentation $T_2$ of $X$ on $H_2$ such that $\langle T_1(x)\xi, \eta \rangle = \langle \xi, T_2(x)\eta \rangle$ for all $x \in X$, $\xi \in H_1$, and $\eta \in H_2$, and that the $\mathbb{C}$-valued functions $x \mapsto \langle T_1(x)\xi, \eta \rangle$ on $X$ are continuous for all $\xi \in H_1$, $\eta \in H_2$.

Two LSR's $T = \langle T_1, T_2 \rangle$ on $H = \langle H_1, H_2 \rangle$ and $T' = \langle T'_1, T'_2 \rangle$
on $H' = \langle H'_1, H'_2 \rangle$ are called equivalent if there exists a pair $\Phi = \langle \psi_1, \psi_2 \rangle$ of linear isomorphisms $\psi_1$ of $H_1$ onto $H'_1$ and $\psi_2$ of $H_2$ onto $H'_2$ such that $\langle \psi_1(\xi), \psi_2(\eta) \rangle = \langle \xi, \eta \rangle$ for all $\xi \in H_1$, $\eta \in H_2$ and that $\psi_1 T_1(x) \psi_1^\dagger = T_1(x)$, $\psi_2 T_2(x) \psi_2^\dagger = T_2(x)$ for all $x \in X$.

A LSR $T = \langle T_1, T_2 \rangle$ of $X$ on $H = \langle H_1, H_2 \rangle$ is called irreducible if every $T_1$-invariant non-trivial subspace of $H_1$ is $\sigma(H)$-dense in $H_1$, or equivalently, if every $T_2$-invariant non-trivial subspace of $H_2$ is $\sigma(H)$-dense in $H_2$.

Let $G$ be a locally compact unimodular group, and $L(G)$ the algebra of all continuous functions on $G$ with compact supports, with multiplication defined by convolution. For every compact subset $C$ of $G$, denote by $L_C(G)$ the normed space of all continuous functions on $G$ whose supports are contained in $C$ with supremum norm. Then $L(G)$ is, as the inductive limit of $\{L_C(G); C$ is a compact subset of $G\}$, a topological algebra. A LSR $T = \langle T_1, T_2 \rangle$ of $G$ on $H = \langle H_1, H_2 \rangle$ is called integrable with respect to $L(G)$ if, for every function $f \in L(G)$, there exist linear operators $T_1(f)$ on $H_1$ and $T_2(f)$ on $H_2$ such that

$$\int_G \langle T_1(x) \xi, \eta \rangle f(x) dx = \langle T_1(f) \xi, \eta \rangle = \langle \xi, T_2(f) \eta \rangle$$

for all $\xi \in H_1$, $\eta \in H_2$, where $dx$ denotes a Haar measure on $G$. For a compact subgroup $K$ of $G$, it is called integrable with respect to $L(K)$ if the restriction of $T$ on $K$ is integrable with respect to $L(K)$. 

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§2. Decomposability of LSR's

Let \( \mathcal{T} \) be a measure space with a \( \sigma \)-finite measure \( \mu \). Suppose there is given, for almost every \( \tau \in \mathcal{T} \), a linear system 
\[ F_\tau = \langle F_1^\tau, F_2^\tau \rangle. \]
Two functions \( \zeta, \zeta' \), defined for almost all \( \tau \in \mathcal{T} \) with its values \( \zeta(\tau), \zeta'(\tau) \) in \( F_i^\tau \) (i = 1 or 2), are identified if \( \zeta(\tau) = \zeta'(\tau) \) for almost all \( \tau \in \mathcal{T} \). Let \( F_1 \) be a vector space of functions (or, strictly speaking, equivalence classes of functions with respect to this identification) \( \xi \) on \( \mathcal{T} \) with its values \( \xi(\tau) \) in \( F_1^\tau \), and \( F_2 \), similarly, a vector space of functions \( \eta \) on \( \mathcal{T} \) with its values \( \eta(\tau) \) in \( F_1^\tau \). When we consider each element \( \xi \in F_1 \) as an equivalence class, we shall denote by \( \hat{\xi} \) a representative function in \( \xi \). Similarly we shall denote by \( \hat{\eta} \) a representative function in \( \eta \in F_2 \). For a such pair \( F_1, F_2 \), we give the following three definitions.

