A METHOD OF CLASSIFYING EXPANSIVE SINGULARITIES

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Introduction

To study singularities is in a sense to study the classification of germs of varieties. It is therefore important to give a method of classification. The purpose of this paper is to show the classification of a class of germs of varieties, which will be called expansive singularities in this paper, is included in that of Lie algebras of formal vector fields. As a matter of course, the classification of the latter does not seem easy. However, note that such a Lie algebra is given by an inverse limit of finite dimensional Lie algebras of polynomial vector fields truncated at the order $k$, $k \geq 0$. Therefore such Lie algebras can be understood by step by step method in the order $k$.

Let $\mathbb{C}^n$ be the Cartesian product of $n$ copies of complex numbers $\mathbb{C}$ with natural coordinate system $(x_1, \ldots, x_n)$. By $\mathcal{O}$, we mean the ring of all convergent power series in $x_1, \ldots, x_n$ centered at the origin $0$. Let $V$ be a germ of variety in $\mathbb{C}^n$ at $0$, and $\mathcal{J}(V)$ the ideal of $V$ in $\mathcal{O}$ (cf. [2] pp. 86-7 for the definitions). Two germs $V, V'$ are called bi-holomorphically equivalent if there is a germ of holomorphic diffeomorphism $\varphi$ such that $\varphi(0) = 0$ and $\varphi(V) = V'$.

Let $\mathfrak{X}$ be the Lie algebra of all germs of holomorphic vector fields at $0$, and $\mathfrak{X}(V)$ the subalgebra defined by

$$\mathfrak{X}(V) = \left\{ u \in \mathfrak{X} : u \mathcal{J}(V) \subseteq \mathcal{J}(V) \right\}.$$
$\mathcal{X}(V)$ is then an $O$-module. If there are $v_1, \ldots, v_s$, linearly independent at $0$, then Corollary 3.4 of [9] shows that $V$ is bi-holomorphically equivalent to the direct product $\mathbb{C}^s \times W$, where $W \subset \mathbb{C}^{n-s}$. Thus, for the structure of singularities we have only to consider the germ $W$.

Taking this fact into account, we may restrict our concern to the varieties such that all $u \in \mathcal{X}(V)$ vanishes at $0$, which we assume throughout this paper, i.e. $\mathcal{X}(V)(0) = \{0\}$.

$u \in \mathcal{X}(V)$ ($u(0) = 0$) is called a semi-simple expansive vector field, if after a suitable bi-holomorphic change of variables at $0$, $u$ can be written in the form

$$u = \sum_{i=1}^{s} \hat{\mu}_i v_i \partial / \partial v_i,$$

where $\hat{\mu}_1, \ldots, \hat{\mu}_s$ lie in the same open half-plane in $\mathbb{C}$ about the origin. (See also §2.4 for a justification of this definition.) The origin $0$ is called to be an expansive singularity, if $\mathcal{X}(V)$ contains a semi-simple expansive vector field. If $V$ is given by the locus of zeros of a weighted homogeneous polynomial, then $V$ has an expansive singularity at $0$. The advantage of existence of such a vector field $u$ is that one can extend through $\exp tu$ a germ $V$ to a subvariety $\tilde{V}$ in $\mathbb{C}^n$. In this paper we restrict our concern to the germs of varieties with expansive singularities at the origin.

For such $\mathcal{X}(V)$, we set $\mathcal{X}_k(V) = \{ u \in \mathcal{X}(V) : j^k u = 0 \}$, where $j^k u$ is the $k$-th jet at $0$. Since $\mathcal{X}(V) = \mathcal{X}_0(V)$, $\mathcal{X}_k(V)$ is a finite codimensional ideal of $\mathcal{X}(V)$ such that $[\mathcal{X}_k(V), \mathcal{X}_\ell(V)] \subset \mathcal{X}_{k+\ell}(V)$ and $\bigcap \mathcal{X}_k(V) = \{0\}$. We denote by $\mathcal{G}(V)$ the inverse limit of $\{ \mathcal{X}(V)/\mathcal{X}_k(V) \}_{k=0}^\infty$ with the inverse limit topology. Since $\mathcal{X}(V)/\mathcal{X}_k(V)$ is finite dimensional, $\mathcal{G}(V)$ is a Fréchet space such that the Lie bracket product $\lbrack , \rbrack : \mathcal{G}(V) \times \mathcal{G}(V) \rightarrow \mathcal{G}(V)$ is
continuous. Namely, $\mathfrak{g}(V)$ is a Frechet-Lie algebra. It is obvious that $\mathfrak{g}(V)$ is a Lie algebra of formal vector fields, where a formal vector field $u$ is a vector field $u = \sum_{i=1}^{n} u_i \partial / \partial x_i$ such that each $u_i$ is a formal power series in $x_1, \ldots, x_n$ without constant terms. The statement to be proved in this paper is as follows:

Theorem I Let $V, V'$ be germs of varieties with expansive singularities at the origins of $\mathbb{C}^n$, $\mathbb{C}^{n'}$ respectively. Notations and assumptions being as above, $V$ and $V'$ are bi-holomorphically equivalent, if and only if $\mathfrak{g}(V)$ and $\mathfrak{g}(V')$ are isomorphic as topological Lie algebras.

By the above result, we see especially that any isomorphism $\Phi$ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$ preserves orders, that is, $\Phi \mathfrak{g}_k(V) = \mathfrak{g}_k(V')$ for every $k$. Hence, to classify $\mathfrak{g}(V)$ is to classify the inverse system $\{ \mathcal{X}(V)/\mathcal{X}_k(V) \}_{k \geq 0}$. Note that $\mathcal{X}(V)/\mathcal{X}_k(V)$ is an extension of $\mathcal{X}(V)/\mathcal{X}_{k-1}(V)$ with an abelian kernel $\mathcal{X}_{k-1}(V)/\mathcal{X}_k(V)$. Such extensions can be classified by representations and second cohomologies (cf. [6]).

The proof of the above theorem is divided into several steps as follows:

Step 1. We define the concept of Cartan subalgebras and prove the conjugacy of Cartan subalgebras.

Step 2. Using the assumption that $V$ (resp. $V'$) has an expansive singularity at 0, we prove that there is a Cartan subalgebra $\mathfrak{g}$ of $\mathfrak{g}(V)$ such that $\mathfrak{g} \subseteq \mathcal{X}(V)$ (resp. $\mathfrak{g}' \subseteq \mathcal{X}(V')$). By a suitable biholomorphic change of variables, every element of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) can be changed simultaneously into a normal form, which is a polynomial vector field. Moreover, every eigenvector with respect to $\text{ad}(\mathfrak{g})$ is a polynomial vector field.
Step 3. Now, suppose there is an isomorphism $\hat{\phi}$ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$. Then, by definition $\hat{\phi}(\xi')$ is a Cartan subalgebra of $\mathfrak{g}(V')$. Hence by Steps 1, 2 we may assume that $\hat{\phi}(\xi') \subset \mathcal{L}(V')$. Thus, considering the eigenspace decomposition of $\mathfrak{g}(V)$, $\mathfrak{g}(V')$ with respect to $\text{ad}(\xi')$ $\text{ad}(\xi')$ respectively, we see that $\hat{\phi}$ induces an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$, where $\mathfrak{g}$ (resp. $\mathfrak{g}'$) is the totality of $u \in \mathfrak{g}(V)$ (resp. $\mathfrak{g}(V')$) which can be expressed as a polynomial vector field with respect to the local coordinate system which normalizes $\xi$ (resp. $\xi'$).

