

Openness Condition for Filtered Complexes  
and  
A Comparison Theorem

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In this note, we will prove an abstract comparison theorem concerning the completion of a filtered complex of abelian groups under a certain hypothesis of "openness" for the differentiation, and its extension to the case of an inductive system of filtered complexes. The latter is a generalization of a key lemma of N. Sasakura [3].

We use the notation and terminology of A. Grothendieck [1].

1. Statement of the results.

Let  $K^\bullet$  be a filtered complex of abelian groups :

$$K^\bullet \supset \dots \supset F^p K^\bullet \supset F^{p+1} K^\bullet \supset \dots \quad (p \in \mathbb{Z}).$$

By definition, the completion  $K^{\wedge}$  of  $K^\bullet$  is the projective limit  $\varprojlim_p K^\bullet / F^p K^\bullet$ . As a type of completion of the cohomology  $H^i(K)$  ( $i \in \mathbb{Z}$ ), we take the projective limit  $\varprojlim_p H^i(K / F^p K)$ . Then, by the universal property of  $\varprojlim_p$ , there exists a canonical functorial homomorphism

$$\psi^i : H^i(K^{\wedge}) \longrightarrow \varprojlim_p H^i(K / F^p K)$$

for each degree  $i \in \mathbb{Z}$ .

We say that  $K^*$  satisfies the openness condition  $(B_i)$ , if there exists a mapping  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that

$$F^{f(p)} K^i \cap B^i(K) \subset d^{i-1}(F^p K^{i-1})$$

for all  $p \in \mathbb{Z}$ . The condition  $(B_i)$  is nothing but the openness of the differentiation  $d^{i-1} : K^{i-1} \rightarrow B^i(K)$ ,  $B^i(K)$  being regarded as endowed with the filtration induced by that of  $K^i$ . As a weaker condition, we say that  $K^*$  satisfies the weak openness condition  $(WB_i)$ , if there exists a mapping  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that

$$F^{f(p)} K^i \cap B^i(K) \subset \bigcap_q (F^q K^i \cap B^i(K) + d^{i-1}(F^p K^{i-1}))$$

for all  $p \in \mathbb{Z}$ . In topological terms, the condition  $(WB_i)$  is equivalent to saying that  $d^{i-1} : K^{i-1} \rightarrow B^i(K)$  maps every neighborhood of the zero to a subset whose closure is a neighborhood of the zero. We will prove the following

Theorem I. Let  $K^*$  be a filtered complex of abelian groups. Assume that  $K^*$  satisfies the weak openness condition  $(WB_i)$  for a degree  $i \in \mathbb{Z}$ . Then the canonical homomorphism

$$\psi^i : H^i(K^\wedge) \longrightarrow \varprojlim_p H^i(K/F^p K)$$

is an isomorphism.

We remark that the (weak) openness condition plays as a substitute for the Artin-Rees theorem in the case of modules of finite type over a commutative Noetherian ring.

We will extend this comparison theorem to the case of an inductive system of filtered complexes.

Let  $(K_\alpha^*, u_{\beta\alpha})$  be an inductive system of filtered complexes of abelian groups indexed by a directed set  $(u_{\beta\alpha} : K_\alpha^* \rightarrow K_\beta^* \text{ for } \alpha \leq \beta)$

$\alpha \leq \beta$ ). We say that  $(K_\alpha)_\alpha$  satisfies the openness condition  $(B_i^*)$ , if, for every index  $\alpha$ , there exist a  $\beta \geq \alpha$  and a mapping  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that

$$u_{\beta\alpha}(F^{f(p)}K_\alpha^i \cap B^i(K_\alpha)) \subset d^{i-1}(F^p K_\beta^{i-1})$$

for all  $p \in \mathbb{Z}$ . We say that  $(K_\alpha)_\alpha$  satisfies the weak openness condition  $(WB_i^*)$ , if, for every index  $\alpha$ , there exist a  $\beta \geq \alpha$  and a mapping  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that

$$u_{\beta\alpha}(F^{f(p)}K_\alpha^i \cap B^i(K_\alpha)) \subset \bigcap_q (u_{\beta\alpha}(F^q K_\alpha^i \cap B^i(K_\alpha)) + d^{i-1}(F^p K_\beta^{i-1})),$$

for all  $p \in \mathbb{Z}$ . In this case, our result is

Theorem II. Let  $(K_\alpha, u_{\beta\alpha})$  be an inductive system of filtered complexes of abelian groups indexed by a directed set. Assume that  $(K_\alpha)_\alpha$  satisfies the weak openness condition  $(WB_i^*)$  for a degree  $i \in \mathbb{Z}$ . Then the canonical homomorphism

$$\psi^i = \varinjlim_\alpha \psi_\alpha^i : H^i(\varinjlim_\alpha \hat{K}_\alpha) \longrightarrow \varinjlim_\alpha \varprojlim_p H^i(K_\alpha / F^p K_\alpha)$$

is an isomorphism.

