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A METHOD OF CLASSIFYING EXPANSIVE SINGULARITIES

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Introduction

To study singularities is in a sense to study the classification of germs of varieties. It is therefore important to give a method of classification. The purpose of this paper is to show the classification of a class of germs of varieties, which will be called expansive singularities in this paper, is included in that of Lie algebras of formal vector fields. As a matter of course, the classification of the latter does not seem easy. However, note that such a Lie algebra is given by an inverse limit of finite dimensional Lie algebras of polynomial vector fields truncated at the order k, $k \geqq 0$. Therefore such Lie algebras can be understood by step by step method in the order k.

Let $\mathbb{C}^n$ be the Cartesian product of n copies of complex numbers $\mathbb{C}$ with natural coordinate system $(x_1,\cdots,x_n)$. By $\mathcal{O}$, we mean the ring of all convergent power series in $x_1,\cdots,x_n$ centered at the origin 0. Let V be a germ of variety in $\mathbb{C}^n$ at 0, and $\mathcal{J}(V)$ the ideal of V in $\mathcal{O}$ (cf.[2] pp86-7 for the definitions). Two germs V, V' are called bi-holomorphically equivalent if there is a germ of holomorphic diffeomorphism φ such that $\varphi(0) = 0$ and $\varphi(V) = V'$

Let $\mathfrak{X}$ be the Lie algebra of all germs of holomorphic vector fields at 0, and $\mathfrak{X}(V)$ the subalgebra defined by

$$\mathfrak{X}(V) = \{u \in \mathfrak{X} \;;\quad u\mathcal{J}(V) \subset \mathcal{J}(V)\}.$$

$\mathfrak{X}(V)$ is then an $\mathcal{O}$-module. If there are $v_1, \cdots, v_s$, linearly independent at 0, then Corollary 3,4 of [9] shows that V is bi-holomorphically equivalent to the direct product $\mathbb{C}^s \times W$, where $W \subset \mathbb{C}^{n-s}$. Thus, for the structure of singularities we have only to consider the germ W. Taking this fact into account, we may restrict our concern to the varieties such that all $u \in \mathfrak{X}(V)$ vanishes at 0, which we assume throughout this paper, i.e. $\mathfrak{X}(V)(0) = \{0\}$.

$u \in \mathfrak{X}(V)$ ($u(0) = 0$) is called a semi-simple expansive vector field, if after a suitable bi-holomorphic change of variables at 0, u can be written in the form

$$(1) \qquad u = \sum_{i=1}^{n} \hat{\mu}_i y_i \, \partial/\partial y_i ,$$

where $\hat{\mu}_1, \cdots, \hat{\mu}_n$ lie in the same open half-plane in $\mathbb{C}$ about the origin. (See also §2.A for a justification of this definition.) The origin 0 is called to be an expansive singularlity, if $\mathfrak{X}(V)$ contains a semi-simple expansive vector field. If V is given by the locus of zeros of a weighted homogeneous polynomial, then V has an expansive singularlity at 0. The advantage of existence of such a vector field u is that one can extend through exp tu a germ V to a subvariety $\tilde{V}$ in $\mathbb{C}^n$. In this paper we restrict our concern to the germs of varieties with expansive singularities at the origin.

For such $\mathfrak{X}(V)$, we set $\mathfrak{X}_k(V) = \{u \in \mathfrak{X}(V) \; ; \; j^k u = 0\}$, where $j^k u$ is the k-th jet at 0. Since $\mathfrak{X}(V) = \mathfrak{X}_0(V)$, $\mathfrak{X}_k(V)$ is a finite codimensional ideal of $\mathfrak{X}(V)$ such that $[\mathfrak{X}_k(V), \mathfrak{X}_\ell(V)] \subset \mathfrak{X}_{k+\ell}(V)$ and $\bigcap \mathfrak{X}_k(V) = \{0\}$. We denote by $\mathfrak{g}(V)$ the inverse limit of $\{ \mathfrak{X}(V)/\mathfrak{X}_k(V) \}_{k \geq 0}$ with the inverse limit topology. Since $\mathfrak{X}(V)/\mathfrak{X}_k(V)$ is finite dimensional, $\mathfrak{g}(V)$ is a Frechet space such that the Lie bracket product $[\ ,\ ] : \mathfrak{g}(V) \times \mathfrak{g}(V) \longmapsto \mathfrak{g}(V)$ is continuous. Namely, $\mathfrak{g}(V)$ is a Frechet-Lie algebra. It is obvious that $\mathfrak{g}(V)$ is a Lie algebra of formal vector fields, where a formal vector field u is a vector field $u = \sum_{i=1}^{n} u_i \partial/\partial x_i$ such that each u_i is a formal power series in $x_1, \cdots, x_n$ without constant terms. The statement to be proved in this paper is as follows :

Theorem I Let V, V' be germs of varieties with expansive singularities at the origins of $\mathbb{C}^n$, $\mathbb{C}^{n'}$ respectively. Notations and assumptions being as above, V and V' are bi-holomorphically equivalent, if and only if $\mathfrak{g}(V)$ and $\mathfrak{g}(V')$ are isomorphic as topological Lie algebras.

By the above result, we see especially that any isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$ preserves orders, that is, $\Phi\mathfrak{g}_k(V) = \mathfrak{g}_k(V')$ for every k. Hence, to classify $\mathfrak{g}(V)$ is to classify the inverse system $\{\mathcal{X}(V)/\mathcal{X}_k(V)\}_{k \geq 0}$. Note that $\mathcal{X}(V)/\mathcal{X}_k(V)$ is an extension of $\mathcal{X}(V)/\mathcal{X}_{k-1}(V)$ with an abelian kernel $\mathcal{X}_{k-1}(V)/\mathcal{X}_k(V)$. Such extensions can be classified by representations and second cohomologies (cf.[6]).

The proof of the above theorem is devided into several steps as follows :

Step 1. We define the concept of Cartan subalgebras and prove the conjugacy of Cartan subalgebras.

Step 2. Using the assumption that V (resp. V') has an expansive singularity at 0, we prove that there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(V)$ such that $\mathfrak{h} \subset \mathcal{X}(V)$ (resp. $\mathfrak{h}' \subset \mathcal{X}(V')$). By a suitable bi-holomorphic change of variables, every element of $\mathfrak{h}$ (resp. $\mathfrak{h}'$) can be changed simultaneously into a normal form, which is a polynomial vector field. Moreover, every "eigenvector with respect to ad($\mathfrak{h}$) is a polynomial vector field.

Step 3. Now, suppose there is an isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$. Then, by definition $\Phi(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}(V')$. Hence by Steps 1, 2 we may assume that $\Phi(\mathfrak{h}) \subset \mathfrak{X}(V')$. Thus, considering the eigenspace decomposition of $\mathfrak{g}(V)$, $\mathfrak{g}(V')$ with respect to $\mathrm{ad}(\mathfrak{h})$ $\mathrm{ad}(\mathfrak{h}')$ respectively, we see that Φ induces an isomorphism of $\mathfrak{P}$ onto $\mathfrak{P}'$, where $\mathfrak{P}$ (resp. $\mathfrak{P}'$) is the totality of $u \in \mathfrak{g}(V)$ (resp. $\mathfrak{g}(V')$) which can be expressed as a polynomial vector field with respect to the local coordinate system which normalizes $\mathfrak{h}$ (resp. $\mathfrak{h}'$).

Step 4. From isomorphism $\Phi : \mathfrak{P} \longmapsto \mathfrak{P}'$, we conclude by the same procedure as in [5] that there is a bi-holomorphic diffeomorphism φ of $\mathbb{C}^n$ onto $\mathbb{C}^{n'}$ such that $\varphi(0) = 0$ and $d\varphi\,\mathfrak{P} = \mathfrak{P}'$. The main idea of making such φ is roughly in the fact that every maximal subalgebra of $\mathfrak{P}$ corresponts to a point. However, since $\mathfrak{P}(0) = \{0\}$, the situation is much more difficult than that of [1]. Existence of expansive vector field plays an important role at this step as well as in the above steps.

Step 5. Recapturing V from the Lie algebra $\mathfrak{P}$, we can conclude $\varphi(V) = V'$.

The theorem is proved by this way. Note that the converse is trivial.

§1 Conjugacy of Cartan subalgebras

We denote a formal power series f in a form $f = \sum_{|\alpha|\geq 0} a_\alpha x^\alpha$, where $a_\alpha \in \mathbb{C}$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. We denote by $\mathfrak{F}$ the Lie algebra of all formal vector fields and $\mathfrak{F}_k$ the subalgebra

$$\{u \in \mathfrak{F} \ ; \ u = \sum_{i=1}^{n} \sum_{|\alpha|>k} a_{i,\alpha} x^\alpha \partial/\partial x_i \}$$

$\mathfrak{F}$ is then regarded as the inverse limit of the system $\{\mathfrak{F}/\mathfrak{F}_k \ ; \ p_k\}$, where $p_k : \mathfrak{F}/\mathfrak{F}_{k+1} \longmapsto \mathfrak{F}/\mathfrak{F}_k$ is the natural projection. We denote by $\tilde{p}_k$ the projection of $\mathfrak{F}$ onto $\mathfrak{F}/\mathfrak{F}_k$. p_k and $\tilde{p}_k$ are sometimes called forgetful mappings. Since $\mathfrak{F}/\mathfrak{F}_k$ is a finite dimensional vector space over $\mathbb{C}$, $\mathfrak{F}$ is a Frechet space, and the Lie bracket product is continuous.

Let $\mathfrak{g}$ be a closed Lie subalgebra of $\mathfrak{F}$, and $\mathfrak{g}_k = \mathfrak{F}_k \cap \mathfrak{g}$. The closedness of $\mathfrak{g}$ implies that $\mathfrak{g}$ is the inverse limit of the system $\{\mathfrak{g}/\mathfrak{g}_k \ ; \ p_k\}_{k\geq 0}$. In this paper, we restrict our concern to a closed subalgebra $\mathfrak{g}$ of $\mathfrak{F}_0$. For any subalgebra $\mathfrak{S}$ of $\mathfrak{g}$, we denote by $\mathfrak{n}(\mathfrak{S})$ the normalizer of $\mathfrak{S}$, i.e. $\mathfrak{n}(\mathfrak{S}) = \{u \in \mathfrak{g} \ ; [u, \mathfrak{S}] \subset \mathfrak{S}\}$, and by $\mathfrak{g}^{(0)}(\mathfrak{S})$ the 0-eigenspace of $\mathrm{ad}(\mathfrak{S})$, i.e. $\mathfrak{g}^{(0)}(\mathfrak{S})$ is the totality of $v \in \mathfrak{g}$ satisfying that there are non-negative integers m_k, $k \geq 0$, (depending on v) such that $\mathrm{ad}(S)^{m_k} v \in \mathfrak{g}_k$ for all $S \in \mathfrak{S}$ and for all $k \geq 0$, where $\mathrm{ad}(u)v = [u,v]$. If $\mathfrak{S}$ is nilpotent, then $\mathfrak{g}^{(0)}(\mathfrak{S}) \supset \mathfrak{n}(\mathfrak{S})$. Therefore, if $\mathfrak{g}^{(0)}(\mathfrak{S}) = \mathfrak{S}$, then $\mathfrak{n}(\mathfrak{S}) = \mathfrak{S}$. The converse is also true if $\dim \mathfrak{g}^{(0)}(\mathfrak{S}) < \infty$ (cf.[6]).