Step 4. From isomorphism $\hat{\phi} : \mathfrak{g} \rightarrow \mathfrak{g}'$, we conclude by the same procedure as in [5] that there is a bi-holomorphic diffeomorphism $\varphi$ of $\mathbb{C}^n$ onto $\mathbb{C}^{n'}$ such that $\varphi(0) = 0$ and $d\varphi \varphi = \varphi'$. The main idea of making such $\varphi$ is roughly in the fact that every maximal subalgebra of $\mathfrak{g}$ corresponds to a point. However, since $\mathfrak{g}(0) = \{0\}$, the situation is much more difficult than that of [1]. Existence of expansive vector field plays an important role at this step as well as in the above steps.

Step 5. Recapturing $V$ from the Lie algebra $\mathfrak{g}$, we can conclude $\mathfrak{g}(V) = V'$.

The theorem is proved by this way. Note that the converse is trivial.
§ 1 Conjugacy of Cartan subalgebras

We denote a formal power series \( f \) in a form \( f = \sum_{k=0}^{\infty} a_k x^k \), where \( a_k \in \mathcal{C}, \quad \alpha = (\alpha_1, \ldots, \alpha_n), \quad |\alpha| = \alpha_1 + \ldots + \alpha_n \) and \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \).

We denote by \( \mathcal{F} \) the Lie algebra of all formal vector fields and \( \mathcal{F}_k \) the subalgebra

\[
\{ u \in \mathcal{F} ; \quad u = \sum_{i=1}^{\infty} \sum_{k \geq k \geq 0} a_{i,k} x_i^k \partial x_i \}
\]

\( \mathcal{F} \) is then regarded as the inverse limit of the system \( \{ \mathcal{F}/\mathcal{F}_k ; p_k \} \), where \( p_k : \mathcal{F}/\mathcal{F}_{k+1} \rightarrow \mathcal{F}/\mathcal{F}_k \) is the natural projection. We denote by \( \bar{p}_k \) the projection of \( \mathcal{F} \) onto \( \mathcal{F}/\mathcal{F}_k \). \( p_k \) and \( \bar{p}_k \) are sometimes called forgetful mappings. Since \( \mathcal{F}/\mathcal{F}_k \) is a finite dimensional vector space over \( \mathcal{C} \), \( \mathcal{F} \) is a Frechet space, and the Lie bracket product is continuous.

Let \( \mathcal{G} \) be a closed Lie subalgebra of \( \mathcal{F} \), and \( \mathcal{F}_k = \mathcal{F}_k \cap \mathcal{F} \).

The closedness of \( \mathcal{G} \) implies that \( \mathcal{G} \) is the inverse limit of the system \( \{ \mathcal{G}/\mathcal{F}_k ; p_k \} \) for any \( \mathcal{F}_k \geq 0 \). In this paper, we restrict our concern to a closed subalgebra \( \mathcal{G} \) of \( \mathcal{G}_0 \). For any subalgebra \( \mathcal{H} \) of \( \mathcal{G} \), we denote by \( \mathcal{N}(\mathcal{H}) \) the normalizer of \( \mathcal{H} \), i.e. \( \mathcal{N}(\mathcal{H}) = \{ u \in \mathcal{G} ; [u, \mathcal{H}] \subset \mathcal{H} \} \), and by \( \mathcal{G}^{(0)}(\mathcal{H}) \) the 0-eigenspace of \( \text{ad}(\mathcal{H}) \), i.e. \( \mathcal{G}^{(0)}(\mathcal{H}) \) is the totality of \( v \in \mathcal{G} \) satisfying that there are non-negative integers \( m_k, k \geq 0 \), (depending on \( v \)) such that \( \text{ad}(s)^{m_k} \cdot v \in \mathcal{G}_k \) for all \( s \in \mathcal{G} \) and for all \( k \geq 0 \), where \( \text{ad}(u)v = [u,v] \). If \( \mathcal{H} \) is nilpotent, then \( \mathcal{G}^{(0)}(\mathcal{H}) \subset \mathcal{N}(\mathcal{H}) \). Therefore, if \( \mathcal{G}^{(0)}(\mathcal{H}) = \mathcal{H} \), then \( \mathcal{N}(\mathcal{H}) = \mathcal{H} \). The converse is also true if \( \dim \mathcal{G}^{(0)}(\mathcal{H}) < \infty \) (cf. [6]).

A subalgebra \( \mathcal{G} \) of \( \mathcal{G} \) is called a Cartan subalgebra of \( \mathcal{G} \), if the following conditions are satisfied:

1. \( \mathcal{G} \) is a closed subalgebra of \( \mathcal{G} \) such that \( \mathcal{F}_k \mathcal{G} \) is a nilpotent
subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ for every $k \geq 0$.

$\mathfrak{g}_0 = \mathfrak{g}^{(0)}(\mathfrak{g})$.

Note that if $\dim \mathfrak{g} \leq \omega$ above $\mathfrak{p}_k$ is a usual Cartan subalgebra. The statement to be proved in this chapter is as follows:

**Proposition A** Let $\mathfrak{g}$ be a closed subalgebra of $\mathfrak{f}_0$. Then, there exists a Cartan subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$. For Cartan subalgebras $\mathfrak{g}_0$, $\tilde{\mathfrak{g}}_0$ of $\mathfrak{g}$, there is an inner automorphism $A$ of $\mathfrak{g}$ such that $A\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{g}}_0$.

1.A. Automorphisms of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a closed Lie subalgebra of $\mathfrak{f}_0$, and $\mathfrak{g}_k = \mathfrak{g} \cap \mathfrak{f}_k$. For every $u \in \mathfrak{g}$, the adjoint action $\text{ad}(u)$ leaves each $\mathfrak{g}_k$ invariant, hence $\text{ad}(u)$ induces a linear mapping $a_k(u)$ of $\mathfrak{g}/\mathfrak{g}_k$ into itself. $\text{ad}(u)$ is then regarded as the inverse limit of the system $\{a_k(u)\}_{k \in \omega}$. Define a linear mapping $e^{t \cdot \text{ad}(u)} : \mathfrak{g} \rightarrow \mathfrak{g}$ by the inverse limit of $\{e^{t \cdot a_k(u)}\}_{k \in \omega}$. Since $\text{ad}(u)$ is a derivation of $\mathfrak{g}$, $e^{t \cdot \text{ad}(u)}$ is a one parameter family of automorphisms of $\mathfrak{g}$. The group $\mathcal{O}(\mathfrak{g})$ generated by $\{e^{\text{ad}(u)} ; u \in \mathfrak{f}\}$ is called the group of inner automorphisms of $\mathfrak{g}$. The purpose of this section is to investigate the structure of $\mathcal{O}(\mathfrak{g})$.

