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Sperical Setions of a Homogeneous Vector Bundle

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In this note, we show some results on the integral representation of eigensections of invariant differential operators on a homogeneous vector bundle over a riemannian symmetric space. First we determine the structure of the algebra $\mathbb{D}$ of invariant differential operators on a homogeneous vector bundle, define the notions of eigensections and spherical sections corresponding to a given finite-dimensional representation of the algebra $\mathbb{D}$ and obtaine the dimension formula and an integral representation of spherical sections. The notion of spherical sections is a generalization of that of zonal spherical functions. The dimension formula of the space of spherical sections will be crucial for the problem of Poisson integral representation of eigensections, which I would like to solve during this summer.

Precisely speaking, let G be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$ of finite center, K a maximal compact subgroup with Lie algebra $\mathfrak{k}$, $G = KAN$ an Iwasawa decomposition with split torus A and $g = K(g)e^{H(g)}n(g)$ $(g \in G,\ K(g) \in K,\ e^{H(g)} \in A,\ n(g) \in N)$ the decomposition of g in G corresponding to $G = KAN$, where $H(g)$ is an element in the

Lie algebra $\mathfrak{a}$ of A. Let E denote the homogeneous vector bundle over G/K associated to a given irreducible unitary representation τ of K and $\mathbb{D}_\tau$ denote the algebra of invariant differential operators on E. Let $\mathfrak{G}$ (resp. $\mathfrak{K}$) denote the universal enveloping algebra of $\mathfrak{g}_c$ (resp. $\mathfrak{k}_c$) and $*$ denote the anti-automorphism of $\mathfrak{G}$ defined by $X^* = -X$ $(X \in \mathfrak{g})$. Let $\mathfrak{G}^K$ be the centralizer of K in $\mathfrak{G}$, $\mathfrak{J}_\tau$ the kernel of $d\tau$ in $\mathfrak{K}$ and $\mathfrak{J}_\tau^*$ the set of $z^*(z \in \mathfrak{J}_\tau)$. Then $\mathbb{D}_\tau$ is canonically isomorphic to the algebra $\mathfrak{G}^K/\mathfrak{G}^K \cap \mathfrak{G}\mathfrak{J}_\tau^*$.

Let ρ denote the half sum of the roots corresponding to N. For $\lambda \in \mathfrak{a}_c^*$, put

$$P_{\tau,\lambda}(g) = e^{-(\lambda+\rho)H(g^{-1})}\tau(K(g^{-1})) \qquad (g \in G) .$$

Then there exists an algebra homomorphism, say $\chi_{\tau,\lambda}$, of $\mathbb{D}_\tau$ into $\mathrm{End}_M(V)$ such that

$$\Delta P_{\tau,\lambda}(g) = P_{\tau,\lambda}(g) \circ \chi_{\tau,\lambda}(\Delta) \qquad (\Delta \in \mathbb{D}_\tau) ,$$

where V denotes the representation space of τ and $\mathrm{End}_M(V)$ denotes the set of endomorphisms on V which commute with $\tau(m)(m \in M = Z_K(A)$ = the centralizer of A in K).

For an irreducible representation (σ, V_σ) of M, put $H_\sigma = \mathrm{Hom}_M(V_\sigma, V)$ and $H_{\tau,\sigma} = \mathrm{Hom}_M(V, V_\sigma)$. Then $\chi_{\tau,\lambda}$ defines a representation $\chi_{\tau,\sigma,\lambda}$ of $\mathbb{D}_\tau$ on H_σ by

$$\chi_{\tau,\sigma,\lambda}(\Delta)a = \chi_{\tau,\lambda}(\Delta) \circ a \qquad (a \in H_\sigma) .$$

Let $\mathcal{A}(E)$ denote the space of analytic sections of E. Given a representation (χ, H) of $\mathbb{D}_\tau$, we call $u \in \mathcal{A}(E)$ an eigensection of type χ, if there exists a finite number of $\mathbb{D}_\tau$-invariant subspaces H_i in $\mathcal{A}(E)$ such that as a representation space of $\mathbb{D}_\tau$, each H_i is isomorphic to a quotient representation of (χ, H). Let $\mathcal{A}(E,\chi)$ denote the space of eigensections of type χ.