This Theorem II is a generalization of Prop. 2.1 in [3]

Though we will prove our theorems under the weak openness conditions  $(WB)$  and  $(WB^*)$ , the openness conditions  $(B)$  and  $(B^*)$  might be more useful in applications. That is why we detailed the latter. We remark that our way of proof is valid in a more general setting of categories.

## 2. Proof of Theorem I.

We use the right derived functor of  $\varprojlim_p : (\text{projective systems of abelian groups indexed by } \mathbb{Z}) \longrightarrow (\text{abelian groups})$ , which we denote by  $R^i \varprojlim_p$ . For this derived functor, we refer to R. Hartshorne [2], Chapter I, §4. Note that  $R^i \varprojlim_p = 0$  for  $i \geq 2$ .

Let  $K^\bullet$  be a filtered complex of abelian groups. Applying Proposition (4.4) (loc. cit.) to the projective system  $(K^\bullet / F^p K^\bullet)_p$ , we get an exact sequence

$$(2.1) \quad 0 \longrightarrow R^1 \varprojlim_p H^{i-1}(K/F^p K) \longrightarrow H^i(K^\wedge) \xrightarrow{\psi^i} \varprojlim_p H^i(K/F^p K) \longrightarrow 0$$

for each degree  $i \in \mathbb{Z}$ . (A rapid way to derive this exact sequence is to compare the two spectral sequences which converge to the hypercohomology  $R^i \varprojlim_p K^\bullet / F^p K^\bullet$ .) This leads us to the study of the kernel of  $\psi^i$ .

We propose to replace  $R^1 \varprojlim_p H^{i-1}(K/F^p K)$  by an abelian group which represents how far the differentiation  $d^{i-1} : K^{i-1} \longrightarrow B^i(K)$  is from openness. First, note that the short exact sequence of complexes

$$0 \longrightarrow F^p K^\bullet \longrightarrow K^\bullet \longrightarrow K^\bullet / F^p K^\bullet \longrightarrow 0$$

induces the cohomology exact sequence

$$(2.2) \quad \dots \longrightarrow H^i(F^p K) \longrightarrow H^i(K) \longrightarrow H^i(K/F^p K) \longrightarrow H^{i+1}(F^p K) \longrightarrow \dots$$

for each  $p$ . We set

$$(2.3) \quad L^i(K)_p = \text{Ker } (H^i(F^p K) \longrightarrow H^i(K))$$

and

$$F^p H^i(K) = \text{Im } (H^i(F^p K) \longrightarrow H^i(K)).$$

Then, from the long exact sequence (2.2), we get an exact sequence

$$(2.4) \quad 0 \longrightarrow H^{i-1}(K)/F^p H^{i-1}(K) \longrightarrow H^{i-1}(K/F^p K) \longrightarrow L^i(K)_p \longrightarrow 0$$

for each  $i \in \mathbb{Z}$ . We regard (2.4) as an exact sequence of projective systems indexed by  $p$ . Since the projective system in the second term of (2.4) consists of epimorphisms, its  $R^j \varprojlim_p$  vanish for  $j \geq 1$ . Hence, passing to the limit, the exact sequence (2.4) assures an isomorphism

$$(2.5) \quad R^1 \varprojlim_p H^{i-1}(K/F^p K) \xrightarrow{\sim} R^1 \varprojlim_p L^i(K)_p.$$

With this identification, we get an exact sequence

$$(2.6) \quad 0 \longrightarrow R^1 \varprojlim_p L^i(K)_p \longrightarrow H^i(K^\wedge) \longrightarrow \varprojlim_p H^i(K/F^p K) \longrightarrow 0,$$

in place of (2.1). Note that this exact sequence (2.6) is functorial in  $K$ .