Let $\hat{\mathfrak{g}}$ be the ring of all formal power series $\sum_{k \in \mathbb{Z}} a_k x^k$ and $\hat{\mathfrak{g}}_k$ the ideal given by $\hat{\mathfrak{g}}_k = \{ \sum_{k \leq k+1} a_k x^k \}$. $\hat{\mathfrak{g}}/\hat{\mathfrak{g}}_k$ is then a finite dimensional algebra over $\mathbb{C}$. We denote by $\tilde{\mathfrak{g}}_k, \pi_k$ the projections $\hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_k, \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_{k+1} \rightarrow \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_k$ respectively. Every $u \in \mathfrak{f}_0$ acts naturally on $\hat{\mathfrak{g}}$ as a derivation such that $u \hat{\mathfrak{g}}_k \subseteq \hat{\mathfrak{g}}_k$ for every $k$. Conversely, $u \in \mathfrak{f}_0$ can be characterised by the above property. Every $u \in \mathfrak{f}_0$ induces, therefore, a derivation $u^{(k)}$ of the algebra $\hat{\mathfrak{g}}/\hat{\mathfrak{g}}_k$ and $u^{(k)}$ is canonically identified with $\tilde{\mathfrak{g}}_k u$. Conversely,
for every derivation $\delta$ of $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ such that $\delta \, \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \subset \hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ there is an element $u \in \mathcal{F}_o$ such that $\delta = \tilde{\nu}_k u$.

Since a derivation $u : \hat{\mathcal{O}} \to \hat{\mathcal{O}}$ can be regarded as an inverse limit of derivations $\{ \tilde{\nu}_k u : \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \to \hat{\mathcal{O}}/\hat{\mathcal{O}}_k \}$, we define an automorphism $\exp u$ of $\hat{\mathcal{O}}$ by an inverse limit of $\{ e^{\tilde{\nu}_k u} \}$. We denote by $G'$ the group generated by $\{ \exp u : u \in \mathcal{F}_k \}$.

Define an automorphism $\text{Ad}(\exp u)$ of $\mathcal{F}_k$ by

$$
(2) \quad (\text{Ad}(\exp u)v)f = (\exp u)v(\exp^{-1}u)f, \quad f \in \hat{\mathcal{O}}.
$$

Since $d/dt \big|_{t=0} (\exp tu)f = uf$, we see easily that

$$
(3) \quad \frac{d}{dt} \text{Ad}(\exp tu)v = \{ u, \text{Ad}(\exp tu)v \}.
$$

On the other hand, $e^{t \cdot \text{ad}(u)}$ satisfies the same differential equation. Thus, by uniqueness, we obtain

$$
(4) \quad \text{Ad}(\exp u) = e^{\text{ad}(u)}.
$$

Especially, if $\mathcal{F}_k$ is a closed Lie subalgebra of $\mathcal{F}_o$, then $\text{Ad}(\exp u) \mathcal{F}_k = \mathcal{F}_k$ for every $u \in \mathcal{F}_k$. Since $e^{\text{ad}(u)} e^{\text{ad}(v)} = \text{Ad}(\exp u \cdot \exp v)$, we obtain that $O(\mathcal{F}_k) = \{ \text{Ad}(g) : g \in G' \}$.

Let $G^{(k)}$ be the group generated by $\{ e^{\tilde{\nu}_k u} : u \in \mathcal{F}_k \}$. Since $\hat{\mathcal{O}}/\hat{\mathcal{O}}_k$ is finite dimensional, $G^{(k)}$ is a Lie group with Lie algebra $\mathcal{F}_k/\mathcal{F}_k$. For every integer $l$ such that $l \leq k$, the group $G^{(k)}$ leaves $\mathcal{F}_l/\mathcal{F}_k$ invariant. Hence $\{ G^{(k)} \}_{k \geq 0}$ forms an inverse system. We denote by $G$ the inverse limit. Obviously, $G'$ is a subgroup of $G$.

However, note that if a sequence $(u_0, u_1, \ldots, u_n, \ldots)$ satisfies $u_l \in \mathcal{F}_k$ for every $l \geq 0$, then $\exp u_0 \cdot \exp u_1 \cdot \ldots \cdot \exp u_n \cdots$ is an element of $G$. Since $G^{(k)}$ is a Lie group, $G$ is a topological group under the inverse limit topology. The purpose of the remainder of this section is to show $G = G'$ and that $G$ is a Frechet-Lie group with
Lie algebra $\mathfrak{g}$.

Let $G^{(k)}_1$, $k \geq 1$, be the group generated by \( \{ e^{\tilde{P}_k u} : u \in \mathfrak{g} \} \), and $G_1$ the inverse limit of \( \{ G^{(k)}_1 \} \) for $k \geq 1$.

1.1 Lemma $\exp$ is a bijective mapping of $\mathfrak{g}_1$ onto $G_1$.

Proof. Let $\exp_k$ be the exponential mapping of $\mathfrak{g}_1/\mathfrak{g}_k$ into $G^{(k)}_1$, i.e. $\exp_k u = e^{\tilde{P}_k u}$. Since $\exp : \mathfrak{g}_1 \rightarrow G_1$ is defined by the inverse limit of $\{ \exp_k \}$, we have only to show that $\exp_k : \mathfrak{g}_1/\mathfrak{g}_k \rightarrow G^{(k)}_1$ is bijective. Since $\mathfrak{g}_1/\mathfrak{g}_k = \tilde{P}_1 \mathfrak{g}_1$ is a nilpotent Lie algebra, we see that $\exp_k$ is regular and surjective (cf. [3] p 229). However, the derivation $\tilde{P}_k u : \hat{\mathfrak{g}}_k \rightarrow \hat{\mathfrak{g}}_k$ is expressed by a triangular matrix with zeros in the diagonal. Therefore, one can define $\log(1 + N)$ by $\sum_{n=1}^\infty (-1)^{n-1}N^n/n$, which gives the inverse of $\exp_k$. Thus $\exp_k$ is bijective.

1.2 Corollary $G' = G$.

Proof. We have only to show $G' \supseteq G$. Since $G^{(1)} = G/G_1$ is generated by $\{ \tilde{p}_1 u : u \in \mathfrak{g} \}$, every $g \in G$ can be written in the form $g = \exp u_1 \cdot \exp u_2 \cdots \cdot \exp u_m \cdot h$, where $u_1, \ldots, u_m \in \mathfrak{g}$ and $h \in G_1$. Thus, the above lemma shows $G \subseteq G'$.

We next prove that $G$ is a Frechet-Lie group. Although such a structure of $G$ has no direct relevance to our present purpose, there is an advantage of making analogies easy from the theory of finite dimensional Lie groups.