**DEFINITION 1.** A pair \( F_1, F_2 \) will be called summable if \( \tau \mapsto \langle \xi(\tau), \eta(\tau) \rangle \) is a \( \mathbb{C} \)-valued summable function on \( \mathcal{T} \) for every \( \xi \in F_1, \eta \in F_2 \).

**DEFINITION 2.** A pair \( F_1, F_2 \) will be called regular if, for every function \( \phi \in L^\infty(\mathcal{T}, \mu) \), \( \xi \in F_1 \) implies \( \phi \xi \in F_1 \), and \( \eta \in F_2 \) implies \( \phi \eta \in F_2 \), where \( \phi \xi(\tau) = \phi(\tau) \xi(\tau), \phi \eta(\tau) = \phi(\tau) \eta(\tau) \).

**DEFINITION 3.** A pair \( F_1, F_2 \) will be called saturating if, for arbitrary complete systems of representative functions \( \{ \xi; \xi \in F_1 \} \) and \( \{ \eta; \eta \in F_2 \} \) there exists a linear isomorphism between the vector spaces \( F_1 \) and \( F_2 \).
\( \varepsilon \in F_1 \) and \( \{ \hat{\xi}; \xi \in F_2 \} \), the set \( \{ \xi(\tau); \xi \in F_1 \} \) is \( \sigma(F^T) \)-dense in \( F_1^T \) and \( \{ \hat{\eta}(\tau); \eta \in F_2 \} \) is \( \sigma(F^T) \)-dense in \( F_2^T \) for almost all \( \tau \in \mathcal{T} \).

**Lemma 1.** Let \( F_1 \), \( F_2 \) be a regular and saturating pair, then there exist \( \xi_0 \in F_1 \) and \( \eta_0 \in F_2 \) such that \( \xi_0(\tau) \neq 0 \) and \( \eta_0(\tau) \neq 0 \) for almost all \( \tau \in \mathcal{T} \).

Let \( F_1 \), \( F_2 \) be a regular saturating summable pair. Then the bilinear form

\[
\langle \xi, \eta \rangle = \int_{\mathcal{T}} \langle \xi(\tau), \eta(\tau) \rangle \, d\mu(\tau)
\]

gives a duality between \( F_1 \) and \( F_2 \). We shall call the linear system \( F = \langle F_1, F_2 \rangle \) with this duality a direct integral of \( F^T \), and denote it by

\[
F = \langle F_1, F_2 \rangle = \int_{\mathcal{T}} \langle F_1^T, F_2^T \rangle \, d\mu(\tau).
\]

**Definition 4.** Let \( X \) be a topological group or a topological algebra. A LSR \( U = \langle U_1, U_2 \rangle \) of \( X \) on a linear system \( E = \langle E_1, E_2 \rangle \) is called decomposable if the following three conditions are satisfied.

1. The linear system \( E = \langle E_1, E_2 \rangle \) is isomorphic to a direct integral \( F = \langle F_1, F_2 \rangle = \int_{\mathcal{T}} \langle F_1^T, F_2^T \rangle \, d\mu(\tau) \).

2. For almost all \( \tau \in \mathcal{T} \), irreducible LSR's \( V^T = \langle V_1^T, V_2^T \rangle \) are defined on \( F^T = \langle F_1^T, F_2^T \rangle \).

3. Denote by \( V_1(x) \xi, V_2(x) \eta \) the functions defined by \( [V_1(x) \xi](\tau) = V_1^T(x) \xi(\tau), [V_2(x) \eta](\tau) = V_2^T(x) \eta(\tau) \). Then \( \xi \in F_1, \eta \in F_2 \) implies \( V_1(x) \xi \in F_1, V_2(x) \eta \in F_2 \) for all \( x \in X \), and there exists
an isomorphism $\Phi = \langle \Phi_1, \Phi_2 \rangle$ of $E$ onto $F$ such that

$$V_1(x) = \Phi_1 U_1(x) \Phi_1^{-1}, \quad V_2(x) = \Phi_2 U_2(x) \Phi_2^{-1}$$

for all $x \in X$.