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a <u>Cartan subalgebra of</u> $\mathfrak{g}$, if the following conditions are satisfied :

($\mathfrak{h}$,1) $\mathfrak{h}$ is a closed subalgebra of $\mathfrak{g}$ such that $\tilde{p}_k \mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ for every $k \geq 0$.

$(\mathfrak{h},2)$ $\mathfrak{h} = \mathfrak{g}^{(0)}(\mathfrak{h})$.

Note that if $\dim \mathfrak{g} < \infty$ above $\mathfrak{h}$ is a usual Cartan subalgebra. The statement to be proved in this chapter is as follows :

Proposition A <u>Let</u> $\mathfrak{g}$ <u>be a closed subalgebra of</u> $\mathfrak{F}_o$. <u>Then, there exists a Cartan subalgebra</u> $\mathfrak{h}$ <u>of</u> $\mathfrak{g}$. <u>For Cartan subalgebras</u> $\mathfrak{h}$, $\hat{\mathfrak{h}}$ <u>of</u> $\mathfrak{g}$, <u>there is an inner automorphism</u> A <u>of</u> $\mathfrak{g}$ <u>such that</u> $A\mathfrak{h} = \hat{\mathfrak{h}}$.

1.A. Automorphisms of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a closed Lie subalgebra of $\mathfrak{F}_o$, and $\mathfrak{g}_k = \mathfrak{g} \cap \mathfrak{F}_k$. For every $u \in \mathfrak{g}$ the adjoint action $ad(u)$ leaves each $\mathfrak{g}_k$ invariant, hence $ad(u)$ induces a linear mapping $a_k(u)$ of $\mathfrak{g}/\mathfrak{g}_k$ into itself. $ad(u)$ is then regarded as the inverse limit of the system $\{a_k(u)\}_{k\geq o}$. Define a linear mapping $e^{t\cdot ad(u)} : \mathfrak{g} \mapsto \mathfrak{g}$ by the inverse limit of $\{e^{t\cdot a_k(u)}\}_{k\geq o}$. Since $ad(u)$ is a derivation of $\mathfrak{g}$, $e^{t\cdot ad(u)}$ is a one parameter family of automorphisms of $\mathfrak{g}$. The group $\mathcal{O}\!\mathcal{l}(\mathfrak{g})$ generated by $\{e^{ad(u)};\ u \in \mathfrak{g}\}$ is called the group of <u>inner</u> automorphisms of $\mathfrak{g}$. The purpose of this section is to investigate the structure of $\mathcal{O}\!\mathcal{l}(\mathfrak{g})$.

Let $\hat{\mathcal{O}}$ be the ring of all formal power series $\sum_{|\alpha|\geq 0} a_\alpha x^\alpha$ and $\hat{\mathcal{O}}_k$ the ideal given by $\hat{\mathcal{O}}_k = \{\sum_{|\alpha|\geq k+1} a_\alpha x^\alpha\}$. $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ is then a finite dimensional algebra over $\mathbf{C}$. We denote by $\tilde{\pi}_k$, π_k the projections $\hat{\mathcal{O}} \mapsto \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$, $\hat{\mathcal{O}}/\hat{\mathcal{O}}_{k+1} \mapsto \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ respectively. Every $u \in \mathfrak{F}_o$ acts naturally on $\hat{\mathcal{O}}$ as a derivation such that $u\hat{\mathcal{O}}_k \subset \hat{\mathcal{O}}_k$ for every k. Conversely, $u \in \mathfrak{F}_o$ can be characterised by the above property. Every $u \in \mathfrak{F}_o$ induces, therefore, a derivation $u^{(k)}$ of the algebra $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ and $u^{(k)}$ is canonically identified with $\tilde{p}_k u$. Conversely, for every derivation δ of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ such that $\delta\,\hat{\mathcal{O}}_o/\hat{\mathcal{O}}_k \subset \hat{\mathcal{O}}_o/\hat{\mathcal{O}}_k$ there is an element $u \in \mathfrak{F}_o$ such that $\delta = \tilde{p}_k u$.

Since a derivation $u : \hat{\mathcal{O}} \mapsto \hat{\mathcal{O}}$ can be regarded as an inverse limit of derivations $\{ \tilde{p}_k u : \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \mapsto \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \}$, we define an automorphism $\exp u$ of $\hat{\mathcal{O}}$ by an inverse limit of $\{ e^{\tilde{p}_k u} \}$. We denote by G' the group generated by $\{ \exp u \; ; \; u \in \mathfrak{g} \}$.

Define an automorphism $\mathrm{Ad}(\exp u)$ of $\mathfrak{F}$ by

(2) $\qquad (\mathrm{Ad}(\exp u)v)f = (\exp u)v(\exp{-u})f, \quad f \in \hat{\mathcal{O}}$.

Since $(d/dt)_{t=o}(\exp tu)f = uf$, we see easily that

(3) $\qquad \frac{d}{dt}\,\mathrm{Ad}(\exp tu)v = [\, u, \mathrm{Ad}(\exp tu)v \,]$.

On the other hand, $e^{t\cdot \mathrm{ad}(u)}$ satisfies the same differential equation. Thus, by uniqueness, we obtain

(4) $\qquad \mathrm{Ad}(\exp u) = e^{\mathrm{ad}(u)}$.

Especially, if $\mathfrak{g}$ is a closed Lie subalgebra of $\mathfrak{F}_o$, then $\mathrm{Ad}(\exp u)\mathfrak{g} = \mathfrak{g}$ for every $u \in \mathfrak{g}$. Since

$$e^{\mathrm{ad}(u)}e^{\mathrm{ad}(v)} = \mathrm{Ad}(\exp u\cdot \exp v),$$

we obtain that $\mathfrak{A}(\mathfrak{g}) = \{ \mathrm{Ad}(g) \; ; \; g \in G' \}$.

Let $G^{(k)}$ be the group generated by $\{ e^{\tilde{p}_k u} \; ; \; u \in \mathfrak{g} \}$. Since $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ is finite dimensional, $G^{(k)}$ is a Lie group with Lie algebra $\mathfrak{g}/\mathfrak{g}_k$. For every integer ℓ such that $\ell \leq k$, the group $G^{(k)}$ leaves $\mathfrak{g}_\ell/\mathfrak{g}_k$ invariant. Hence $\{ G^{(k)} \}_{k \geq o}$ forms an inverse system. We denote by G the inverse limit. Obviously, G' is a subgroup of G. However, note that if a sequence $(u_o, u_1, \cdots, u_n, \cdots)$ satisfies $u_\ell \in \mathfrak{g}_\ell$ for every $\ell \geq 0$, then $\exp u_o \cdot \exp u_1 \cdot \cdots \cdot \exp u_n \cdot \cdots$ is an element of G. Since $G^{(k)}$ is a Lie group, G is a topological group under the inverse limit topology. The purpose of the remainder of this section is to show G = G' and that G is a Frechet-Lie group with

Lie algebra $\mathfrak{g}$.

Let $G_1^{(k)}$, $k \geqq 1$, be the group generated by $\{e^{\tilde{p}_k u} \; ; \; u \in \mathfrak{g}_1\}$, and G_1 the inverse limit of $\{G_1^{(k)}\}_{k \geq 1}$.

1.1 Lemma exp <u>is a bijective mapping of</u> $\mathfrak{g}_1$ <u>onto</u> G_1.

Proof. Let $\exp_k$ be the exponential mapping of $\mathfrak{g}_1/\mathfrak{g}_k$ into $G_1^{(k)}$, i.e. $\exp_k u = e^{\tilde{p}_k u}$. Since $\exp : \mathfrak{g}_1 \mapsto G_1$ is defined by the inverse limit of $\{\exp_k\}$, we have only to show that $\exp_k : \mathfrak{g}_1/\mathfrak{g}_k \mapsto G_1^{(k)}$ is bijective. Since $\mathfrak{g}_1/\mathfrak{g}_k = \tilde{p}_1\mathfrak{g}_1$ is a nilpotent Lie algebra, we see that $\exp_k$ is regular and surjective (cf. [3] p 229). However, the derivation $\tilde{p}_k u : \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \mapsto \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ is expressed by a triangular matrix with zeros in the diagonal. Therefore, one can define $\log(1 + N)$ by $\sum_{n=1}^{\infty} (-1)^{n-1} N^n/n$, which gives the inverse of $\exp_k$. Thus $\exp_k$ is bijective.

1.2 Corollary $G' = G$.

Proof. We have only to show $G' \supset G$. Since $G^{(1)} = G/G_1$ is generated by $\{\tilde{p}_1 u \; ; \; u \in \mathfrak{g}\}$, every $g \in G$ can be written in the form $g = \exp u_1 \cdot \exp u_2 \cdots \exp u_m \cdot h$, where $u_1, \cdots, u_m \in \mathfrak{g}$ and $h \in G_1$. Thus, the above lemma shows $G \subset G'$.

We next prove that G is a Frechet-Lie group. Although such a structure of G has no direct relevance to our present purpose, there is an advantage of making analogies easy from the theory of finite dimensional Lie groups.

Let $\sigma : \tilde{p}_1\mathfrak{g} \mapsto \mathfrak{g}$ be a linear mapping such that $\tilde{p}_1 \sigma \tilde{u} = \tilde{u}$ for $\tilde{u} \in \tilde{p}_1\mathfrak{g}$. It is not hard to see that $\xi(u) = \exp \sigma \tilde{p}_1 u \cdot \exp(u - \sigma \tilde{p}_1 u)$ gives a homeomorphism of an open neighborhood U of 0 of $\mathfrak{g}$ onto an open neighborhood $\tilde{U}$ of the identity e of G. Since G is a topological group, there is an open neighborhood V of 0 of $\mathfrak{g}$ such that

$\xi(V)^{-1} = \xi(V)$, $\xi(V)^2 \subset \xi(U)$. We set $\eta(u,v) = \xi^{-1}(\xi(u)\xi(v))$ and $i(u) = \xi^{-1}(\xi(u)^{-1})$ for $u, v \in V$. We have next to prove the differentiability of η and i. However, the differentiability is defined by inverse limits of differentiable mappings, hence that of η and i are trivial in our case. Thus, we get the following:

1.3 Lemma G is a Frechet-Lie group with Lie algebra $\mathfrak{g}$.

1.B. Simultaneous normalization and eigenspace decomposition

For any $u \in \mathfrak{F}_o$, the linear mapping $u^{(k)} : \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \mapsto \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ splits uniquely into a sum of semi-simple part $u_s^{(k)}$ and nilpotent part $u_N^{(k)}$ such that $[u_s^{(k)}, u_N^{(k)}] = 0$. Using eigenspace decomposition of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$, we see that $u_s^{(k)}$ is also a derivation of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ hence so is $u_N^{(k)}$. For $u^{(k+1)}$, we have that $[p_k u_s^{(k+1)}, p_k u_N^{(k+1)}] = 0$, $p_k u_N^{(k+1)}$ is nilpotent, and that $p_k u_s^{(k+1)}$ is semi-simple by considering eigenspace decomposition of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_{k+1}$. Therefore, $p_k u_s^{(k+1)} = u_s^{(k)}$ and $p_k u_N^{(k+1)} = u_N^{(k)}$. Hence, taking inverse limit, we get formal vector fields u_s, u_N which will be called the <u>semi-simple part</u> and the <u>nilpotent part</u> of u respectively. A formal vector field is called to be <u>semi-simple</u> if it has no nilpotent part.