Let $\varphi : \tilde{P}_1 \mathfrak{g} \rightarrow \mathfrak{g}$ be a linear mapping such that $\tilde{P}_1 \varphi \tilde{u} = \hat{u}$ for $\tilde{u} \in \tilde{P}_1 \mathfrak{g}$. It is not hard to see that $\xi(u) = \exp \varphi \tilde{P}_1 u \cdot \exp (u - \xi \tilde{P}_1 u)$ gives a homeomorphism of an open neighborhood $U$ of 0 of $\mathfrak{g}$ onto an open neighborhood $\tilde{U}$ of the identity $e$ of $G$. Since $G$ is a topological group, there is an open neighborhood $V$ of 0 of $\mathfrak{g}$ such that
\( \xi(V)^{-1} = \xi(V), \quad \xi(V)^2 \subset \xi(U) \). We set \( \eta(u,v) = \xi^{-1}(\xi(u)\xi(v)) \) and \( i(u) = \xi^{-1}(\xi(u)^{-1}) \) for \( u, v \in V \). We have next to prove the differentiability of \( \eta \) and \( i \). However, the differentiability is defined by inverse limits of differentiable mappings, hence that of \( \eta \) and \( i \) are trivial in our case. Thus, we get the following:

1.3 Lemma G is a Frechet-Lie group with Lie algebra \( \mathfrak{g} \).

1.3. Simultaneous normalization and eigenspace decomposition

For any \( u \in \mathfrak{g}_0 \), the linear mapping \( u^{(k)} : \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_k \rightarrow \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_k \) splits uniquely into a sum of semi-simple part \( u^{(k)}_S \) and nilpotent part \( u^{(k)}_N \) such that \([u^{(k)}_S, u^{(k)}_N] = 0\). Using eigenspace decomposition of \( \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_k \), we see that \( u^{(k)}_S \) is also a derivation of \( \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_k \) hence so is \( u^{(k)}_N \). For \( u^{(k+1)} \), we have that \([p^{(k+1)}_S, p^{(k+1)}_N] = 0\), \( p^{(k+1)}_S \) is nilpotent, and that \( p^{(k+1)}_S \) is semi-simple by considering eigenspace decomposition of \( \hat{\mathfrak{g}}/\hat{\mathfrak{g}}_{k+1} \). Therefore, \( p^{(k+1)}_S = u^{(k)}_S \) and \( p^{(k+1)}_N = u^{(k)}_N \). Hence, taking inverse limit, we get formal vector fields \( u_S, u_N \) which will be called the semi-simple part and the nilpotent part of \( u \) respectively. A formal vector field is called to be semi-simple if it has no nilpotent part.

Let \( \mathfrak{g}^k \) be a nilpotent subalgebra of \( \mathfrak{g}_0/\mathfrak{g}_k \) for an arbitrarily fixed \( k \). Set \( \mathfrak{g}^k_S = \{ u^{(k)} \in \mathfrak{g}^k : u^{(k)} \in \mathfrak{g}_k \} \), and denote by \( p^k \) the forgetful projection of \( \mathfrak{g}_0/\mathfrak{g}_k \) onto \( \mathfrak{g}_0/\mathfrak{g}_1 \), that is, \( p^k = p^k_{1}p^k_{4} \cdots p^k_{k-1} \). Since \( p^k_{1} \mathfrak{g}^k \) is a nilpotent subalgebra of \( \mathfrak{g}_0/\mathfrak{g}_1 \), there is a basis \( \{ f^{(1)}_1, \ldots, f^{(1)}_n \} \) of \( \hat{\mathfrak{g}}_0/\hat{\mathfrak{g}}_1 \) such that every \( u^{(1)} \in p^k_{1} \mathfrak{g}^k \) is represented by an upper triangular matrix. Let \( \{ \mu_{j}^{1}(u^{(1)}), \ldots, \mu_{n}^{1}(u^{(1)}) \} \) be the diagonal part. \( \mu_{j}^{1} \) is then a linear mapping of \( p^k_{1} \mathfrak{g}^k \) into \( \mathfrak{g} \) for every \( j \), which one may regard as a
linear mapping of $\mathcal{O}^k$ into $\mathcal{C}$. Since $u^{(1)}_s$ is the semi-simple part of $u^{(1)}$, it must satisfy
\[
(5) \quad u^{(1)}_s f_j^{(1)} = \mu_j^{(1)}(u^{(1)}) f_j^{(1)}.
\]
By a simple linear algebra, we see that there are $f_1^{(k)}, \ldots, f_n^{(k)} \in \hat{\mathcal{O}}_0 / \hat{\mathcal{O}}_k$ such that
\[
(5) \quad u^{(k)}_s f_j^{(k)} = \mu_j^{(k)}(u^{(k)}) f_j^{(k)}, \quad \pi_k^{(k)} f_j^{(k)} = f_j^{(k)} \quad (1 \leq j \leq n)
\]
for every $u^{(k)} \in \mathcal{O}_k$, where $\pi_k^{(k)}$ is the forgetful projection of $\hat{\mathcal{O}}_0 / \hat{\mathcal{O}}_k$ onto $\hat{\mathcal{O}}_0 / \hat{\mathcal{O}}_k$, that is, $\pi_k^{(k)} = \pi_{k-1} \cdots \pi_1$.

Since $f_j^{(k)} \in \hat{\mathcal{O}}_0 / \hat{\mathcal{O}}_k$, $f_j^{(k)}$ is expressed in the form
\[
(7) \quad f_j^{(k)} = \sum_{\alpha \in \mathbb{N}^{\leq k}} a_j^{(k)} x^\alpha.
\]
Set $y_j = \sum_{\alpha \in \mathbb{N}^{\leq k}} a_j^{(k)} x^\alpha$. Since $f_1^{(1)}, \ldots, f_n^{(1)}$ are linearly independent, these give a formal change of variables and every $u^{(k)}_s$ can be written in the form
\[
(8) \quad u^{(k)}_s = \sum_{i=1}^n \mu_i^{(k)}(u^{(k)}) y_i \partial / \partial y_i.
\]
Since $[\mathcal{O}_s^k, \mathcal{O}_s^k] = 0$, because $\mathcal{O}_s^k$ is nilpotent, every $u^{(k)} \in \mathcal{O}_s^k$ should be written in the form
\[
(9) \quad u^{(k)} = \sum_{i=1}^n \sum_{\alpha, \lambda \in \mathbb{N}^{\leq k}} a_i^{(k)} \alpha y_i^{\alpha} \partial / \partial y_i
\]
where $\langle \alpha, \lambda \rangle = \alpha_1^{(k)} + \cdots + \alpha_n^{(k)}$. It should be noted that the semi-simple part $u^{(1)}_s$ of $u^{(1)}$ has been changed into a linear diagonal vector field such as (8).

Let $\mathcal{O}_s^{k+1}$ be another nilpotent subalgebra of $\mathcal{O}_0 / \mathcal{O}_s^{k+1}$ such that $P_k \mathcal{O}_s^{k+1} \subset \mathcal{O}_s^{k+1}$, and let $\mathcal{O}_s^{k+1} = \{u^{(k+1)}_s ; u^{(k+1)} \in \mathcal{O}_s^{k+1}\}$.