The LSR $U = \langle U_1, U_2 \rangle$ is called finite-dimensionally decomposable if, in addition, $F^\tau = \langle F_1^\tau, F_2^\tau \rangle$ are finite-dimensional for almost all $\tau \in $.

§3. Spherical LSR's of $L^0(\delta)$ and canonical LSR's of $G$

Let $G$ be a locally compact unimodular group, $K$ a compact subgroup of $G$, and $\delta$ an equivalence class of irreducible representations of $K$. The normalized trace of $\delta$ will be denoted by $\chi_\delta$, and the normalized Haar measure on $K$ will be denoted by $du$.

For a LSR $T = \langle T_1, T_2 \rangle$ of $G$ on $H = \langle H_1, H_2 \rangle$ which is integrable with respect to $L(G)$ and $L(K)$, we define continuous projections $P_1(\delta), P_2(\delta)$ on $H_1, H_2$ respectively by

$$\int_K \langle T_1(u) \xi, \eta \rangle \overline{\chi_\delta(u)} du = \langle P_1(\delta) \xi, \eta \rangle = \langle \xi, P_2(\delta) \eta \rangle.$$

Put $H_1(\delta) = P_1(\delta)H_1, H_2(\delta) = P_2(\delta)H_2$, then $H(\delta) = \langle H_1(\delta), H_2(\delta) \rangle$ is a linear system with the duality $\langle \ , \ \rangle$ restricted from $H$.

For every function $f \in L(\delta) = \overline{\chi_\delta} \ast L(G) \ast \overline{\chi_\delta}$, the space $H_1(\delta)$ is invariant under $T_1(f)$, and $H_2(\delta)$ is invariant under $T_2(f)$. Hence we obtain a LSR $\tilde{T} = \langle \tilde{T}_1, \tilde{T}_2 \rangle$ of $L(\delta)$ on $H(\delta) = \langle H_1(\delta), H_2(\delta) \rangle$ where $\tilde{T}_1(f) = T_1(f) \big|_{H_1(\delta)}$ and $\tilde{T}_2(f) = T_2(f) \big|_{H_2(\delta)}$ for each $f \in L(\delta)$. If $T$ is irreducible, then $\tilde{T}$ is also irreducible.

Now we fix a unitary matricial representation $u \mapsto D(u)$ of $K$ which belongs to $\delta$. We shall denote by $d$ its degree and by
\[ d \int_K \langle T_1(u) \xi, \eta \rangle d_i^i(u) du = \langle P_1^i(\delta) \xi, \eta \rangle = \langle \xi, P_2^i(\delta) \eta \rangle. \]

Put \( H_1^i(\delta) = P_1^i(\delta) H_1 \), \( H_2^i(\delta) = P_2^i(\delta) H_2 \), then the pairs \( H_1^i(\delta) = \langle H_1^i(\delta), H_2^i(\delta) \rangle \) are linear systems with the dualities restricted from \( H \). Since \( H_1^i(\delta) \) and \( H_2^i(\delta) \) are invariant under \( T_1(f) \) and \( T_2(f) \) respectively for all functions \( f \in L^0(\delta) = \{ f^o ; f \in L(\delta) \} \),

where \( f^o(x) = \int_K f(ux^{-1}) d\mu \), we obtain \( d \) LSR's of the algebra \( L^0(\delta) \) on \( H^i(\delta) = \langle H_1^i(\delta), H_2^i(\delta) \rangle \) for \( i = 1, \ldots, d \). These LSR's are mutually equivalent. A LSR \( U = \langle U_1, U_2 \rangle \) of \( L^0(\delta) \) will be called a spherical LSR corresponding to \( T = \langle T_1, T_2 \rangle \) if it is equivalent to these LSR's of \( L^0(\delta) \).