Let $\mathcal{S}^k$ be a nilpotent subalgebra of $\mathfrak{F}_o/\mathfrak{F}_k$ for an arbitrarily fixed k. Set $\mathcal{S}_s^k = \{u_s^{(k)} ; u^{(k)} \in \mathcal{S}^k\}$, and denote by p_k^ℓ the forgetful projection of $\mathfrak{F}_o/\mathfrak{F}_k$ onto $\mathfrak{F}_o/\mathfrak{F}_\ell$ that is, $p_k^\ell = p_\ell p_{\ell+1} \cdots p_{k-1}$. Since $p_k^1 \mathcal{S}^k$ is a nilpotent subalgebra of $\mathfrak{F}_o/\mathfrak{F}_1$, there is a basis $(f_1^{(1)}, \cdots, f_n^{(1)})$ of $\hat{\mathcal{O}}_o/\hat{\mathcal{O}}_1$ such that every $u^{(1)} \in p_k^1 \mathcal{S}^k$ is represented by an upper triangular matrix. Let $(\mu_1(u^{(1)}), \cdots, \mu_n(u^{(1)}))$ be the diagonal part. μ_j is then a linear mapping of $p_k^1 \mathcal{S}^k$ into $\mathbb{C}$ for every j, which one may regard as a linear mapping of $\mathcal{S}^k$ into $\mathbb{C}$. Since $u_s^{(1)}$ is the semi-simple part of $u^{(1)}$, it must satisfy

(5) $$u_s^{(1)} f_j^{(1)} = \mu_j(u^{(1)}) f_j^{(1)}.$$

By a simple linear algebra, we see that there are $f_1^{(k)}, \cdots, f_n^{(k)} \in \hat{\mathcal{O}}_o/\hat{\mathcal{O}}_k$ such that

(6) $$u_s^{(k)} f_j^{(k)} = \mu_j(u^{(k)}) f_j^{(k)}, \quad \pi_k^\ell f_j^{(k)} = f_j^{(\ell)} \quad (1 \leq j \leq n)$$

for every $u^{(k)} \in \mathcal{S}^k$, where π_k^ℓ is the forgetful projection of $\hat{\mathcal{O}}_o/\hat{\mathcal{O}}_k$ onto $\hat{\mathcal{O}}_o/\hat{\mathcal{O}}_\ell$, that is, $\pi_k^\ell = \pi_\ell \cdot \pi_{\ell+1} \cdots \pi_{k-1}$.

Since $f_j^{(k)} \in \hat{\mathcal{O}}_o/\hat{\mathcal{O}}_k$, $f_j^{(k)}$ is expressed in the form

(7) $$f_j^{(k)} = \sum_{0<|\alpha|\leq k} a_{j,\alpha} x^\alpha .$$

Set $y_j = \sum_{0<|\alpha|\leq k} a_{j,\alpha} x^\alpha$. Since $f_1^{(1)}, \cdots, f_n^{(1)}$ are linearly independent, these give a formal change of variables and every $u_s^{(k)}$ can be written in the form

(8) $$u_s^{(k)} = \sum_{i=1}^{n} \mu_i(u^{(k)}) y_i \partial/\partial y_i .$$

Since $[\mathcal{S}_s^k, \mathcal{S}^k] = 0$, because $\mathcal{S}^k$ is nilpotent, every $u^{(k)} \in \mathcal{S}^k$ should be written in the form

(9) $$u^{(k)} = \sum_{i=1}^{n} \sum_{\substack{\langle\alpha,\mu\rangle=\mu_i \\ 0<|\alpha|\leq k}} a_{i,\alpha} y^\alpha \partial/\partial y_i$$

where $\langle\alpha,\mu\rangle = \alpha_1\mu_1 + \cdots + \alpha_n\mu_n$. It should be noted that the semi-simple part $u_s^{(k)}$ of $u^{(k)}$ has been changed into a linear diagonal vector field such as (8).

Let $\mathcal{S}^{k+1}$ be another nilpotent subalgebra of $\mathcal{F}_o/\mathcal{F}_{k+1}$ such that $p_k\mathcal{S}^{k+1} \subset \mathcal{S}^k$, and let $\mathcal{S}_s^{k+1} = \{u_s^{(k+1)} ; u^{(k+1)} \in \mathcal{S}^{k+1}\}$. Since $p_{k+1}^1 \mathcal{S}^{k+1} \subset p_k^1 \mathcal{S}^k$, the equality (5) holds also for every $u^{(1)} \in p_{k+1}^1 \mathcal{S}^{k+1}$ and the equality (6) does for every $p_k\mathcal{S}^{k+1}$. By a simple linear algebra, we see that there are $f_1^{(k+1)}, \cdots, f_n^{(k+1)} \in \hat{\mathcal{O}}_o/\hat{\mathcal{O}}_{k+1}$ such that

(10) $u_s^{(k+1)} f_j^{(k+1)} = \mu_j(u^{(k+1)}) f_j^{(k+1)}, \quad \pi_k f_j^{(k+1)} = f_j^{(k)}.$

Note that $f_j^{(k+1)} = f_j^{(k)} + \sum_{|\alpha|=k+1} a_{j,\alpha} x^{\alpha}$. Hence by putting

(11) $y_j = \sum_{0<|\alpha|\leq k+1} a_{j,\alpha} x^{\alpha}$

instead of (7), we get the same equations as (8) and (9) with respect to $\mathcal{S}^k$. Moreover we have

(12) $u_s^{(k+1)} = \sum_{i=1}^{n} \mu_i(u^{(k+1)}) y_i \partial/\partial y_i$,

(13) $u^{(k+1)} = \sum_{i=1}^{n} \sum_{\substack{\langle\alpha,\mu\rangle=\mu_i \\ 0<|\alpha|\leq k+1}} a_{i,\alpha} y^{\alpha} \partial/\partial y_i$

for every $u^{(k+1)} \in \mathcal{S}^{k+1}$. Especially we obtain the following :

1.4 Lemma <u>Notations and assumptions being as above, the forgetful projection</u> $p_k : \mathcal{S}_s^{k+1} \mapsto \mathcal{S}_s^k$ <u>is injective</u>.

Let $\{\mathcal{S}^k\}_{k\geq 1}$ be a series of nilpotent subalgebras $\mathcal{S}^k$ of $\mathfrak{F}_o/\mathfrak{F}_k$ such that $p_k \mathcal{S}^{k+1} \subset \mathcal{S}^k$ for every $k \geq 1$. We denote by $\mathcal{S}$ the inverse limit, and set $\mathcal{S}_s = \{u_s \ ; \ u \in \mathcal{S}\}$. Note that $\dim \mathcal{S}_s^k \leq n$ for every $k \geq 1$. Thus, there is an integer k_o such that $p_k : \mathcal{S}_s^{k+1} \mapsto \mathcal{S}_s^k$ is bijective for every $k \geq k_o$. By a method of inverse limit, we see that there is a formal change of variables

(14) $y_j = f_j(x_1, \cdots, x_n) \quad 1 \leq j \leq n, \quad f_j \in \hat{\mathcal{O}}_o$

such that (8) and (9) hold for every $u^{(k)} \in \mathcal{S}^k$ $(k \geq 1)$, and

(15) $u_s = \sum_{i=1}^{n} \mu_i(u) y_i \partial/\partial y_i$,

(16) $u = \sum_{i=1}^{n} \sum_{\langle\alpha,\mu\rangle=\mu_i} a_{i,\alpha} y^{\alpha} \partial/\partial y_i$

for every $u \in \mathcal{S}$.

Now, let $\mathfrak{g}$ be a closed subalgebra of $\mathfrak{F}_o$, and suppose the above $\mathcal{S}^k$'s are subalgebras of $\mathfrak{g}/\mathfrak{g}_k$ respectively. Hence, the inverse limit $\mathcal{S}$ is a closed subalgebra of $\mathfrak{g}$. We next consider the eigen-space decomposition of $\mathfrak{g}$ with respect to $\mathrm{ad}(\mathcal{S})$. Since

$ad(u) : \mathfrak{F}_o \longmapsto \mathfrak{F}_o$ leaves $\mathfrak{g}$ invariant for every $u \in \mathcal{S}$, and $[ad(u), ad(u_s)] = 0$, we see that $ad(u_s) : \mathfrak{F}_o \longmapsto \mathfrak{F}_o$ is the semi-simple part of $ad(u)$ and hence $ad(u_s)\mathfrak{g} \subset \mathfrak{g}$. Therefore, we have only to consider the eigenspace decomposition with respect to $ad(\mathcal{S}_s)$.

For a linear mapping λ of $\tilde{p}_1\mathcal{S}_s$ into $\mathbb{C}$, i.e. $\lambda \in (\tilde{p}_1\mathcal{S}_s)^*$, we denote by $\mathfrak{F}_\lambda$ the subspace

$$\{u \in \mathfrak{F}_o ; u = \sum_{i=1}^{n} \sum_{\langle\alpha,\mu\rangle-\mu_i=\lambda} a_{i,\alpha} y^\alpha \partial/\partial y_i\}.$$

Note that $\mathfrak{F}_\lambda = \{0\}$ for allmost all $\lambda \in (\tilde{p}_1\mathcal{S}_s)^*$ except countably many λ's. By $\Pi(\mathcal{S})$ we denote the set of all $\lambda \in (\tilde{p}_1\mathcal{S}_s)^*$ such that $\mathfrak{F}_\lambda \neq \{0\}$. If $\tilde{p}_1\mathcal{S}_s = \{0\}$, then we set $\Pi(\mathcal{S}) = 0$, because all μ_j's are zeros.

1.5 Lemma <u>If</u> $\tilde{p}_1\mathcal{S}_s = 0$, <u>then</u> $\mathfrak{g}^{(o)}(\mathcal{S}) = \mathfrak{g}$.

Proof. By (16), every $u \in \mathcal{S}$ can be written in the form $u = u_1 + u_2$ such that

$$u_1 = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_j^i y_j \partial/\partial y_i , \quad u_2 = \sum_{i=1}^{n} \sum_{|\alpha|\geq 2} a_{i,\alpha} y^\alpha \partial/\partial y_i .$$

The reason for the shape of u_1 is that the linear part of u is an upper triangular matrix. Therefore, for every $k \geq 1$, there is an integer m_k such that $ad(u)^{m_k} \mathfrak{F}_o \subset \mathfrak{F}_k$ for every $u \in \mathcal{S}$. This means $\mathfrak{g} = \mathfrak{g}^{(o)}(\mathcal{S})$ by definition.

Now, we set $\mathfrak{g}^{(\lambda)}(\mathcal{S}) = \mathfrak{g} \cap \mathfrak{F}_\lambda$ for every $\lambda \in \Pi(\mathcal{S})$.

1.6 Lemma <u>Every</u> $u \in \mathfrak{g}$ <u>can be rearranged in the form</u>

$$u = \sum_{\lambda\in\Pi(\mathcal{S})} u_\lambda , \quad u_\lambda \in \mathfrak{F}_\lambda$$

<u>Moreover, every</u> u_λ <u>is contained in</u> $\mathfrak{g}^{(\lambda)}(\mathcal{S})$.