Since $P_k^{(k+1)} \mathcal{O}_s^{k+1} \subset P_k^{(k+1)} \mathcal{O}_s^{k+1}$ and the equality (5) holds also for every $u^{(1)} \in P_k^{(k+1)} \mathcal{O}_s^{k+1}$ and the equality (6) does for every $P_k \mathcal{O}_s^{k+1}$. By a simple linear algebra, we see that there are $f_1^{(k+1)}, \ldots, f_n^{(k+1)} \in \hat{\mathcal{O}}_0 / \hat{\mathcal{O}}_{k+1}$ such that
\[
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\( u^{(k+1)}_{s} f^{(k+1)}_{j} = \mu_{j}(u^{(k+1)}) f^{(k+1)}_{j}, \quad \Pi_{k} f^{(k+1)}_{j} = f^{(k)}_{j}. \)

Note that \( f^{(k+1)}_{j} = f^{(k)}_{j} + \sum_{|a|=k+1} a_{j,\alpha} x^{\alpha}. \) Hence by putting \( y_{j} = \sum_{0 \leq |w| \leq k+1} a_{j,\alpha} x^{\alpha} \) instead of (7), we get the same equations as (8) and (9) with respect to \( \mathcal{A}^{k}. \) Moreover we have

\[
\begin{align*}
(12) \quad u^{(k+1)}_{s} &= \sum_{i=1}^{n} \mu_{i}(u^{(k+1)}) y_{i} \partial / \partial y_{i}, \\
(13) \quad u^{(k+1)} &= \sum_{i=1}^{n} \sum_{|a_{\mu}| \leq k+1} a_{i,\alpha} y^{\alpha} \partial / \partial y_{i}
\end{align*}
\]

for every \( u^{(k+1)} \in \mathcal{A}^{k+1}. \) Especially we obtain the following:

1.4 Lemma Notations and assumptions being as above, the forgetful projection \( p_{k} : \mathcal{A}^{k+1}_{S} \rightarrow \mathcal{A}^{k}_{S} \) is injective.

Let \( \{ \mathcal{A}^{k}_{S} \}_{k \geq 1} \) be a series of nilpotent subalgebras \( \mathcal{A}^{k} \) of \( \mathcal{A}_{S}^{k} \) such that \( p_{k} \mathcal{A}^{k+1} \subset \mathcal{A}^{k} \) for every \( k \geq 1. \) We denote by \( \mathcal{A} \) the inverse limit, and set \( \mathcal{A}_{S} = \{ u_{S} ; u \in \mathcal{A} \}. \) Note that \( \dim \mathcal{A}_{S}^{k} \leq n \) for every \( k \geq 1. \) Thus, there is an integer \( k_{0} \) such that \( p_{k} : \mathcal{A}^{k+1}_{S} \rightarrow \mathcal{A}^{k}_{S} \) is bijective for every \( k \geq k_{0}. \) By a method of inverse limit, we see that there is a formal change of variables

\[
\begin{align*}
(14) \quad y_{j} &= f_{j}(x_{1}, \ldots, x_{n}) \quad 1 \leq j \leq n, \quad f_{j} \in \mathcal{A}_{S}^{0} \\
\text{such that (8) and (9) hold for every } u^{(k)} \in \mathcal{A}^{k} (k \geq 1), \text{ and}
\end{align*}
\]

\[
\begin{align*}
(15) \quad u_{S} &= \sum_{i=1}^{n} \mu_{i}(u) y_{i} \partial / \partial y_{i}, \\
(16) \quad u &= \sum_{i=1}^{n} \sum_{|a_{\mu}| = k} a_{i,\alpha} y^{\alpha} \partial / \partial y_{i}
\end{align*}
\]

for every \( u \in \mathcal{A}^{k}_{S}. \)

Now, let \( \mathcal{G} \) be a closed subalgebra of \( \mathcal{A}_{S}, \) and suppose the above \( \mathcal{A}^{k}_{S} \)'s are subalgebras of \( \mathcal{G} / \mathcal{A}^{k}_{S} \) respectively. Hence, the inverse limit \( \mathcal{G} \) is a closed subalgebra of \( \mathcal{A}. \) We next consider the eigenspace decomposition of \( \mathcal{G} \) with respect to \( \text{ad}(\mathcal{G}). \) Since
ad(u) : \mathcal{F}_o \rightarrow \mathcal{F}_o leaves \mathcal{G} invariant for every \( u \in \mathcal{G} \), and \([\text{ad}(u), \text{ad}(u^*_s)] = 0\), we see that \( \text{ad}(u^*_s) : \mathcal{F}_o \rightarrow \mathcal{F}_o \) is the semi-simple part of \( \text{ad}(u) \) and hence \( \text{ad}(u^*_s) \mathcal{G} \subseteq \mathcal{G} \). Therefore, we have only to consider the eigenspace decomposition with respect to \( \text{ad}(\mathcal{G}_s) \).

For a linear mapping \( \lambda \) of \( \tilde{\mathcal{P}}_1 \mathcal{G}_s \) into \( \mathcal{C} \), i.e. \( \lambda \in (\tilde{\mathcal{P}}_1 \mathcal{G}_s)^* \), we denote by \( \mathcal{F}_\lambda \) the subspace

\[
\{ u \in \mathcal{F}_o : u = \sum_{i=1}^{n} \sum_{i'=1}^{m} a_{i,i'} y_i^* \partial / \partial y_{i'} \}.
\]

Note that \( \mathcal{F}_\lambda = \{0\} \) for almost all \( \lambda \in (\tilde{\mathcal{P}}_1 \mathcal{G}_s)^* \) except countably many \( \lambda \)'s. By \( \mathcal{F}(\mathcal{G}) \) we denote the set of all \( \lambda \in (\tilde{\mathcal{P}}_1 \mathcal{G}_s)^* \) such that \( \mathcal{F}_\lambda \neq \{0\} \). If \( \tilde{\mathcal{P}}_1 \mathcal{G}_s = \{0\} \), then we set \( \mathcal{F}(\mathcal{G}) = \{0\} \), because all \( \lambda_j \)'s are zeros.

1.5 Lemma If \( \tilde{\mathcal{P}}_1 \mathcal{G}_s = \{0\} \), then \( \mathcal{G}^{(o)}(\mathcal{G}) = \mathcal{G} \).

Proof. By (16), every \( u \in \mathcal{G} \) can be written in the form \( u = u_1 + u_2 \) such that

\[
u_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} y_i \partial / \partial y_j, \quad u_2 = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} y_i \partial / \partial y_j.
\]

The reason for the shape of \( u_1 \) is that the linear part of \( u \) is an upper triangular matrix. Therefore, for every \( k \geq 1 \), there is an integer \( m_k \) such that \( \text{ad}(u) m_k \mathcal{F}_o \subseteq \mathcal{F}_k \) for every \( u \in \mathcal{G} \). This means \( \mathcal{G} = \mathcal{G}^{(o)}(\mathcal{G}) \) by definition.

Now, we set \( \mathcal{G}^{(\lambda)}(\mathcal{G}) = \mathcal{G} \cap \mathcal{F}_\lambda \) for every \( \lambda \in \mathcal{F}(\mathcal{G}) \).