For a linear system \( E = \langle E_1, E_2 \rangle \), we shall denote by \( E_1^d \) the vector space of all column vectors \( \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_d \end{pmatrix} \) with \( \xi_i \in E_1 \), and by \( E_2^d \) the vector space of all column vectors \( \eta \) whose components are in \( E_2 \). Then \( E^d = \langle E_1^d, E_2^d \rangle \) is a linear system with the duality \( \langle \xi, \eta \rangle = \sum_{i=1}^d \langle \xi_i, \eta_i \rangle \).

**Lemma 2.** Let \( U = \langle U_1, U_2 \rangle \) be a LSR of \( L^0(\delta) \) on \( E = \langle E_1, E_2 \rangle \) which satisfies one of the following conditions,

(a) \( U \) is a spherical LSR corresponding to a LSR of \( G \),

(b) \( U \) is irreducible and finite-dimensional.

Then there exists a unique LSR \( \tilde{U} = \langle \tilde{U}_1, \tilde{U}_2 \rangle \) of \( L(\delta) \) on \( E^d = \langle E_1^d, E_2^d \rangle \),
\( E_1 \) such that
\[
\tilde{U}_1(\varepsilon_k \ast f)\xi = D(k) \left( \begin{array}{c} U_1(f) \xi_1 \\ \vdots \\ U_1(f) \xi_d \end{array} \right), \quad \tilde{U}_2(\varepsilon_k \ast f)\eta = tD(k) \left( \begin{array}{c} U_2(f) \eta_1 \\ \vdots \\ U_2(f) \eta_d \end{array} \right)
\]
for all \( k \in K \) and \( f \in L^0(\delta) \), where \( \varepsilon_k \ast f(x) = f(k^x) \) and \( tD(k) \) is the transposed matrix of \( D(k) \), and right hand sides are formal products of matrices.

Let \( U = \langle U_1 , U_2 \rangle \) be a finite-dimensional irreducible LSR of \( L^0(\delta) \) on a linear system \( E = \langle E_1 , E_2 \rangle \), and \( \tilde{U} = \langle \tilde{U}_1 , \tilde{U}_2 \rangle \) the LSR of \( L(\delta) \) on \( E^d = \langle E_1^d , E_2^d \rangle \) which is given in Lemma 2. Then it is not difficult to show that \( \tilde{U} = \langle \tilde{U}_1 , \tilde{U}_2 \rangle \) is irreducible. Choose non zero vectors \( \xi_0 \in E_1^d \) and \( \eta_0 \in E_2^d \) arbitrarily, and put
\[
\mathfrak{M}_1 = \{ f \in L(G) ; \tilde{U}_1(\chi_0 \ast g \ast f \ast \chi_0)\xi_0 = 0 \text{ for all } g \in L(G) \}, \\
\mathfrak{M}_2 = \{ g \in L(G) ; \tilde{U}_2(\chi_0 \ast g \ast f \ast \chi_0)\eta_0 = 0 \text{ for all } f \in L(G) \}.
\]
Then \( \mathfrak{M}_1 \) is a closed maximal left ideal and \( \mathfrak{M}_2 \) is a closed maximal right ideal in \( L(G) \). Now put
\[
H_1 = \frac{L(G)}{\mathfrak{M}_1}, \quad H_2 = \frac{L(G)}{\mathfrak{M}_2}.
\]
Denoting by \([f]_1\) the coset in \( H_1 \) which contains \( f \) and by \([g]_2\) the coset in \( H_2 \) which contains \( g \), the pair \( H = \langle H_1 , H_2 \rangle \) is a linear system with the duality
\[
\langle [f]_1 , [g]_2 \rangle = \langle \tilde{U}_1(\chi_0 \ast g \ast f \ast \chi_0)\xi_0 , \eta_0 \rangle.
\]
Then the LSR \( T = \langle T_1 , T_2 \rangle \) of \( G \) on \( H = \langle H_1 , H_2 \rangle \), defined by
\[
T_1(x)[f]_1 = [\varepsilon_x \ast f]_1, \quad T_2(x)[g]_2 = [g \ast \varepsilon_x]_2,
\]
is irreducible, and is called a canonical LSR of \( G \) corresponding to \( U \). Of course it depends on the choice of \( \xi_0 \) and \( \eta_0 \), but it is
unique up to equivalence.