Proof. Since the first assertion is trivial, we have only to show the second one. Since $\Pi(\mathcal{S})$ is a countable set, there is $v_o \in \mathcal{S}_s$ such that $\lambda(v_o^{(1)}) \neq \lambda'(v_o^{(1)})$ for any $\lambda, \lambda' \in \Pi(\mathcal{S})$ such that $\lambda \neq \lambda'$. For every k, let $u^{(k)}$ be the truncation of $u \in \mathfrak{g}$ at the order k. $u^{(k)}$ is canonically identified with $\tilde{p}_k u$. $u^{(k)}$ can be rearranged in the form $u^{(k)} = \sum_{\lambda\in\Pi(\mathfrak{s})} u_\lambda^{(k)}$, where each $u_\lambda^{(k)}$ is the truncation of u_λ at the order k. Since $\mathfrak{g}/\mathfrak{g}_k$ is finite dimensional, only finite number of $u_\lambda^{(k)}$'s do not vanish. Apply $\operatorname{ad}(v_o^{(k)})^\ell$ to $u^{(k)}$. Since $\operatorname{ad}(\mathfrak{s}_s)\mathfrak{g} \subset \mathfrak{g}$, we have

$$\operatorname{ad}(v_o^{(k)})^\ell u^{(k)} = \sum_{\lambda\in\Pi(\mathfrak{s})} (v_o)^\ell u_\lambda^{(k)} \in \mathfrak{g}/\mathfrak{g}_k$$

Hence, considering Vandermonde's matrix, we get $u_\lambda^{(k)} \in \mathfrak{g}/\mathfrak{g}_k$. Thus, taking inverse limit, we get $u_\lambda \in \mathfrak{g}$, hence the desired result.

1.7 Corollary $\tilde{p}_k\,\mathfrak{g}^{(o)}(\mathfrak{s})$ <u>is the zero-eigenspace of</u> $\operatorname{ad}(\tilde{p}_k\mathfrak{s}) : \mathfrak{g}/\mathfrak{g}_k \mapsto \mathfrak{g}/\mathfrak{g}_k$.

Proof. It is trivial that $\tilde{p}_k\,\mathfrak{g}^{(o)}(\mathfrak{s})$ is contained in the zero-eigenspace of $\operatorname{ad}(\tilde{p}_k\mathfrak{s})$, for $[\mathfrak{s}_s, \mathfrak{g}^{(o)}(\mathfrak{s})] = \{0\}$. Thus, we have only to show the converse. The zero-eigenspace of $\operatorname{ad}(\tilde{p}_k\mathfrak{s})$ is equal to that of $\operatorname{ad}(\tilde{p}_k\mathfrak{s}_s)$, that is, the space of all $v^{(k)} \in \mathfrak{g}/\mathfrak{g}_k$ such that $[\tilde{p}_k\mathfrak{s}_s, v^{(k)}] = \{0\}$. Thus, $v^{(k)}$ should be written in the form (9). Let $v \in \mathfrak{g}$ be an element such that such that $\tilde{p}_k v = v^{(k)}$, and let $v = \sum_{\lambda\in\Pi(\mathfrak{s})} v_\lambda$ be the decomposition in accordance with the above lemma. Then it is clear that $\tilde{p}_k v_o = v^{(k)}$. Since $v_o \in \mathfrak{g}^{(o)}(\mathfrak{s})$, we get the desired result.

1.C Existence and conjugacy of Cartan subalgebras

Let $\mathfrak{g}$ be a closed subalgebra of $\mathfrak{F}_o$. If $\mathfrak{g}/\mathfrak{g}_1 = \{0\}$, then $\mathfrak{g}/\mathfrak{g}_k$ is nilpotent for every $k \geq 1$, for $[\mathfrak{g}_k, \mathfrak{g}_\ell] \subset \mathfrak{g}_{k+\ell}$. Therefore, by 1.5 Lemma, we see that $\mathfrak{g}$ itself is the only Cartan subalgebra of $\mathfrak{g}$. Thus, the conjugacy is trivial in this case.

Now, suppose $\mathfrak{g}/\mathfrak{g}_1 \neq \{0\}$, and let $\mathfrak{h}^1$ be a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_1$.

1.8 Lemma *Let* $\mathfrak{h}^1, \cdots, \mathfrak{h}^k$ *be a series of Cartan subalgebras of* $\mathfrak{g}/\mathfrak{g}_1, \cdots, \mathfrak{g}/\mathfrak{g}_k$ *respectively such that* $p_{\ell-1}\mathfrak{h}^{\ell} = \mathfrak{h}^{\ell-1}$ *for* $2 \leqq \ell \leqq k$. *Then, there is a Cartan subalgebra* $\mathfrak{h}^{k+1}$ *of* $\mathfrak{g}/\mathfrak{g}_{k+1}$ *such that* $p_k \mathfrak{h}^{k+1} = \mathfrak{h}^k$.

Proof. Let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_{k+1}$. We prove at first that $p_k\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$. Since $\mathfrak{h}'$ is nilpotent, so is $p_k\mathfrak{h}'$. Let $\mathfrak{h}'_s = \{u_s^{(k+1)};\ u^{(k+1)} \in \mathfrak{h}'\}$, and let $v^{(k)}$ be an element of the zero-eigenspace of $p_k\mathfrak{h}'$. Then, $[v^{(k)}, p_k\mathfrak{h}'_s] = \{0\}$ and hence $v^{(k)}$ can be written in the form (9). Let $v^{(k+1)}$ be an element of $\mathfrak{g}/\mathfrak{g}_{k+1}$ such that $p_k v^{(k+1)} = v^{(k)}$. Using the eigenspace decomposition of $\mathfrak{g}/\mathfrak{g}_{k+1}$ with respect to $\mathrm{ad}(\mathfrak{h}'_s)$, we see that $v^{(k+1)} = \sum_{\lambda \in \Pi(\mathfrak{h}')} v_\lambda^{(k+1)}$. Note that this decomposition is given by only rearranging of the terms of $v^{(k+1)}$ (cf.1.6 Lemma). Hence it is clear that $p_k v_o^{(k+1)} = v^{(k)}$. $v_o^{(k+1)}$ is an element of the zero-eigenspace of $\mathfrak{h}'_s$. However, since $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_{k+1}$ we get $v_o^{(k+1)} \in \mathfrak{h}'$. Thus, $v^{(k)} \in p_k\mathfrak{h}'$. Hence $p_k\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$.

By the well-known conjugacy of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_k$, there is an inner automorphism A such that $A(p_k\mathfrak{h}') = \mathfrak{h}^k$. Since there is a natural projection of $G^{(k+1)}$ onto $G^{(k)}$ (cf. 1.A), there is an inner automorphism A' of $\mathfrak{g}/\mathfrak{g}_{k+1}$ which induces naturally A. Thus, by setting $A'\mathfrak{h}' = \mathfrak{h}^{k+1}$, $\mathfrak{h}^{k+1}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_{k+1}$ such that $p_k\mathfrak{h}^{k+1} = \mathfrak{h}^k$.

By the above lemma, we have a series $\{\mathfrak{h}^k\}_{k \geqq 1}$ of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_k$ such that $p_k\mathfrak{h}^{k+1} = \mathfrak{h}^k$. Let $\mathfrak{h}$ be the inverse limit of $\mathfrak{h}^k$.

1.9 Lemma *Notations and assumptions being as above,* $\mathfrak{h}$ *is a Cartan*

<u>subalgebra of $\mathfrak{g}$.</u>

Proof. Since $\tilde{p}_k \mathfrak{h} = \mathfrak{h}^k$, $\tilde{p}_k \mathfrak{h}$ is a nilpotent subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ for every $k \geq 1$. By 1.7 Corollary, $\tilde{p}_k \mathfrak{g}^{(o)}(\mathfrak{h})$ is the zero-eigenspace of $\mathrm{ad}(p_k \mathfrak{h})$. Since $\tilde{p}_k \mathfrak{h} = \mathfrak{h}^k$ is a Cartan subalgebra, we have $\tilde{p}_k \mathfrak{g}^{(o)}(\mathfrak{h}) = \mathfrak{h}^k$ and hence $\mathfrak{g}^{(o)}(\mathfrak{h}) = \mathfrak{h}$. Thus, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

We next consider the converse of the above lemma.

1.10 Lemma <u>Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then, $\tilde{p}_k \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ for every</u> $k \geq 1$.

Proof. By 1.7 Corollary, the zero-eigenspace of $\mathrm{ad}(\tilde{p}_k \mathfrak{h})$ is equal to $\tilde{p}_k \mathfrak{g}^{(o)}(\mathfrak{h})$. Since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, we see $\tilde{p}_k \mathfrak{g}^{(o)}(\mathfrak{h}) = \tilde{p}_k \mathfrak{h}$. Thus, $\tilde{p}_k \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$.

As in 1.A, we denote by $G^{(k)}$ the Lie group generated by $\{e^{\tilde{p}_k u} ; u \in \mathfrak{g}\}$. Let $\pi_k : G^{(k+1)} \mapsto G^{(k)}$ be the natural projection. We shall next prove the conjugacy of Cartan subalgebras, which completes the proof of Proposition A. Let $\mathfrak{h}$, $\hat{\mathfrak{h}}$ be Cartan subalgebras of $\mathfrak{g}$. By the argument in the first part of this section, we may assume $\mathfrak{g}/\mathfrak{g}_1 \neq \{0\}$. Since $\tilde{p}_1 \mathfrak{h}$, $\tilde{p}_1 \hat{\mathfrak{h}}$ are Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_1$, there is $g_1 \in G^{(1)}$ such that $\mathrm{Ad}(g_1)(\tilde{p}_1 \mathfrak{h}) = \tilde{p}_1 \hat{\mathfrak{h}}$. Therefore, one may assume without loss of generality that $\tilde{p}_1 \mathfrak{h} = \tilde{p}_1 \hat{\mathfrak{h}}$. Let $G_\ell^{(k)}$ be the Lie group generated by $\{e^{\tilde{p}_k u} ; u \in \mathfrak{g}_\ell\}$ for any ℓ, $\ell \leq k$.

1.11 Lemma <u>Let $\mathfrak{h}$, $\hat{\mathfrak{h}}$ be Cartan subalgebras of $\mathfrak{g}$ such that</u> $\tilde{p}_k \mathfrak{h} = \tilde{p}_k \hat{\mathfrak{h}}$. <u>Then, there is</u> $g_{k+1} \in G_k^{(k+1)}$ <u>such that</u> $\mathrm{Ad}(g_{k+1})(\tilde{p}_{k+1} \mathfrak{h}) = \tilde{p}_{k+1} \hat{\mathfrak{h}}$.

Proof. Since $\tilde{p}_k \mathfrak{h} = \tilde{p}_k \hat{\mathfrak{h}}$, $\tilde{p}_{k+1} \mathfrak{h}$ and $\tilde{p}_{k+1} \hat{\mathfrak{h}}$ are Cartan subalgebras of $p_k^{-1} \tilde{p}_k \mathfrak{h} = p_k^{-1} \tilde{p}_k \hat{\mathfrak{h}}$. Let

$$p_k^{-1}\tilde{p}_k\mathfrak{h} = \tilde{p}_{k+1}\mathfrak{h} \oplus \sum_{\lambda\neq 0}\mathfrak{g}'_\lambda, \qquad p_k^{-1}p_k = \tilde{p}_{k+1}\hat{\mathfrak{h}} \oplus \sum_{\lambda\neq 0}\mathfrak{g}''_\lambda$$

be the eigenspace decompositions with respect to $ad(\tilde{p}_{k+1}\mathfrak{h})$ and $ad(\tilde{p}_{k+1}\hat{\mathfrak{h}})$ respectively. Since $p_k\tilde{p}_{k+1}\mathfrak{h} = p_k\tilde{p}_{k+1}\hat{\mathfrak{h}} = p_k\mathfrak{h}$, we see that $\sum\mathfrak{g}'_\lambda \subset \mathfrak{g}_k/\mathfrak{g}_{k+1}$ and $\sum\mathfrak{g}''_\lambda \subset \mathfrak{g}_k/\mathfrak{g}_{k+1}$. It is well-known (cf. [6] pp59-66) that there are $v_1,\cdots,v_m \in \sum_{\lambda\neq 0}\mathfrak{g}'_\lambda$, $w_1,\cdots,w_\ell \in \sum_{\lambda\neq 0}\mathfrak{g}''_\lambda$ such that

$$Ad(\exp v_1)\cdots Ad(\exp v_m)Ad(\exp w_1)\cdots Ad(\exp w\ )\tilde{p}_{k+1}\mathfrak{h} = \tilde{p}_{k+1}\hat{\mathfrak{h}}$$

Since $\exp v_i$, $\exp w_j \in G_k^{(k+1)}$, we see that there is $g_{k+1} \in G_k^{(k+1)}$ such that $Ad(g_{k+1})(\tilde{p}_{k+1}\mathfrak{h}) = \tilde{p}_{k+1}\hat{\mathfrak{h}}$.