1.6 Lemma Every \( u \in \mathcal{G} \) can be rearranged in the form

\[
u = \sum_{\lambda \in \mathcal{F}(\mathcal{G})} u_\lambda, \quad u_\lambda \in \mathcal{G}^{(\lambda)}.
\]

Moreover, every \( u_\lambda \) is contained in \( \mathcal{G}^{(\lambda)}(\mathcal{G}) \).

Proof. Since the first assertion is trivial, we have only to show the second one. Since \( \mathcal{F}(\mathcal{G}) \) is a countable set, there is \( v_0 \in \mathcal{G}_s \) such that \( \lambda(v_0^{(1)}) \neq \lambda'(v_0^{(1)}) \) for any \( \lambda, \lambda' \in \mathcal{F}(\mathcal{G}) \) such that \( \lambda \neq \lambda' \). For every \( k \), let \( u(k) \) be the truncation of \( u \in \mathcal{G} \) at the
order \( k \), \( u^{(k)} \) is canonically identified with \( \tilde{p}_k u \). \( u^{(k)} \) can be rearranged in the form 
\[
\sum_{\lambda \in \Pi(\mathfrak{g})} u_{\lambda}^{(k)} = u^{(k)}
\]
where each \( u_{\lambda}^{(k)} \) is the truncation of \( u_{\lambda} \) at the order \( k \). Since \( \mathfrak{g}/\mathfrak{g}_k \) is finite dimensional, only finite number of \( u^{(k)}_{\lambda} \)'s do not vanish. Apply \( \text{ad}(v^{(k)}_o) \) to \( u^{(k)} \). Since \( \text{ad}(\mathfrak{g}_S) \mathfrak{g} \subset \mathfrak{g} \), we have
\[
\text{ad}(v^{(k)}_o)u^{(k)} = \sum_{\lambda \in \Pi(\mathfrak{g})} \left( v^{(k)}_o \right) u_{\lambda}^{(k)} \in \mathfrak{g}/\mathfrak{g}_k
\]
Hence, considering Vandermonde's matrix, we get \( u^{(k)}_{\lambda} \in \mathfrak{g}/\mathfrak{g}_k \). Thus, taking inverse limit, we get \( u_{\lambda} \in \mathfrak{g} \), hence the desired result.

1.7 Corollary \( \tilde{p}_k \mathfrak{g}^{(o)}(\mathfrak{g}) \) is the zero-eigenspace of \( \text{ad}(\tilde{p}_k \mathfrak{g}) \) :
\( \mathfrak{g}/\mathfrak{g}_k \mapsto \mathfrak{g}/\mathfrak{g}_k \).
Proof. It is trivial that \( \tilde{p}_k \mathfrak{g}^{(o)}(\mathfrak{g}) \) is contained in the zero-eigenspace of \( \text{ad}(\tilde{p}_k \mathfrak{g}) \), for \( [\mathfrak{g}_S, \mathfrak{g}^{(o)}(\mathfrak{g})] = \{0\} \). Thus, we have only to show the converse. The zero-eigenspace of \( \text{ad}(\tilde{p}_k \mathfrak{g}) \) is equal to that of \( \text{ad}(\tilde{p}_k \mathfrak{g}_S) \), that is, the space of all \( v^{(k)} \in \mathfrak{g}/\mathfrak{g}_k \) such that \( [\tilde{p}_k \mathfrak{g}_S, v^{(k)}] = \{0\} \). Thus, \( v^{(k)} \) should be written in the form (9). Let \( v \in \mathfrak{g} \) be an element such that such that \( \tilde{p}_k v = v^{(k)} \), and let \( v = \sum_{\lambda \in \Pi(\mathfrak{g})} v_{\lambda} \) be the decomposition in accordance with the above lemma. Then it is clear that \( \tilde{p}_k v_o = v^{(k)} \). Since \( v_o \in \mathfrak{g}^{(o)}(\mathfrak{g}) \), we get the desired result.

1.8 Existence and conjugacy of Cartan subalgebras

Let \( \mathfrak{g} \) be a closed subalgebra of \( \tilde{\mathfrak{g}}_o \). If \( \mathfrak{g}/\mathfrak{g}_1 \neq \{0\} \), then \( \mathfrak{g}/\mathfrak{g}_k \) is nilpotent for every \( k \geq 1 \), for \( [\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \mathfrak{g}_{k+l} \). Therefore, by 1.5 Lemma, we see that \( \mathfrak{g} \) itself is the only Cartan subalgebra of \( \mathfrak{g} \). Thus, the conjugacy is trivial in this case.

Now, suppose \( \mathfrak{g}/\mathfrak{g}_1 \neq \{0\} \), and let \( \mathfrak{g}_1^1 \) be a Cartan subalgebra of \( \mathfrak{g}/\mathfrak{g}_1 \).
1.8 Lemma Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ be a series of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_1, \ldots, \mathfrak{g}/\mathfrak{g}_k$ respectively such that $p_{k-1} \mathfrak{g}_d = \mathfrak{g}_d$ for $2 \leq k \leq k$. Then, there is a Cartan subalgebra $\mathfrak{g}_d^{k+1}$ of $\mathfrak{g}/\mathfrak{g}_k$ such that $p_{k} \mathfrak{g}_d^{k+1} = \mathfrak{g}_d^{k}$.

Proof. Let $\mathfrak{g}_d'$ be a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$. We prove at first that $p_{k} \mathfrak{g}_d'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$. Since $\mathfrak{g}_d'$ is nilpotent, so is $p_{k} \mathfrak{g}_d'$. Let $\mathfrak{g}_d' = \{u_{(k+1)} \mathfrak{g}_d, u_{(k+1)} \mathfrak{g}_d'\}$, and let $v(k)$ be an element of the zero-eigenspace of $p_{k} \mathfrak{g}_d'$. Then, $[v(k), p_{k} \mathfrak{g}_d'] = 0$ and hence $v(k)$ can be written in the form (9). Let $v^{(k+1)}$ be an element of $\mathfrak{g}/\mathfrak{g}_k$ such that $p_{k} v^{(k+1)} = v(k)$. Using the eigenspace decomposition of $\mathfrak{g}/\mathfrak{g}_k$ with respect to $\text{ad}(\mathfrak{g}_d')$, we see that $v^{(k+1)} = \sum \lambda \in \mathfrak{g}_d' v_{\lambda}^{(k+1)}$. Note that this decomposition is given by only rearranging of the terms of $v^{(k+1)}$ (cf. 1.6 Lemma). Hence it is clear that $p_{k} v_{\lambda}^{(k+1)} = v_{\lambda}^{(k+1)}$ is an element of the zero-eigenspace of $\mathfrak{g}_d'$. However, since $\mathfrak{g}_d'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ we get $v_{\lambda}^{(k+1)} \in \mathfrak{g}_d'$. Thus, $v(k) \in p_{k} \mathfrak{g}_d'$. Hence $p_{k} \mathfrak{g}_d'$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$.