§4. Decomposability of a homogeneous LSR of G

Let G, K, and \( \delta \) be the same as in §3. Let \( T = \langle T_1, T_2 \rangle \) be a LSR of G on a linear system \( H = \langle H_1, H_2 \rangle \). Under the condition of integrability with respect to \( L(K) \), it is called G-homogeneous with respect to \( \delta \) if every \( T_1 \)-invariant subspace of \( H_1 \) containing \( H_1(\delta) \) is \( \sigma(H) \)-dense in \( H_1 \), and if every \( T_2 \)-invariant subspace of \( H_2 \) containing \( H_2(\delta) \) is \( \sigma(H) \)-dense in \( H_2 \).

Suppose there exist \( \sigma(H) \)-dense \( T_1 \)- or \( T_2 \)-invariant subspaces \( H_1', H_2' \) of \( H_1, H_2 \) respectively, then \( H' = \langle H_1', H_2' \rangle \) is a linear system with the duality restricted from \( H \). We shall call the LSR \( T' = \langle T_1', T_2' \rangle \), where \( T_1' = T_1|_{H_1'} \) and \( T_2' = T_2|_{H_2'} \), a dense contraction of \( T \) on \( H' \).

THEOREM. Assume that G is second countable. Let \( T = \langle T_1, T_2 \rangle \) be a LSR of G on \( H = \langle H_1, H_2 \rangle \), which is integrable with respect to \( L(G), L(K) \), and is G-homogeneous with respect to \( \delta \). Suppose the corresponding spherical LSR of \( L^\delta(\delta) \) is finite-dimensionally decomposable, then there exists a decomposable dense contraction \( T' \) of \( T \) on \( H' = \langle H_1', H_2' \rangle \) which is integrable with respect to \( L(G) \) and \( L(K) \) and satisfies \( H_1'(\delta) = H_1(\delta), H_2'(\delta) = H_2(\delta) \).

Let's sketch the outline of the proof. Let \( U = \langle U_1, U_2 \rangle \) be the corresponding spherical LSR of \( L^\delta(\delta) \) on a linear system \( E = \langle E_1, E_2 \rangle \). For simplicity we consider as follows.
(1) \( E = \langle E_1, E_2 \rangle = \int_J \langle E_1^\perp, E_2^\perp \rangle \, d\mu(\tau). \)

(2) For almost all \( \tau \in J \), finite-dimensional irreducible LSR's \( U^\perp = \langle U_1^\perp, U_2^\perp \rangle \) of \( L^0(\delta) \) are defined on \( E^\perp = \langle E_1^\perp, E_2^\perp \rangle \).

(3) For every \( \xi \in E_1 \), we have \( [U_1(f) \xi](\tau) = U_1^\perp(f) \xi(\tau) \), and for every \( \eta \in E_2 \), we have \( [U_2(f) \eta](\tau) = U_2^\perp(f) \eta(\tau) \).

Consider the algebras \( \mathcal{A}(G) = L^\infty(J, \mu) \otimes \mathcal{L}(G) \) and \( \mathcal{A}(\delta) = L^\infty(J, \mu) \otimes \mathcal{L}(\delta) \). Let \( \tilde{U} = \langle \tilde{U}_1, \tilde{U}_2 \rangle \) be the LSR of \( L(\delta) \) which is given in Lemma 2 for \( U \). For every element \( \alpha = \sum_i \phi_i \otimes f_i \in \mathcal{A}(\delta) \), we define

\[
\pi_1(\alpha) \xi = \sum_i \tilde{U}_1(f_i) \phi_i \xi, \quad \pi_2(\alpha) \eta = \sum_i \tilde{U}_2(f_i) \phi_i \eta
\]