Let G_k be the subgroup of G generated by $\{e^u ; u \in \mathfrak{g}_k\}$ For Cartan subalgebras $\mathfrak{h}$, $\hat{\mathfrak{h}}$ of $\mathfrak{g}$, the above lemma shows that there are elements $g_1, g_2, \cdots, g_k, \cdots$ such that $g_k \in G_k$ and

$$Ad(g_1)Ad(g_2)\cdots Ad(g_k)\hat{\mathfrak{h}} = \mathfrak{h} \mod \mathfrak{g}_{k+1}.$$

Note that $g_1g_2\cdots g_k\cdots \in G$, hence putting $g = g_1g_2\cdots g_k\cdots$, we see $Ad(g)\mathfrak{h} = \hat{\mathfrak{h}}$. This shows the conjugacy of Cartan subalgebras. Proposition A is thereby proved.

§2 Cartan subalgebras at expansive singularities

2.A Semi-simple expansive vector fields

In this section, notations are as in the introduction. A germ of holomorphic vector field $u \in \mathfrak{X}(V)$ is called <u>expansive</u>, if the eigenvalues of the linear term of u at 0 lie in the same open half plane in $\mathbb{C}$ about the origin. u is called to be <u>semi-simple expansive</u> if u is expansive and semi-simple as a formal vector field. The purpose of this section is to show the following :

2.1 Lemma <u>Let</u> $u \in \mathfrak{X}(V)$ <u>be a semi-simple expansive vector field. Then, there is a germ</u> $y_j = f_j(x_1, \cdots, x_n)$, $1 \leq j \leq n$, <u>of biholomorphic change of variables such that</u> u <u>can be written in the form</u>

$$u = \sum_{i=1}^{n} \hat{\mu}_i y_i \partial/\partial y_i$$

Proof. By a suitable change of variables $y_j = \sum_{0<|\alpha|\leq k} a_{j,\alpha} x^{\alpha}$ such as in (7), we have that u can be written in the form

$$u = \sum_{i=1}^{n} \hat{\mu}_i y_i \partial/\partial y_i + w, \quad w \in \mathfrak{X}_k(V)$$

for sufficiently large k. For the proof that u is linearizable, it is enough to show that there are holomorphic functions $f_1, \cdots, f_n$ in $y_1, \cdots, y_n$ such that $uf_j = \hat{\mu}_j f_j$ $(1 \leq j \leq n)$ and $f_j = y_j$ + higher order terms. Set $f_j = y_j + g_j$ and consider the equation $u(y_j + g_j) = \hat{\mu}_j(y_j + g_j)$. Then we get

$$(17) \qquad (u - \hat{\mu}_j)\, g_j = - w y_j .$$

Since k is sufficiently large, we have

$$(18) \qquad \lim_{t \to \infty} e^{-t(u - \hat{\mu}_j)} w\, y_j = 0$$

and

$$(19) \qquad - \int_0^{\infty} e^{-t(u - \hat{\mu}_j)} w\, y_j \, dt$$

exists as a germ of holomorphic functions (cf. [5]). Set $g_j = -\int_0^\infty e^{-t(u-\hat{\mu}_j)} w\, y_j\, dt$. Then,

$$(u-\hat{\mu}_j)g_j = \int_0^\infty \frac{d}{dt} e^{-t(u-\hat{\mu}_j)} w\, y_j\, dt = [\, e^{-t(u-\hat{\mu}_j)} w\, y_j]_0^\infty = -w\, y_j .$$

2.B Lie algebras containing semi-simple expansive vector fields.

Let $\mathfrak{g}$ be a closed subalgebra of $\mathfrak{F}_o$ such that $\mathfrak{g}$ contains a semi-simple expansive vector field X.

2.2 Lemma <u>Let</u> X <u>be a semi-simple expansive vector field in</u> $\mathfrak{g}$. <u>Then, there is a Cartan subalgebra</u> $\mathfrak{h}$ <u>of</u> $\mathfrak{g}$ <u>containing</u> X.

Proof. By the same proof as in the above lemma, we see that X can be linearizable by a suitable formal change of variables, and hence we may assume that X can be written in the form $X = \sum_{i=1}^{n} \hat{\mu}_i y_i \partial/\partial y_i$, $\mathrm{Re}\,\hat{\mu}_i > 0$. Let $\mathfrak{g}^{(0)}(X) = \{u \in \mathfrak{g}\ ;\ [X,u] = 0\}$. Since every $u \in \mathfrak{g}^{(0)}(X)$ can be written in the form

$$(20) \qquad u = \sum_{i=1}^{n} \sum_{\langle\alpha,\hat{\mu}\rangle = \hat{\mu}_i} a_{i,\alpha}\, y^\alpha \partial/\partial y_i ,$$

we see that $\mathfrak{g}^{(0)}(X)$ is a finite dimensional Lie subalgebra of $\mathfrak{g}$. Since $\mathrm{ad}(X) : \mathfrak{g}^{(0)}(X) \longmapsto \mathfrak{g}^{(0)}(X)$ is of diagonal type, there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^{(0)}(X)$ containing X. We shall show that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. For that purpose we have only to show $\mathfrak{g}^{(0)}(\mathfrak{h}) = \mathfrak{h}$. Since $X \in \mathfrak{h}$, we see $\mathfrak{g}^{(0)}(\mathfrak{h}) \subset \mathfrak{g}^{(0)}(X)$ and hence $\mathfrak{g}^{(0)}(\mathfrak{h})$ is the zero-eigenspace of $\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{g}^{(0)}(X)$. However since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}^{(0)}(X)$, we have $\mathfrak{h} = \mathfrak{g}^{(0)}(\mathfrak{h})$.

2.3 Corollary <u>If</u> $\mathfrak{g}$ <u>has a semi-simple expansive vector field, then every Cartan subalgebra</u> $\mathfrak{h}$ <u>of</u> $\mathfrak{g}$ <u>is finite dimensional and</u> $\mathfrak{g}^{(\lambda)}(\mathfrak{h})$ <u>is finite dimensional for every</u> $\lambda \in \Pi(\mathfrak{h})$.

Proof. By the above lemma, there is a finite dimensional Cartan subalgebra of $\mathfrak{g}$. However by Proposition A it implies that all Cartan subalgebras are finite dimensional and every Cartan subalgebra contains a semi-simple expansive vector field. Note that

$$\mathfrak{F}_\lambda = \{u \in \mathfrak{F}_o ; u = \sum_{i=1}^{n} \sum_{\langle\alpha,\mu\rangle-\mu_i=\lambda} a_{i,\alpha} y^\alpha \partial/\partial y_i$$

Since $\mathfrak{h}$ contains an expansive vector field, we see that $\dim \mathfrak{F}_\lambda < \infty$ and hence $\dim \mathfrak{g}^{(\lambda)}(\mathfrak{h}) < \infty$.

2.4 Corollary <u>Notations being as in the introduction, if</u> $\mathfrak{X}(V)$ <u>contains a semi-simple expansive vector field</u> X, <u>then there is a Cartan subalgebra</u> $\mathfrak{h}$ <u>of</u> $\mathfrak{g}(V)$ <u>such that</u> $\mathfrak{h} \subset \mathfrak{X}(V)$. <u>Moreover, for that</u> $\mathfrak{h}$, $\mathfrak{g}^{(\lambda)}(\mathfrak{h})$ <u>is contained in</u> $\mathfrak{X}(V)$ <u>for every</u> $\lambda \in \Pi(\mathfrak{h})$.

Proof. Since $X \in \mathfrak{X}(V)$, 2.1 Lemma shows that X can be written in the form $X = \sum_{i=1}^{n} \hat{\mu}_i y_i \partial/\partial y_i$ by a suitable biholomorphic change of variables. Therefore, every $u \in \mathfrak{g}^{(\lambda)}(\mathfrak{h})$ is contained in $\mathfrak{X}(V)$, because u is a polynomial vector field in $y_1, \cdots, y_n$.

2.C Isomorphisms of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$.

Let V, V' be germs of varieties in $\mathbb{C}^n$, $\mathbb{C}^{n'}$ respectively. Suppose there is a bicontinuous isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$.

2.5 Lemma <u>Let</u> $\mathfrak{h}$ <u>be a Cartan subalgebra of</u> $\mathfrak{g}(V)$. <u>Then, so is</u> $\Phi(\mathfrak{h})$ of $\mathfrak{g}(V')$.

Proof. Set $\mathfrak{h}' = \Phi(\mathfrak{h})$. Since $\Phi : \mathfrak{g}(V) \mapsto \mathfrak{g}(V')$ is continuous, for every k' there is an integer $k = k(k')$ such that $\Phi(\mathfrak{g}_k(V)) \subset \mathfrak{g}_{k'}(V')$. Thus, $\tilde{p}_{k'} \mathfrak{h}'$ is a nilpotent subalgebra of $\mathfrak{g}(V')/\mathfrak{g}_{k'}(V')$ and $\mathfrak{g}^{(o)}(\mathfrak{h}') \supset \Phi(\mathfrak{g}^{(o)}(\mathfrak{h}))$. Thus, replacing Φ by Φ^{-1}, we get the desired result.

Now, suppose that V and V' have expansive singularities at the origins respectively. By 2.4 Corollary, $\mathfrak{X}(V)$ and $\mathfrak{X}(V')$ contain Cartan subalgebras of $\mathfrak{g}(V)$ and $\mathfrak{g}(V')$ respectively.

2.6 Corollary <u>Assumptions being as above, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}(V)$ contained in $\mathfrak{X}(V)$. Suppose there is a bicontinuous isomorphism Φ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$. Then, there is a bicontinuous isomorphism Ψ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$ such that $\Psi(\mathfrak{h}) \subset \mathfrak{X}(V')$, that is, $\Psi(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}(V')$ contained in $\mathfrak{X}(V')$.</u>

Proof. By the above lemma, $\Phi(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}(V')$. By 2,4 Corollary, there is a Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{g}(V')$ contained in $\mathfrak{X}(V')$. By Proposition A, there is $g \in G$ such that $Ad(g)\Phi(\mathfrak{h}) = \mathfrak{h}'$. Note that $Ad(g) : \mathfrak{g}(V') \mapsto \mathfrak{g}(V')$ is a bicontinuous isomorphism. Thus, $\Psi = Ad(g)\Phi$ is the desired one.