By the well-known conjugacy of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_k$, there is an inner automorphism $A$ such that $A(p_{k} \mathfrak{g}_d') = \mathfrak{g}_d^k$. Since there is a natural projection of $G(k+1)$ onto $G(k)$ (cf. 1.4), there is an inner automorphism $A'$ of $\mathfrak{g}/\mathfrak{g}_k$ which induces naturally $A$. Thus, by setting $A' \mathfrak{g}_d' = \mathfrak{g}_d^{k+1}$, $\mathfrak{g}_d^{k+1}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_k$ such that $p_{k} \mathfrak{g}_d^{k+1} = \mathfrak{g}_d^k$.

By the above lemma, we have a series $\{\mathfrak{g}_d^k\}_{k \geq 1}$ of Cartan subalgebras of $\mathfrak{g}/\mathfrak{g}_k$ such that $p_{k} \mathfrak{g}_d^{k+1} = \mathfrak{g}_d^k$. Let $\mathfrak{g}_d$ be the inverse limit of $\mathfrak{g}_d^k$.

1.9 Lemma Notations and assumptions being as above, $\mathfrak{g}_d$ is a Cartan
subalgebra of $\mathfrak{g}$. 

Proof. Since $\tilde{p}_{k}^{o} = \tilde{p}_{k}^{k}$, $\tilde{p}_{k}^{o}$ is a nilpotent subalgebra of $\mathfrak{g}/\mathfrak{g}_{k}$ for every $k \geq 1$. By 1.7 Corollary, $\tilde{p}_{k}^{o}{\mathfrak{o}}(\tilde{p}_{j}^{k})$ is the zero-eigenspace of $\text{ad}(\tilde{p}_{k}^{o})$. Since $\tilde{p}_{k}^{o} = \tilde{p}_{k}^{k}$ is a Cartan subalgebra, we have $\tilde{p}_{k}^{o}{\mathfrak{o}}(\tilde{p}_{j}^{k}) = \tilde{p}_{j}^{k}$ and hence $\mathfrak{g}_{k}{\mathfrak{o}}(\tilde{p}_{j}^{k}) = \tilde{p}_{j}^{k}$. Thus, $\tilde{p}_{j}^{k}$ is a Cartan subalgebra of $\mathfrak{g}_{k}$. 

We next consider the converse of the above lemma.

1.10 Lemma Let $\tilde{p}_{j}^{k}$ be a Cartan subalgebra of $\mathfrak{g}_{j}$. Then, $\tilde{p}_{k}^{j}$ is a Cartan subalgebra of $\mathfrak{g}_{k}$ for every $k \geq 1$.

Proof. By 1.7 Corollary, the zero-eigenspace of $\text{ad}(\tilde{p}_{k}^{j})$ is equal to $\tilde{p}_{k}^{o}{\mathfrak{o}}(\tilde{p}_{j}^{k})$. Since $\tilde{p}_{j}^{k}$ is a Cartan subalgebra of $\mathfrak{g}_{j}$, we see $\tilde{p}_{k}^{o}{\mathfrak{o}}(\tilde{p}_{j}^{k}) = \tilde{p}_{k}^{j}$. Thus, $\tilde{p}_{k}^{j}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{g}_{k}$.

As in 1.A, we denote by $G^{(k)}$ the Lie group generated by $\{ e^{\tilde{p}_{k}u} : u \in \mathfrak{g}_{j} \}$. Let $\pi_{k} : G^{(k+1)} \rightarrow G^{(k)}$ be the natural projection. We shall next prove the conjugacy of Cartan subalgebras, which completes the proof of Proposition A. Let $\tilde{p}_{j}^{k}$, $\tilde{p}_{j}^{l}$ be Cartan subalgebras of $\mathfrak{g}_{j}$. By the argument in the first part of this section, we may assume $\mathfrak{g}_{j}/\mathfrak{g}_{1} \not\simeq \{ 0 \}$. Since $\tilde{p}_{j}^{j}$, $\tilde{p}_{j}^{l}$ are Cartan subalgebras of $\mathfrak{g}_{j}/\mathfrak{g}_{1}$, there is $g_{1} \in G^{(1)}$ such that $\text{Ad}(g_{1})(\tilde{p}_{j}^{j}) = \tilde{p}_{j}^{j}$. Therefore, one may assume without loss of generality that $\tilde{p}_{j}^{j} = \tilde{p}_{j}^{1}$. Let $G_{k}^{(k)}$ be the Lie group generated by $\{ e^{\tilde{p}_{k}u} : u \in \mathfrak{g}_{j} \}$ for any $j$, $l \leq k$.

1.11 Lemma Let $\tilde{p}_{j}^{k}$, $\tilde{p}_{j}^{l}$ be Cartan subalgebras of $\mathfrak{g}_{j}$ such that $\tilde{p}_{k}^{j} = \tilde{p}_{k}^{1}$. Then, there is $g_{k+1} \in G_{k}^{(k+1)}$ such that $\text{Ad}(g_{k+1})(\tilde{p}_{k+1}^{j}) = \tilde{p}_{k+1}^{j}$. 

Proof. Since $\tilde{p}_{k}^{j} = \tilde{p}_{k}^{1}$, $\tilde{p}_{k+1}^{j}$ and $\tilde{p}_{k+1}^{j}$ are Cartan subalgebras of $\mathfrak{p}_{k}^{j}$. Let $\mathfrak{p}_{k}^{j} = \mathfrak{p}_{k}^{1}$.
\[ p_k^{-1} \rho_\lambda \rho_\mu = \hat{\rho}_k \rho_\mu \oplus \sum_{\lambda \in \mu} g_\lambda', \quad p_k^{-1} \rho_\lambda = \hat{\rho}_k \rho_\mu \oplus \sum_{\lambda \in \mu} g_\lambda'' \]

be the eigenspace decompositions with respect to \( \text{ad}(\hat{\rho}_k) \) and \( \text{ad}(\hat{\rho}_k \rho_\mu) \) respectively. Since \( p_k \rho_\mu \rho_\mu + 1 \rho_\mu = p_k \rho_\mu \rho_\mu + 1 \rho_\mu = p_k \rho_\mu \), we see that \( \sum g_\lambda' \subset g_\lambda' / g_\lambda'_{k+1} \) and \( \sum g_\lambda'' \subset g_\lambda'' / g_\lambda''_{k+1} \). It is well-known (cf. [6] pp59-66) that there are \( v_1, \ldots, v_m \in \sum g_\lambda' \), \( w_1, \ldots, w_k \in \sum g_\lambda'' \) such that

\[ \text{Ad}(\exp v_1) \cdots \text{Ad}(\exp v_m) \text{Ad}(\exp w_1) \cdots \text{Ad}(\exp w_k) \hat{p}_k + 1 \rho_\mu = \hat{p}_k + 1 \rho_\mu \]

Since \( \exp v_1, \exp w_j \in g_{k+1} \), we see that there is \( g_{k+1} \in g_{k+1} \)

such that

\[ \text{Ad}(g_{k+1})(\hat{p}_k + 1 \rho_\mu) = \hat{p}_k + 1 \rho_\mu \].