(\( \xi \in E_1^d, \eta \in E_2^d \)). By Lemma 1, there exist \( \xi_0 \in E_1 \) and \( \eta_0 \in E_2 \) such that \( \xi_0(\tau) \neq 0 \) and \( \eta_0(\tau) \neq 0 \) for almost all \( \tau \in J \). We put

\[
\xi_0 = \begin{pmatrix} \xi_0 \\ \vdots \\ 0 \end{pmatrix} \in E_1^d, \quad \eta_0 = \begin{pmatrix} \eta_0 \\ \vdots \\ 0 \end{pmatrix} \in E_2^d.
\]

Then, using the second countability of \( G \), we can prove the following

**Lemma 3.** The subspace \( \{\pi_1(\alpha) \xi_0 : \alpha \in \mathcal{A}(\delta)\} \) is \( \sigma(E^d) \)-dense in \( E_1^d \), and \( \{\pi_2(\alpha) \eta_0 : \alpha \in \mathcal{A}(\delta)\} \) is \( \sigma(E^d) \)-dense in \( E_2^d \).

Let \( B(\ , \ ) \) be a bilinear form on \( \mathcal{A}(G) \times \mathcal{A}(G) \) defined by

\[
B(\alpha, \beta) = \sum_{i,j} \langle \tilde{U}_1(\overline{g}_{i}\star f_i \star \overline{g}_{j}) \phi_i \xi_0, \psi_j \eta_0 \rangle
\]

for \( \alpha = \sum_i \phi_i \otimes f_i \) and \( \beta = \sum_j \psi_j \otimes g_j \). Now we put

\[
\mathcal{M}_1 = \{ \alpha \in \mathcal{A}(G) : B(\alpha, \beta) = 0 \text{ for all } \beta \in \mathcal{A}(G) \},
\]

\[
\mathcal{M}_2 = \{ \beta \in \mathcal{A}(G) : B(\alpha, \beta) = 0 \text{ for all } \alpha \in \mathcal{A}(G) \}.
\]
Then the pair $\mathfrak{f} = \langle \mathfrak{f}_1, \mathfrak{f}_2 \rangle$, $\mathfrak{f}_1 = \mathcal{A}(G) / \mathcal{M}_1$, $\mathfrak{f}_2 = \mathcal{A}(G) / \mathcal{M}_2$, is a linear system with the duality

$$\langle [\alpha]_1, [\beta]_2 \rangle = \mathbb{B}(\alpha, \beta).$$

Now we construct a LSR $S = \langle S_1, S_2 \rangle$ of $G$ on $\mathfrak{f} = \langle \mathfrak{f}_1, \mathfrak{f}_2 \rangle$ by

$$S_1(x)[\alpha]_1 = [\varepsilon_x \ast \alpha]_1, \quad S_2(x)[\beta]_2 = [\beta \ast \varepsilon_x]_2$$

for every $x \in G$.

Since the LSR $\tilde{U} = \langle \tilde{U}_1, \tilde{U}_2 \rangle$ of $L(\delta)$ on $E^d = \langle E^d_1, E^d_2 \rangle$ is equivalent to $\tilde{T} = \langle \tilde{T}_1, \tilde{T}_2 \rangle$ on $H(\delta) = \langle H_1(\delta), H_2(\delta) \rangle$, there exists an isomorphism $\psi = \langle \psi_1, \psi_2 \rangle$ of $E^d$ onto $H(\delta)$ such that $\tilde{T}_1(f) = \psi_1 U_1(f) \psi_1^\dagger$, $\tilde{T}_2(f) = \psi_2 U_2(f) \psi_2^\dagger$ for all $f \in L(\delta)$. For every element $\alpha = \sum_i \phi_i \otimes f_i \in \mathcal{A}(G)$, we put

$$\phi_1([\alpha]_1) = \sum_i T_1(f_i) \psi_1(\phi_i \varepsilon_\delta), \quad \phi_2([\alpha]_2) = \sum_i T_2(f_i) \psi_2(\phi_i \varepsilon_\delta).$$