In the remainder of this section, we assume that there is a bicontinuous isomorphism $\Phi : \mathfrak{g}(V) \mapsto \mathfrak{g}(V')$ such that $\Phi(\mathfrak{h}) = \mathfrak{h}'$ where $\mathfrak{h}$, $\mathfrak{h}'$ are Cartan subalgebras of $\mathfrak{g}(V)$, $\mathfrak{g}(V')$ respectively such that $\mathfrak{h} \subset \mathfrak{X}(V)$ and $\mathfrak{h}' \subset \mathfrak{X}(V')$. By 2.3-4 Corollaries, there is a local coordinate system $(y_1, \cdots, y_n)$, related biholomorphically to the original one such that every $\mathfrak{g}^{(\lambda)}(\mathfrak{h})$ is a finite dimensional space of polynomial vector fields in $y_1, \cdots, y_n$. We choose such a local coordinate system $(z_1, \cdots, z_{n'})$ for $\mathfrak{g}(V')$. Let $\mathfrak{p}(V; y_1, \cdots, y_n)$ (resp. $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$) be the totality of $u \in \mathfrak{g}(V)$ (resp. $\mathfrak{g}(V')$) such that u can be expressed as a polynomial vector field in $y_1, \cdots, y_n$ (resp. $z_1, \cdots, z_{n'}$). $\mathfrak{p}(V; y_1, \cdots, y_n)$ and $\mathfrak{p}(V'; z_1, \cdots, z_{n'})$ are Lie subalgebras of $\mathfrak{X}(V)$, $\mathfrak{X}(V')$ respectively. Since $\mathfrak{g}^{(\lambda)}(\mathfrak{h}) \subset \mathfrak{p}(V; y_1, \cdots, y_n)$ for every $\lambda \in \Pi(\mathfrak{h})$, we get the following :

2.7 Corollary <u>Notations and assumptions being as above, the above isomorphism</u> $\Phi : \mathfrak{g}(V) \mapsto \mathfrak{g}(V')$ <u>induces an isomorphism of</u> $\wp(V;y_1,\cdots,y_n)$ <u>onto</u> $\wp(V';z_1,\cdots,z_{n'})$.

Proof. Note that $\Phi(\mathfrak{g}^{(\lambda)}(\xi)) = \mathfrak{g}^{(\lambda)}(\xi')$, because $\mathfrak{g}^{(\lambda)}(\xi)$ is an eigenspace of $ad(\xi)$. Every $u \in \wp(V;y_1,\cdots,y_n)$ can be written in the form $u = \sum_{\lambda\in\Pi(\xi)} u_\lambda$, but the summation in this case is a finite sum. Since $\Phi(u) = \sum_{\lambda\in\Pi(\xi)} \Phi(u_\lambda)$ and $\Phi(u_\lambda) \in \mathfrak{g}^{(\lambda)}(\xi')$, we see that $\Phi(u) \in \wp(V';z_1,\cdots,z_{n'})$. Replacing Φ by Φ^{-1}, we get the desired result.

Let $\mathbb{C}[y_1,\cdots,y_n]$ be the ring of all polynomials in $y_1,\cdots,y_n$. Then, since $\mathfrak{g}(V)$ is an $\hat{\mathcal{O}}$-module, $\wp(V;y_1,\cdots,y_n)$ is a $\mathbb{C}[y_1,\cdots,y_n]$-module.

3 Theorem of Pursell-Shanks' type

In this chapter, we consider two Lie algebras $\wp(V;y_1,\cdots,y_n)$ and $\wp(V';z_1,\cdots,z_{n'})$ of polynomial vector fields such that they are $\mathbb{C}[y_1,\cdots,y_n]$ and $\mathbb{C}[z_1,\cdots,z_{n'}]$-module respectively and that there is an isomorphism Φ of $\wp(V;y_1,\cdots,y_n)$ onto $\wp(V';z_1,\cdots,z_{n'})$. The goal is as follows :

Theorem II <u>Notations and assumptions being as above, there is a biholomorphic mapping</u> φ <u>of</u> $\mathbb{C}^n$ <u>onto</u> $\mathbb{C}^{n'}$ <u>such that</u> $d\varphi\ \wp(V;y_1,\cdots y_n) = \wp(V';z_1,\cdots,z_{n'})$. <u>Moreover,</u> $\varphi(V) = V'$ <u>as germs of varieties.</u>

Note at first that Theorem II implies Theorem I in the introduction, for 2.6-7 Corollaries show that an isomorphism between $\mathfrak{g}(V)$ and $\mathfrak{g}(V')$ induces an isomorphism between $\wp(V;y_1,\cdots,y_n)$ and $\wp(V';z_1,\cdots,z_{n'})$.

3.A Characterization of maximal subalgebras

Let $\mathfrak{h}$ be a subalgebra of $\wp(V;y_1,\cdots,y_n)$. We denote by $\mathfrak{h}^{(\infty)}$ the ideal consisting of all $u \in \mathfrak{h}$ such that $ad(v_1)\cdots ad(v_k)u \in \mathfrak{h}$ for every $k \geq 0$ and any $v_1,\cdots, v_k \in \wp(V;y_1,\cdots,y_n)$. Let $V_\wp$ be the set of all points $q \in \mathbb{C}^n$ such that $\wp(V;y_1,\cdots,y_n)$ does not span n-dimensional vector space at q, that is, $\dim \wp(V;y_1,\cdots,y_n)(q) < n$. For a point $p \in \mathbb{C}^n$, let $\wp_p$ be the isotropy subalgebra of $\wp(V;y_1,\cdots,y_n)$ at p, i.e. $\wp_p = \{u \in \wp(V;y_1,\cdots,y_n)\ ;\ u(p) = 0\}$.

3.1 Lemma <u>For a point</u> $p \in \mathbb{C}^n - V_\wp$, $\wp_p$ <u>is a maximal, finite co-dimensional subalgebra such that</u> $\wp_p^{(\infty)} = \{0\}$.

Proof. Since $p \in \mathbb{C}^n - V_\wp$, there are $u_1,\cdots,u_n \in \wp(V;y_1,\cdots,y_n)$

such that $u_j(p) = \partial/\partial y_j|_p$ for $1 \leq j \leq n$. Consider

$$(\mathrm{ad}(u_1)^{\ell_1} \cdots \mathrm{ad}(u_n)^{\ell_n} v)(p) = 0$$

for any $\ell_1, \cdots \ell_n$, and we get easily that $\mathfrak{P}_p^{(\infty)} = \{0\}$.

We next prove the maximality of $\mathfrak{P}_p$. Let $\mathfrak{f}$ be a subalgebra of $\mathfrak{P}(V;y_1,\cdots,y_n)$ such that $\mathfrak{f} \supsetneqq \mathfrak{P}_p$. There is then an element $v \in \mathfrak{f}$ such that $v(p) \neq 0$. By a suitable linear change of variables, we may assume that v is written in the form

$$(21) \qquad v = g\,\partial/\partial y_1 + \sum_{j=2}^{n} h_j\,\partial/\partial y_j, \quad g(p) \neq 0, \ h_j(p) = 0.$$

Let $(p_1,\cdots,p_n)$ be the coordinate of p. Then, $(y_1-p_1)u_j \in \mathfrak{P}_p$ for $1 \leq j \leq n$. Therefore, $[v,(y_1 - p_1)u_j] = v(y_1)u_j + (y_1-p_1)[v,u_j] \in \mathfrak{f}$. Since $v(y_1)(p) = g(p) \neq 0$, we have $\mathfrak{f}(p) = \mathfrak{P}(V;y_1,\cdots,y_n)(p)$ and hence $\mathfrak{f} = \mathfrak{P}(V;y_1,\cdots,y_n)$.

Let $\mathcal{W}_{\mathfrak{P}}$ be the set of all points q such that $\mathfrak{P}_q$ is a maximal subalgebra and $\mathfrak{P}_q^{(\infty)} = \{0\}$. By the above lemma, $\mathcal{W}_{\mathfrak{P}}$ contains $\mathbb{C}^n - V_{\mathfrak{P}}$. The goal of this section is as follows :

3.2 Proposition <u>Let $\mathfrak{g}$ be a maximal, finite codimensional subalgebra of $\mathfrak{P}(V;y_1,\cdots,y_n)$ such that $\mathfrak{g}^{(\infty)} = \{0\}$. Then, there is a unique point $p \in \mathcal{W}_{\mathfrak{P}}$ such that $\mathfrak{g} = \mathfrak{P}_p$.</u>

Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{P}(V;y_1,\cdots,y_n)$, and let $J = \{f \in \mathbb{C}[y_1,\cdots,y_n] \,;\, f\mathfrak{P}(V;y_1,\cdots,y_n) \subset \mathfrak{g}\}$. Obviously, J is an ideal of $\mathbb{C}[y_1,\cdots,y_n]$, for $\mathfrak{P}(V;y_1,\cdots,y_n)$ is a $\mathbb{C}[y_1,\cdots,y_n]$-module.

3.3 Lemma <u>Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{P}(V;y_1,\cdots,y_n)$ such that $\mathbb{C}[y_1,\cdots,y_n]\mathfrak{g} = \mathfrak{P}(V;y_1,\cdots,y_n)$. Then $J\mathfrak{P}(V;y_1,\cdots,y_n)$ is an ideal of $\mathfrak{P}(V;y_1,\cdots,y_n)$ contained in $\mathfrak{g}$.</u>

Proof. By definition $J\wp(V;y_1,\cdots,y_n) \subset \mathfrak{g}$. Since $(uf)v = [u,fv] - f[u,v]$, we have $\mathfrak{g}J \subset J$, hence $(\mathbb{C}[y_1,\cdots,y_n]\mathfrak{g})J \subset J$. By the assumption, we get $\wp(V;y_1,\cdots,y_n)J \subset J$. Therefore, $J\wp(V;y_1,\cdots,y_n)$ is an ideal of $\wp(V;y_1,\cdots,y_n)$.

By the above lemma, we see also that $J\wp(V;y_1,\cdots,y_n) \subset \mathfrak{g}^{(\infty)}$. The next lemma is due to Amamiya [1]. The proof is seen also in [5], however we repeat the proof for the sake of selfcontainedness.

3.4 Lemma <u>Let</u> $\mathfrak{g}$ <u>be a finite codimensional subalgebra of</u> $\wp(V;y_1,\cdots,y_n)$. <u>Then</u>, $J \neq \{0\}$.

Proof. Set $\mathfrak{g}^{(1)} = \{ u \in \mathfrak{g} \; ; \; [u, \wp(V;y_1,\cdots,y_n)] \subset \mathfrak{g} \}$. Since $\operatorname{codim} \mathfrak{g} < \infty$ and $\operatorname{ad}(u)$ for every $u \in \mathfrak{g}$ induces a linear mapping of $\wp(V;y_1,\cdots,y_n)/\mathfrak{g}$ into itself, we see that $\operatorname{codim} \mathfrak{g}^{(1)} < \infty$ and hence in particular $\mathfrak{g}^{(1)} \neq \{0\}$.

Let v be a non-trivial element in $\mathfrak{g}^{(1)}$, and let f be a polynomial such that $vf \neq 0$. Consider a sequence fv, f^2v, $f^3v, \cdots$. Since $\operatorname{codim} \mathfrak{g}^{(1)} < \infty$, there is a polynomial $P(t)$ in t such that $P(f)v \in \mathfrak{g}^{(1)}$.

We next prove that if v and gv are contained in $\mathfrak{g}^{(1)}$, then $(vg)^2 \in J$. For that purpose, let w be an arbitrary element of $\wp(V;y_1,\cdots,y_n)$. Then, we have

$$[v,gw] = (vg)w + g[w,v] \in \mathfrak{g}$$

$$[gv,w] = -(wg)v + g[w,v] \in \mathfrak{g}$$

Hence

$$(vg)w + (wg)v \in \mathfrak{g} \tag{22}$$

for every $w \in \wp(V;y_1,\cdots,y_n)$. Replacing w by $(wg)v$, we have $(vg)(wg)v \in \mathfrak{g}$. Replacing w in (22) by $(vg)w$, we have also

$$(vg)^2w + (vg)(wg)v \in \mathfrak{g}$$

Hence $(vg)^2w \in \mathfrak{g}$. Thus, $(vg)^2 \in J$.