Recall that a projective system  $(M_p, \pi_{pq})$  ( $\pi_{pq} : M_q \longrightarrow M_p$ , for  $p \leq q$ ) indexed by  $\mathbb{Z}$  is said to satisfy the Mittag-Leffler condition (ML) if there exists a mapping  $f : \mathbb{Z} \longrightarrow \mathbb{Z}$  ( $f(p) \geq p$ ) such that  $\text{Im } \pi_{pq} = \text{Im } \pi_{pf(p)}$ , for all  $p$  and  $q \geq f(p)$ . If  $(M_p)_p$  satisfies (ML), then  $R^1 \varprojlim_p M_p = 0$ . (loc. cit.) As a more restrictive condition, we say that  $(M_p)_p$  is essentially zero, if there exists a mapping  $f : \mathbb{Z} \longrightarrow \mathbb{Z}$  such that the homomorphism  $\pi_{pf(p)} : M_{f(p)} \longrightarrow M_p$  is zero for all  $p \in \mathbb{Z}$  (or, equivalently, if  $(M_p)_p$  is isomorphic to zero as a pro-object). It is easy to verify that, if  $(M_p)_p$  is essentially zero, then  $\varprojlim_p M_p = 0$  and  $R^1 \varprojlim_p M_p = 0$ . (The implication

$$(2.7) \quad (M_p)_p \text{ is essentially zero.} \implies (M_p)_p \text{ satisfies (ML).}$$

is clear.) Note that the notion "essentially zero" is stable

under functorial operations.

Returning to the filtered complex  $K^\bullet$ , we focus on the projective system  $(L^i(K)_p)_p$ . By the definition (2.3), we have

$$(2.8) \quad L^i(K)_p = \frac{F^p K^i \cap B^i(K)}{d^{i-1}(F^p K^{i-1})}.$$

Rewrite the two conditions "essentially zero" and (ML) for this  $(L^i(K)_p)_p$ , using the formulae (2.8). Then, we have the following dictionary :

$$(2.9) \quad K^\bullet \text{ satisfies } (B_i) \iff (L^i(K)_p)_p \text{ is essentially zero.}$$

$$(2.10) \quad K^\bullet \text{ satisfies } (WB_i) \iff (L^i(K)_p)_p \text{ satisfies (ML).}$$

((2.9) implies (2.10). cf. (2.7).)

Now, the Theorem I is clear. If  $K^\bullet$  satisfies  $(WB_i)$ , then  $(L^i(K)_p)_p$  satisfies (ML) and  $R^1 \varprojlim_p L^i(K)_p = 0$ . By the exact sequence (2.6), we have an isomorphism

$$\psi^i : H^i(K^\wedge) \xrightarrow{\sim} \varprojlim_p H^i(K/F^p K).$$

### 3. Proof of Theorem II.

Let  $(K_\alpha^\bullet, u_{\beta\alpha})$  be an inductive system of filtered complexes of abelian groups indexed by a directed set. Then, by the exact sequence (2.6), we get an exact sequence of inductive systems indexed by  $\alpha$

$$(3.1) \quad 0 \longrightarrow R^1 \varprojlim_p L^i(K_\alpha) \longrightarrow H^i(K^\wedge) \xrightarrow{\psi_\alpha^i} \varprojlim_p H^i(K_\alpha/F^p K_\alpha) \longrightarrow 0$$

for each degree  $i \in \mathbb{Z}$ .

We modify the definitions of "essentially zero" and (ML) for this case. Let  $(M_{p,\alpha} : \pi_{pq}^\alpha, u_{\beta\alpha}^p)$  be an inductive system, indexed by a

$$\begin{array}{ccc} M_{p,\alpha} & \xrightarrow{u_{\alpha\beta}^p} & M_{p,\beta} \\ \pi_{pq}^\alpha \uparrow & & \uparrow \pi_{pq}^\beta \\ M_{q,\alpha} & \xrightarrow{u_{\alpha\beta}^q} & M_{q,\beta} \end{array} \quad (p \leq q, \alpha \leq \beta)$$

directed set, of projective systems

indexed by  $\mathbb{Z}$ . Here,  $p$  and  $\alpha$  indicate

the indices as a projective system and as an inductive system,

respectively. We say that the system  $(M_{p,\alpha})_{p,\alpha}$  is essentially

zero, if, for every index  $\alpha$ , there exist a  $\beta \geq \alpha$  and a mapping