Let $F_{\sigma,\lambda}$ denote the vector bundle over G/MAN associated to the representation $\sigma \otimes e^{-\lambda+\rho} \otimes 1$ of MAN on V_σ and let $\mathcal{B}(F_{\sigma,\lambda})$ denote the space of $F_{\sigma,\lambda}$-valued hyperfunctions on G/MAN. For $\varphi \otimes a \in \mathcal{B}(F_{\sigma,\lambda}) \otimes H_\sigma$, put

$$P_{\tau,\sigma,\lambda}(\varphi \otimes a) = \int_K P_{\tau,\lambda}(k^{-1}g) a\varphi(k)dk .$$

Then it is easy to see that $P_{\tau,\sigma,\lambda}$ gives a G-linear mapping of $\mathcal{B}(F_{\sigma,\lambda}) \otimes H_\sigma$ into $\mathcal{A}(E,\chi_{\tau,\sigma,\lambda})$.

Let $\mathcal{A}(E,\chi_{\tau,\sigma,\lambda})^\tau$ denote the subspace consisting of the sections in $\mathcal{A}(E,\chi)$ which transform according to τ under the left regular representation of K on $\mathcal{A}(E)$. We identify $V \otimes H_{\tau,\sigma}$ with the subspace of sections in $\mathcal{B}(F_{\sigma,\lambda})$ which transform according to τ under the left regular representation of K on $\mathcal{B}(F_{\sigma,\lambda})$ by

$$V \otimes H_{\tau,\sigma} \ni v \otimes b \longmapsto \varphi_{v,b}(k) = b\tau(k^{-1})v \in \mathcal{B}(F_{\sigma,\lambda})$$

$$(v \in V\ ,\ b \in H_{\tau,\sigma},\ k \in K)\ ,$$

where $\varphi_{v,b}$ is regarded as an element in $\mathcal{B}(F_{\sigma,\lambda})$ by

$$\varphi_{v,b}(kan) = e^{(\lambda-\rho)H(a)}\varphi_{v,b}(k) .$$

Then the mapping

$$v \otimes b \otimes a \longmapsto P_{\tau,\sigma,\lambda}(\varphi_{v,b} \otimes a)$$

$$= \int_K P_{\tau,\lambda}(k^{-1}g)ab\tau(k^{-1})vdk$$

gives a linear mapping of $V \otimes H_{\tau,\sigma} \otimes H_\sigma$ onto $\mathcal{A}(E,\chi_{\tau,\sigma,\lambda})^\tau$. As a corollary, using the "subrepresentation theorem" of Casselman, we have a linear isomorphism

$$\mathcal{A}(E,\chi)^\tau \cong V \otimes (\mathbb{D}_\tau/\mathrm{Ker}\ \chi)^*$$

for any irreducible representation χ of $\mathbb{D}_\tau$.

Now it is very interesting to study when $P_{\tau,\sigma,\lambda}$ will give a G-isomorphism of $\mathcal{B}(F_{\sigma,\lambda}) \otimes H_\sigma$ onto $\mathcal{A}(E,\chi_{\tau,\sigma,\lambda})$. We have the following conjecture:

Under a certain regularity condition on λ, we can construct a boundary value mapping $\beta_{\tau,\sigma,\lambda}$ of $\mathcal{A}(E,\chi_{\tau,\sigma,\lambda})$ into $\mathcal{B}(F_{\sigma,\lambda}) \otimes H_\sigma$ such that

$$\beta_{\tau,\sigma,\lambda} \circ P_{\tau,\sigma,\lambda}(\varphi \otimes a) = \varphi \otimes c(\tau,\lambda)a, \quad \varphi \in \mathcal{B}(F_{\sigma,\lambda}), \quad a \in H_\sigma \quad ,$$

where $c(\tau,\lambda)$ is the generalized c-function.

It seems not so hard to prove it now because almost all necessary lemmas have been proved and we have only to see what differential operators in $\mathbb{D}_\tau$ will be necessary and suitable in order to determine the vector bundle to which the boundary values should belong.

References

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