Set $g = P(f)$. Then, $v,\ gv \in \mathfrak{g}^{(1)}$ and $vg \neq 0$ because of $vf \neq 0$. Thus, we get $J \neq \{0\}$.

3.5 Corollary <u>Let $\mathfrak{g}$ be a maximal finite codimensional subalgebra of $\mathcal{P}(V;y_1,\cdots,y_n)$ such that $\mathfrak{g}^{(\infty)} = \{0\}$. Then, $\mathfrak{g}$ is a $\mathbb{C}[y_1,\cdots,y_n]$-module.</u>

Proof. We have only to show that $\mathbb{C}[y_1,\cdots,y_n]\mathfrak{g} \subsetneqq \mathcal{P}(V;y_1,\cdots,y_n)$, because if so, the maximality of $\mathfrak{g}$ shows that $\mathbb{C}[y_1,\cdots,y_n]\mathfrak{g} = \mathfrak{g}$. Thus, assume that $\mathbb{C}[y_1,\cdots,y_n]\mathfrak{g} = \mathcal{P}(V;y_1,\cdots,y_n)$. Then by the above lemma, we get that $\mathfrak{g}^{(\infty)} \supset J\mathcal{P}(V;y_1,\cdots,y_n) \neq 0$, contradicting the assumption.

Now, we have only to consider a maximal finite codimensional subalgebra $\mathfrak{g}$ of $\mathcal{P}(V;y_1,\cdots,y_n)$ such that $\mathfrak{g}^{(\infty)} = \{0\}$ and $\mathfrak{g}$ is a $\mathbb{C}[y_1,\cdots,y_n]$-module. Let $M_p = \{f \in \mathbb{C}[y_1,\cdots,y_n]\ ;\ f(p) = 0\}$.

3.6 Lemma <u>For a $\mathbb{C}[y_1,\cdots,y_n]$-submodule $\mathfrak{g}$ of $\mathcal{P}(V;y_1,\cdots,y_n)$, if</u>

$$\mathfrak{g} + M_p\mathcal{P}(V;y_1,\cdots,y_n) = \mathcal{P}(V;y_1,\cdots,y_n)$$

<u>for every $p \in \mathbb{C}^n$, then $\mathfrak{g} = \mathcal{P}(V;y_1,\cdots,y_n)$.</u>

Proof. By Nakayama's lemma, we see that for each $p \in \mathbb{C}^n$, there is $f_p \in \mathbb{C}[y_1,\cdots,y_n]$ such that $f_p(p) \neq 0$ and $f_p\mathcal{P}(V;y_1,\cdots,y_n) = \mathfrak{g}$. Since the ideal $\mathfrak{J}$ generated by $\{f_p\ ;\ p \in \mathbb{C}^n\}$ has no common zero, we see that $\mathfrak{J} = \mathbb{C}[y_1,\cdots,y_n]$ and hence there are $f_{p_1}, f_{p_2},\cdots, f_{p_\ell}$, $g_1, g_2,\cdots,g_\ell \in \mathbb{C}[y_1,\cdots,y_n]$ such that $1 = \sum_{j=1}^{\ell} g_j f_{p_j}$. Therefore,

$$\mathcal{P}(V;y_1,\cdots,y_n) = (\sum_{j=1}^{\ell} g_j)\mathfrak{g} \subset \mathfrak{g} .$$

3.7 Corollary <u>Let $\mathfrak{g}$ be a maximal, finite codimensional subalgebra of $\mathcal{P}(V;y_1,\cdots,y_n)$ such that $\mathfrak{g}^{(\infty)} = \{0\}$. Then, there exists uniquely a point $p \in \mathcal{W}_{\mathcal{P}}$ such that $\mathfrak{g} = \mathcal{P}_p$.</u>

Proof. By 3.5 Corollary, $\mathfrak{g}$ is a $\mathbb{C}[y_1,\cdots,y_n]$-module, and hence there is a point $p \in \mathbb{C}^n$ such that $\mathfrak{J} + M_p \mathfrak{P}(V;y_1,\dots,y_n) \subsetneqq \mathfrak{P}(V;y_1,\dots,y_n)$. Thus, $\mathfrak{J} \supset M_p \mathfrak{P}(V;y_1,\dots,y_n)$ by the maximality of $\mathfrak{J}$. It is easy to see that such a point is unique, because $M_p + M_q = \mathbb{C}[y_1,\dots,y_n]$ if $p \neq q$.

If $\mathfrak{P}(V;y_1,\dots,y_n)(p) = \{0\}$, then $M_p \mathfrak{P}(V;y_1,\dots,y_n)$ is an ideal of $\mathfrak{P}(V;y_1,\dots,y_n)$, hence it must be contained in $\mathfrak{J}^{(\infty)}$. Thus, by the assumption, it must be $\{0\}$, contradicting the assumption. Therefore we get $\mathfrak{P}(V;y_1,\dots,y_n)(p) \neq \{0\}$. Now, there is $u \in \mathfrak{P}(V;y_1,\dots,y_n)$ such that $u(p) \neq 0$ and $f \in \mathbb{C}[y_1,\dots,y_n]$ such that $f(p) = 0$ and $(uf)(p) \neq 0$. For every $v \in \mathfrak{P}(V;y_1,\dots,y_n)$, fv is an element of $\mathfrak{J}$. Therefore if u were contained in $\mathfrak{J}$, then $[u,fv] \in \mathfrak{J}$. Thus, $(uf)v \in \mathfrak{J}$. It follows that $(uf)(p)v \in (uf - (uf)(p))v + \mathfrak{J} \subset \mathfrak{J}$. Since $(uf)(p) \neq 0$, we get $v \in \mathfrak{J}$, hence $\mathfrak{J} = \mathfrak{P}(V;y_1,\dots,y_n)$, contradicting the assumption.

By the above argument, we see that $\mathfrak{J} \subset \mathfrak{P}_p$, and hence $\mathfrak{J} = \mathfrak{P}_p$ by the maximality of $\mathfrak{J}$. Since $\mathfrak{J}^{(\infty)} = \{0\}$, we see $p \in \mathcal{W}_{\mathfrak{P}}$ by definition.

This completes the proof of 3.2 Proposition.

3.B A diffeomorphism induced from Φ.

Let $\mathfrak{P}(V';z_1,\dots,z_{n'})$ be another Lie algebra of polynomial vector fields on $\mathbb{C}^{n'}$. Subsets $V_{\mathfrak{P}'}$, $\mathcal{W}_{\mathfrak{P}'}$ are defined by the same way as in $\mathfrak{P}(V;y_1,\dots,y_n)$. Suppose there is an isomorphism Φ of $\mathfrak{P}(V;y_1,\dots,y_n)$ onto $\mathfrak{P}(V';z_1,\dots,z_{n'})$. For a point $p \in \mathcal{W}_{\mathfrak{P}}$, $\mathfrak{P}_p$ is a maximal finite codimensional subalgebra such that $\mathfrak{P}_p^{(\infty)} = 0$. Then, $\Phi(\mathfrak{P}_p)$ has the same property, hence there is a point $\varphi(p) \in \mathcal{W}_{\mathfrak{P}'}$ such that $\Phi(\mathfrak{P}_p) = \mathfrak{P}'_{\varphi(p)}$, where $\mathfrak{P}'_{\varphi(p)}$ is defined by the same manner as in $\mathfrak{P}(V;y_1,\dots,y_n)$. $\varphi : \mathcal{W}_{\mathfrak{P}} \mapsto \mathcal{W}_{\mathfrak{P}'}$ is a bijective mapping. The goal of this section is as follows :

3.8 Proposition <u>Notations and assumptions being as above, assume further that</u> $\mathfrak{P}(V;y_1,\dots,y_n)$ (resp. $\mathfrak{P}(V';z_1,\dots,z_{n'})$) <u>contains a vector field</u> X (resp. X') <u>such that</u> $X = \sum_{j=1}^{n} \hat{\mu}_j y_j \partial/\partial y_j$ (resp. $X' = \sum_{j=1}^{n'} \hat{\mu}'_j z_j \partial/\partial z_j$). <u>Then</u> φ <u>can be extended to a holomorphic diffeomorphism of</u> $\mathbb{C}^n$ <u>onto</u> $\mathbb{C}^{n'}$ <u>such that</u> $\varphi(V_{\mathfrak{P}}) = V_{\mathfrak{P}'}$.

Note that the existence of X and X' are obtained by 2.1 Lemma.

Let $\Psi_{\mathfrak{P}}$ be the totality of $\mathbb{C}$-valued functions f on $\mathcal{W}_{\mathfrak{P}}$ such that fu can be extended to an element of $\mathfrak{P}(V;y_1,\dots,y_n)$ for every $u \in \mathfrak{P}(V;y_1,\dots,y_n)$. Remark that the extension of fu is unique, because $\mathcal{W}_{\mathfrak{P}}$ is dence in $\mathbb{C}^n$. $\Psi_{\mathfrak{P}}$ is a ring and $\mathfrak{P}(V;y_1,\dots,y_n)$ is an $\Psi_{\mathfrak{P}}$-module. For $\mathfrak{P}(V';z_1,\dots,z_{n'})$, we define $\Psi_{\mathfrak{P}'}$ by the same manner as above.

3.9 Lemma <u>Notations and assumptions being as above,</u> φ <u>induces an isomorphism of</u> $\Psi_{\mathfrak{P}'}$ <u>onto</u> $\Psi_{\mathfrak{P}}$.

Proof. Let $f \in \Psi_{\mathfrak{P}'}$ and p an arbitrary point in $\mathcal{W}_{\mathfrak{P}}$. By definition, $f\Phi(u)$ can be extended to an element of $\mathfrak{P}(V';z_1,\dots,z_{n'})$, which will be denoted by the same notation. $f\Phi(u) - f(\varphi(p))\Phi(u) \in \mathfrak{P}'_{\varphi(p)}$, hence $\Phi^{-1}(f\Phi(u) - f(\varphi(p))\Phi(u)) \in \mathfrak{P}_p$, that is, $\Phi^{-1}(f\Phi(u) - f(\varphi(p))\Phi(u))(p) = 0$. Therefore, $\Phi^{-1}(f\Phi(u))(p) = f(\varphi(p))u$, that is, $\Phi^{-1}(f\Phi(u)) = (\varphi^*f)u$. Since the left hand member is contained in $\mathfrak{P}(V;y_1,\dots,y_n)$, we see $\varphi^*f \in \Psi_{\mathfrak{P}}$. It is easy to see that $\varphi^* : \Psi_{\mathfrak{P}'} \mapsto \Psi_{\mathfrak{P}}$ is an isomorphism.

3.10 Lemma <u>Under the same assumption as in the statement of</u> 3.8 Proposition, <u>we have</u> $\Psi_{\mathfrak{P}} = \mathbb{C}[y_1,\dots,y_n]$. <u>Hence</u> φ <u>is a bi-holomorphic diffeomorphism of</u> $\mathbb{C}^n$ <u>onto</u> $\mathbb{C}^{n'}$.