$f : \mathbb{Z} \rightarrow \mathbb{Z}$  ( $f(p) \geq p$ ) such that the homomorphism  $\pi_{pf(p)}^\beta \circ u_{\beta\alpha}^{f(p)}$  is zero for all  $p \in \mathbb{Z}$ . As for (ML), we say that

$(M_{p,\alpha})_{p,\alpha}$  satisfies (ML), if, for every index  $\alpha$ , there exist

a  $\beta \geq \alpha$  and a mapping  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  ( $f(p) \geq p$ ) such that

$\text{Im}(\pi_{pq}^\beta \circ u_{\beta\alpha}^q) = \text{Im}(\pi_{pf(p)}^\beta \circ u_{\beta\alpha}^{f(p)})$  for all  $p$  and  $q \geq f(p)$ . We

need the following

Lemma. a) If the system  $(M_{p,\alpha})_{p,\alpha}$  is essentially zero, then

$$\varprojlim_{\alpha} \varprojlim_p M_p = 0 \quad \text{and} \quad \varprojlim_{\alpha} R^1 \varprojlim_p M_p = 0.$$

b) If the system  $(M_{p,\alpha})_{p,\alpha}$  satisfies (ML), then  $\varprojlim_{\alpha} R^1 \varprojlim_p M_{p,\alpha} = 0$ .

Proof) We assume that  $(M_{p,\alpha})_{p,\alpha}$  satisfies (ML) (resp. is essentially zero). It suffices to show that, for every index  $\alpha$ , there exists a  $\beta \geq \alpha$  such that the homomorphism

$$R^1 \varprojlim_p u_{\beta\alpha}^p : R^1 \varprojlim_p M_{p,\alpha} \longrightarrow R^1 \varprojlim_p M_{p,\beta}$$

is zero for  $i = 1$  (resp.  $i = 0, 1$ ). For any index  $\alpha$ , we take the  $\beta \geq \alpha$  in the definition above, and set

$$N_p = \text{Im} (u_{\beta\alpha}^p : M_{p,\alpha} \longrightarrow M_{p,\beta}).$$

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Then the morphism of projective systems

$$(u_{\beta\alpha}^p)_p : (M_{p,\alpha})_p \longrightarrow (M_{p,\beta})_p$$

is factored through the projective system  $(N_p)_p$ . Passing to the limit, the homomorphism  $R^i \varprojlim_p u_{\beta\alpha}^p$  is also factored through  $R^i \varprojlim_p N_p$ . Since  $(N_p)_p$  satisfies (ML) (resp. is essentially zero) in the sense of  $n^\circ 2$ , we have  $R^i \varprojlim_p N_p = 0$  for  $i = 1$  (resp.  $i = 0, 1$ ). Hence,  $R^i \varprojlim_p u_{\beta\alpha}^p = 0$  for  $i = 1$  (resp.  $i = 0, 1$ ). q.e.d.)

Rewrite the conditions "essentially zero" and (ML) for the system  $(L^i(K_\alpha)_p)_{p,\alpha}$  as we did in  $n^\circ 2$ . Then we have a dictionary similar to (2.9) and (2.10) :

(3.2)  $(K_\alpha)_\alpha$  satisfies  $(B_i^*) \iff (L^i(K_\alpha)_p)_{p,\alpha}$  is essentially zero.

(3.3)  $(K_\alpha)_\alpha$  satisfies  $(WB_i^*) \iff (L^i(K_\alpha)_p)_{p,\alpha}$  satisfies (ML).

If  $(K_\alpha)_\alpha$  satisfies  $(WB_i^*)$ , then the system  $(L^i(K_\alpha)_p)_{p,\alpha}$  satisfies (ML), and we have  $\varinjlim_\alpha R^1 \varprojlim_p L^i(K_\alpha)_p = 0$  by Lemma above. Taking the inductive limit of the sequence (3.1), we get an isomorphism

$$\psi^i = \varinjlim_\alpha \psi_\alpha^i : H^i(\varinjlim_\alpha K_\alpha^\wedge) = \varinjlim_\alpha H^i(K_\alpha^\wedge) \xrightarrow{\sim} \varinjlim_\alpha \varprojlim_p H^i(K_\alpha / F^p K_\alpha).$$

This gives our result.

### References

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