Then $\phi = \langle \phi_1, \phi_2 \rangle$ is a homomorphism of $\mathfrak{f} = \langle \mathfrak{f}_1, \mathfrak{f}_2 \rangle$ into $H = \langle H_1, H_2 \rangle$, and, by Lemma 3, the images $H'_1 = \phi_1(\mathfrak{f}_1)$, $H'_2 = \phi_2(\mathfrak{f}_2)$ are $\sigma(H)$-dense $T_1$- or $T_2$-invariant subspaces of $H_1$, $H_2$ respectively. The dense contraction $T' = \langle T'_1, T'_2 \rangle$ of $T$ on $H' = \langle H'_1, H'_2 \rangle$ is integrable with respect to $L(G)$, $L(K)$, and satisfies $H'_1(\delta) = H_1(\delta)$, $H'_2(\delta) = H_2(\delta)$. Moreover it is equivalent to $S = \langle S_1, S_2 \rangle$.

On the other hand, using vectors $\xi_0(\tau) \in (E^d_1)^d$, $\eta_0(\tau) \in (E^d_2)^d$, we can construct the canonical LSR $T^T = \langle T^T_1, T^T_2 \rangle$ of $G$ on a linear system $H^T = \langle H^T_1, H^T_2 \rangle$ corresponding to $U$ with

$$\langle [f]_1^T, [g]_2^T \rangle = \langle \tilde{U}_1(\tilde{T}_1^T f) \ast g \ast \bar{x}_\delta \varepsilon_x(\tau), \eta_0(\tau) \rangle,$$

$$T^T_1(x)[f]_1^T = [\varepsilon_x \ast f]_1^T, \quad T^T_2(x)[g]_2^T = [g \ast \varepsilon_x]_2^T.$$

**LEMMA 4.** For every function $f \in L(\delta)$, we have

$$\langle \tilde{U}_1(f) \xi, \eta \rangle = \int \langle \tilde{U}_1(f) \xi(\tau), \eta(\tau) \rangle \, d\mu(\tau) \quad (\xi \in E^d_1, \eta \in E^d_2).$$
It follows from Lemma 4 that, for every \( \alpha = \sum_{i} \phi_i \otimes f_i \in \mathcal{A}(G) \),
\[ \beta = \sum_{j} \psi_j \otimes g_j \in \mathcal{A}(G) , \]
\[ \langle [\alpha]_1 , [\beta]_2 \rangle = \int_{G} \langle \sum_{i} \left[ \phi_i(\tau) f_i \right]^T , \sum_{j} \left[ \psi_j(\tau) g_j \right]^T \rangle \ d\mu(\tau) . \]
This means that every \( [\alpha]_1 \in \mathfrak{H}_1 \) can be seen as a function
\[ [\alpha]_1(\tau) = \sum_{i} \left[ \phi_i(\tau) f_i \right]^T \]
on \( G \), and that every \( [\beta]_2 \in \mathfrak{H}_2 \) can be seen as a function
\[ [\beta]_2(\tau) = \sum_{j} \left[ \psi_j(\tau) g_j \right]^T . \]
Then it is easy to verify that the pair \( \mathfrak{H}_1 , \mathfrak{H}_2 \) is regular, summable, and saturating. Thus the LSR \( S = \langle S_1 , S_2 \rangle \) of \( G \) on \( \mathfrak{H} = \langle \mathfrak{H}_1 , \mathfrak{H}_2 \rangle \) is decomposable in the following way:
\[ \mathfrak{H}_2 = \langle \mathfrak{H}_1 , \mathfrak{H}_2 \rangle = \int_{G} \langle H_1^T , H_2^T \rangle \ d\mu(\tau) , \]
\[ \langle S_1(\tau) [\alpha]_1 , [\beta]_2 \rangle = \int_{G} \langle T_1^T(\tau) [\alpha]_1(\tau) , [\beta]_2(\tau) \rangle \ d\mu(\tau) . \]
Since, as is remarked above, \( S \) is equivalent to \( T' \), the theorem follows.