Proof. Obviously $\Psi_{\mathfrak{P}} \supset \mathbb{C}[y_1,\dots,y_n]$. For any $f \in \Psi_{\mathfrak{P}}$, fX is an element of $\mathfrak{P}(V;y_1,\dots,y_n)$. Thus, $fy_1,\dots,fy_n \in \mathbb{C}[y_1,\dots,y_n]$. Hence it is not hard to see $f \in \mathbb{C}[y_1,\cdots,y_n]$.

3.11 Lemma $\varphi(\mathbb{C}^n - V_{\wp}) = \mathbb{C}^{n'} - V_{\wp'}$.

Proof. By the above lemma, we have $n = n'$. Let p be a point of $\mathbb{C}^n - V_{\wp}$. Then $\operatorname{codim} \wp_p = n$, hence $\operatorname{codim} \wp'_{\varphi(p)} = n$, because $\wp'_{\varphi(p)} = \Phi(\wp_p)$. Therefore, we see $\varphi(\mathbb{C}^n - V_{\wp}) = \mathbb{C}^{n'} - V_{\wp'}$.

This completes the proof of 3.8 Proposition.

3.C Recapture of the germ.

Recall that V is a germ of variety with 0 as an expansive singularity. Hence there is $X = \sum_{i=1}^{n} \hat{\mu}_i y_i \partial/\partial y_i \in \mathfrak{X}(V)$ such that $\operatorname{Re}\hat{\mu}_i > 0$ for $1 \leq i \leq n$. Since X is a linear vector field, $\exp tX$ is a bi-holomorphic diffeomorphism of $\mathbb{C}^n$ onto itself. Remark that $(\exp tX)V = V$ as germs of varietis, for $X\mathfrak{J}(V) \subset \mathfrak{J}(V)$ where $\mathfrak{J}(V)$ is the ideal of V in $\mathcal{O}$. Let $\tilde{V} = \bigcup_{t\in\mathbb{R}} (\exp tX)V$. Though V is a germ of variety at 0, the expansive property of X yields that $\tilde{V}$ is a closed subset of $\mathbb{C}^n$ such that $(\exp tX)\tilde{V} = \tilde{V}$. Obviously, $\tilde{V} = V$ as germs of varieties.

In this section, we shall prove that $V_{\wp} = \tilde{V}$, hence $V_{\wp} = V$ as germs of varieties. Let $\hat{\mathfrak{J}}(V)$ be the closure of $\mathfrak{J}(V)$ in $\hat{\mathcal{O}}$. Note that $\mathfrak{g}(V)$ is also the closure of $\mathfrak{X}(V)$ in $\mathfrak{J}_o$. Hence $\mathfrak{g}(V)\hat{\mathfrak{J}}(V) \subset \hat{\mathfrak{J}}(V)$. Recall that $\wp(V;y_1,\cdots,y_n)$ is given by using the eigenspace decomposition of $\mathfrak{g}(V)$ with respect to $\operatorname{ad}(X)$, that is, every $u \in \mathfrak{g}(V)$ can be rearranged in the form $u = \sum u_\lambda$ as in 1.6 Lemma, and $\wp(V;y_1,\cdots,y_n)$ is generated by the u_λ's. Similarly, we decompose $\hat{\mathfrak{J}}(V)$ into eigenspaces of X. Let f be an element of $\hat{\mathfrak{J}}(V)$. Then, f can be rearranged in the form

$$(23) \qquad f = \sum_\nu f_\nu, \quad f_\nu = \sum_{\langle\alpha,\hat{\mu}\rangle=\nu} a_\alpha y^\alpha .$$

Then, f_ν is a polynomial such that $Xf_\nu = \nu f_\nu$. By the same proof as in 1.6 Lemma, we see that $f_\nu \in \hat{\mathfrak{J}}(V)$. We denote by $I_{\wp}$ the ideal of $\mathbb{C}[y_1,\cdots,y_n]$ generated by all f_ν's with $f \in \hat{\mathfrak{J}}(V)$.

3.11 Lemma $I_{\wp} \subset \mathfrak{J}(V)$.

Proof. Let $f \in \mathfrak{J}(V)$. f can be rearranged in the form $f = \sum_{i=1}^{\infty} f_{\nu_i}$, $f_{\nu_i} = \sum_{\langle\alpha,\hat{\mu}\rangle=\nu_i} a_\alpha y^\alpha$. We may assume $0 < \nu_1 < \nu_2 < \cdots < \nu_k < \cdots$. First of all, we shall show $f_{\nu_i} \in \mathfrak{J}(V)$. Note that $e^{\nu_1 t}(\exp -tX) f = \sum e^{-(\nu_j - \nu_1)t} f_{\nu_j} \in \mathfrak{J}(V)$ for $t > 0$. Suppose f is defined on a neighborhood N of 0 in $\mathbb{C}^n$. Then, $(\exp -tX)f$ is defined on $(\exp tX)N$. Note that $\bigcup_{t>0}(\exp tX)N = \mathbb{C}^n$ and $\bigcup_{t>0}(\exp tX)(N \cap V) = \tilde{V}$. Since $e^{\nu_1 t}(\exp -tX)f = 0$ on $(\exp tX)(N \cap V)$, taking $\lim_{t\to\infty}$ we see that $f_{\nu_1} = 0$ on $\tilde{V}$. Since $\tilde{V} = V$ as germs of varieties, we have $f_{\nu_1} \in \mathfrak{J}(V)$. Repeating the same procedure to $f - f_{\nu_1}$, we have $f_{\nu_2} \in \mathfrak{J}(V)$, and so on. Hence $f_{\nu_j} \in \mathfrak{J}(V)$.

Let $f \in \hat{\mathfrak{J}}(V)$. Then, there is a sequence $\{f^{(m)}\}$ in $\mathfrak{J}(V)$ such that $\lim f^{(m)} = f$ in the topology of formal power series. For any eigenvalue ν of $X : \hat{\mathcal{O}} \longmapsto \hat{\mathcal{O}}$, we see $f_\nu^{(m)} \in \mathfrak{J}(V)$, and $\lim_{m\to\infty} f_\nu^{(m)} = f_\nu$ as polynomials, because the degrees of $f_\nu^{(m)}$, f_ν are bounded from above by a number related only to $\hat{\mu}_1,\cdots,\hat{\mu}_n$ and ν. Since $f_\nu^{(m)}|V \equiv 0$, we have $f_\nu|V \equiv 0$, hence $f_\nu \in \mathfrak{J}(V)$. Recall that the f_ν's generate $I_{\wp}$. Thus, we see $I_{\wp} \subset \mathfrak{J}(V)$.

3.12 Lemma <u>Notations and assumptions being as above, a polynomial vector field u with $u(0) = 0$ is contained in $\wp(V;y_1,\cdots,y_n)$ if and only if $uI_{\wp} \subset I_{\wp}$.</u>

Proof. For $u \in \mathfrak{g}(V)$, $f \in \hat{\mathfrak{J}}(V)$, let $u = \sum_\lambda u_\lambda$, $f = \sum_\nu f_\nu$ be the decompositions of eigenvectors with respect to $ad(X)$, X respectively. Then, $u_\lambda \in \wp(V;y_1,\cdots,y_n)$, $f_\nu \in I_{\wp}$. Since $Xu_\lambda f_\nu = [X,u_\lambda]f_\nu + u_\lambda X f_\nu = (\lambda+\nu)u_\lambda f_\nu$, $u_\lambda f_\nu$ is also an eigenvector of X. Since $uf \in \hat{\mathfrak{J}}(V)$, the $u_\lambda f_\nu$'s appear in the eigenspace decomposition of uf, and hence $u_\lambda f_\nu \in I_{\wp}$. Thus, we have $\wp(V;y_1,\cdots,y_n)I_{\wp} \subset I_{\wp}$.

Conversely, if $uI_{\wp} \subset I_{\wp}$ for a polynomial vector field u with $u(0) = 0$. Then, $u\hat{\mathcal{J}}(V) \subset \hat{\mathcal{J}}(V)$ by taking the closure in the formal power series. Note that $u\,\mathcal{J}(V) \subset \mathcal{O} \cap \hat{\mathcal{J}}(V)$. We next prove that $\mathcal{J}(V) = \mathcal{O} \cap \hat{\mathcal{J}}(V)$. For that purpose, we have only to show $\mathcal{J}(V) \supset \mathcal{O} \cap \hat{\mathcal{J}}(V)$, because the converse is trivial. Let $f \in \mathcal{O} \cap \hat{\mathcal{J}}(V)$, and $f = \sum_{\nu} f_{\nu}$ the eigenvector decomposition of f with respect to X. Then, by 3.11 Lemma, we have $f_{\nu} \in I_{\wp} \subset \mathcal{J}(V)$. Thus, $f_{\nu} = 0$ on V, hence $f = 0$ on V. This means $f \in \mathcal{J}(V)$. Thus, $uI_{\wp} \subset I_{\wp}$ yields $u \in \mathfrak{X}(V) \subset \mathfrak{g}(V)$. However u is a polynomial vector field in $y_1,\dots,y_n$, hence $u \in \wp(V;y_1,\dots,y_n)$.

3.13 Lemma $V_{\wp} = V_{I_{\wp}}$: <u>the locus of zeros of</u> $I_{\wp}$.

Proof. Let p be a point in $\mathbb{C}^n - V_{\wp}$. By definition there are $u_1, \dots, u_n \in \wp(V;y_1,\dots,y_n)$ such that $u_1(p),\dots,u_n(p)$ are linearly independent. Assume for a while that $p \in V_{I_{\wp}}$. Since $u_i I_{\wp} \subset I_{\wp}$, we have

$$(u_1^{\ell_1} u_2^{\ell_2} \dots u_n^{\ell_n} f)(p) = 0$$

for every $f \in I_{\wp}$ and any $\ell_1, \ell_2, \cdots, \ell_n$. Thus, $f = 0$, contradicting the fact $I_{\wp} \neq \{0\}$. Therefore, $V_{\wp} \supset V_{I_{\wp}}$.

Conversely, let $p \in \mathbb{C}^n - V_{I_{\wp}}$. There is then $g \in I_{\wp}$ such that $g(p) \neq 0$. By 3.12 Lemma, $g\partial/\partial y_1,\dots, g\partial/\partial y_n \in \wp(V;y_1,\dots,y_n)$, which are linearly independent at p. Hence $p \in \mathbb{C}^n - V_{\wp}$. Thus, $V_{I_{\wp}} \supset V_{\wp}$.

3.14 Lemma $V_{I_{\wp}} = V$ <u>as germs of varieties</u>.

Proof. By 3.11 Lemma, we have $\mathcal{O} I_{\wp} \subset \mathcal{J}(V)$, hence $V_{I_{\wp}} \supset V$. Assume for a while that $V_{I_{\wp}} \supsetneq V$. Then there is $f \in \mathcal{J}(V)$ such that $f \not\equiv 0$ on V. Let $f = \sum_{\nu} f_{\nu}$ be the eigenvector decomposition of f. Then $f_{\nu} \in I_{\wp}$. Therefore $f_{\nu} = 0$ on V, hence $f = 0$ on V contradicting the assuption. Thus, we get $V_{I_{\wp}} = V$ as germs of varieties, and hence $V_{I_{\wp}} = \tilde{V}$.

By the above result, we get that $\varphi : \mathbb{C}^n \mapsto \mathbb{C}^{n'}$ maps $\tilde{V}$ onto $\tilde{V}'$ and $\varphi(V) = V'$ as germs. This implies that $\varphi^*\mathcal{J}(V') = \mathcal{J}(V)$ and hence $d\varphi\mathfrak{X}(V) = \mathfrak{X}(V')$. This completes the proof of Theorem I in the introduction.

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