Let \( g_k \) be the subgroup of \( G \) generated by \( \{ e^u : u \in g_k \} \). For Cartan subalgebras \( f_\lambda \), \( f_\mu \) of \( g_\lambda \), the above lemma shows that there are elements \( g_1, g_2, \ldots, g_k, \ldots \) such that \( g_k \in g_k \) and

\[ \text{Ad}(g_1) \text{Ad}(g_2) \cdots \text{Ad}(g_k)(\hat{f}_\mu) = \hat{f}_\mu \mod g_{k+1} \].

Note that \( g_1 g_2 \cdots g_k \cdots \in G \), hence putting \( g = g_1 g_2 \cdots g_k \cdots \), we see

\[ \text{Ad}(g)(\hat{f}_\mu) = \hat{f}_\mu \].

This shows the conjugacy of Cartan subalgebras.

Proposition A is thereby proved.
§2 Cartan subalgebras at expansive singularities

2.A Semi-simple expansive vector fields

In this section, notations are as in the introduction. A germ of holomorphic vector field \( u \in \mathcal{X}(V) \) is called \textit{expansive}, if the eigenvalues of the linear term of \( u \) at 0 lie in the same open half plane in \( \mathbb{C} \) about the origin. \( u \) is called to be \textit{semi-simple expansive} if \( u \) is expansive and semi-simple as a formal vector field. The purpose of this section is to show the following:

2.1 Lemma \textbf{Let} \( u \in \mathcal{X}(V) \) be a semi-simple expansive vector field. Then, there is a germ \( y_j = f_j(x_1, \ldots, x_n), 1 \leq j \leq n, \) of biholomorphic change of variables such that \( u \) can be written in the form

\[
u = \sum_{i=1}^{n} \hat{\lambda}_j y_i \partial / \partial y_i
\]

Proof. By a suitable change of variables \( y_j = \sum_{\ell=1}^{k} a_{j\ell} y^\ell \) such as in (7), we have that \( u \) can be written in the form

\[
u = \sum_{i=1}^{n} \hat{\lambda}_j y_i \partial / \partial y_i + w, \quad w \in \mathcal{X}_k(V)
\]

for sufficiently large \( k \). For the proof that \( u \) is linearizable, it is enough to show that there are holomorphic functions \( f_1, \ldots, f_n \) in \( y_1, \ldots, y_n \) such that \( uf_j = \hat{\lambda}_j f_j \) (1 \leq j \leq n) and \( f_j = y_j + \text{higher order terms} \). Set \( f_j = y_j + g_j \) and consider the equation \( u(y_j + g_j) = \hat{\lambda}_j (y_j + g_j) \). Then we get

\[
(17) \quad (u - \hat{\lambda}_j) g_j = -wy_j.
\]

Since \( k \) is sufficiently large, we have

\[
(18) \quad \lim_{t \to \infty} e^{-t(u - \hat{\lambda}_j)w} y_j = 0
\]

and

\[
(19) \quad -\int_{0}^{\infty} e^{-t(u - \hat{\lambda}_j)w} y_j \, dt
\]
exists as a germ of holomorphic functions (cf. [5]). Set \( g_j = - \int_0^\infty e^{-t(u - \hat{\mu}_j)^w} y_j \, dt \). Then,

\[
(u - \hat{\mu}_j)g_j = \int_0^\infty \frac{d}{dt} e^{-t(u - \hat{\mu}_j)^w} y_j \, dt = [ e^{-t(u - \hat{\mu}_j)^w} y_j ]_0^\infty = -w y_j.
\]

2.2 Lemma Let \( X \) be a semi-simple expansive vector field in \( \mathcal{G} \). Then, there is a Cartan subalgebra \( \mathcal{H}_x \) of \( \mathcal{G} \) containing \( X \).

Proof. By the same proof as in the above lemma, we see that \( X \) can be linearizable by a suitable formal change of variables, and hence we may assume that \( X \) can be written in the form \( X = \sum_{i=1}^k \hat{\mu}_i y_i \frac{\partial}{\partial y_i} \), \( \text{Re} \, \hat{\mu}_i > 0 \). Let \( \mathcal{G}^{(o)}(X) = \{ u \in \mathcal{G} : [X,u] = 0 \} \). Since every \( u \in \mathcal{G}^{(o)}(X) \) can be written in the form

\[
(20) \quad u = \sum_{i=1}^k \sum_{q, \lambda, \mu} \hat{\mu}_i \, a_{i, q} \, y^q \, \frac{\partial}{\partial y_i},
\]

we see that \( \mathcal{G}^{(o)}(X) \) is a finite dimensional Lie subalgebra of \( \mathcal{G} \).

Since \( \text{ad}(X) : \mathcal{G}^{(o)}(X) \to \mathcal{G}^{(o)}(X) \) is of diagonal type, there is a Cartan subalgebra \( \mathcal{H}_x \) of \( \mathcal{G}^{(o)}(X) \) containing \( X \). We shall show that \( \mathcal{H}_x \) is a Cartan subalgebra of \( \mathcal{G} \). For that purpose we have only to show \( \mathcal{G}^{(o)}(\mathcal{H}_x) = \mathcal{G} \). Since \( X \in \mathcal{H}_x \), we see \( \mathcal{G}^{(o)}(\mathcal{H}_x) \subset \mathcal{G}^{(o)}(X) \) and hence \( \mathcal{G}^{(o)}(\mathcal{H}_x) \) is the zero-eigenspace of \( \text{ad}(\mathcal{H}_x) \) in \( \mathcal{G}^{(o)}(X) \).

However since \( \mathcal{H}_x \) is a Cartan subalgebra of \( \mathcal{G}^{(o)}(X) \), we have \( \mathcal{H}_x = \mathcal{G}^{(o)}(\mathcal{H}_x) \).

2.3 Corollary If \( \mathcal{G} \) has a semi-simple expansive vector field, then every Cartan subalgebra \( \mathcal{H}_x \) of \( \mathcal{G} \) is finite dimensional and \( \mathcal{G}^{(\lambda)}(\mathcal{H}_x) \) is finite dimensional for every \( \lambda \in \mathfrak{g}(\mathcal{H}_x) \).
Proof. By the above lemma, there is a finite dimensional Cartan subalgebra of $\mathfrak{g}$. However by Proposition A it implies that all Cartan subalgebras are finite dimensional and every Cartan subalgebra contains a semi-simple expansive vector field. Note that

$$\mathfrak{g}_\lambda = \{ u \in \mathfrak{g} : u = \sum_{k=1}^n \sum_{\alpha \in \Phi^+} \lambda \alpha \cdot \alpha \cdot \gamma^\alpha / \gamma^\alpha \}$$

Since $\mathfrak{g}_\lambda$ contains an expansive vector field, we see that $\dim \mathfrak{g}_\lambda < \infty$ and hence $\dim \mathfrak{g}^{(\lambda)}(\mathfrak{g}) < \infty$.

2.4 Corollary Notations being as in the introduction, if $\mathfrak{X}(V)$ contains a semi-simple expansive vector field $X$, then there is a Cartan subalgebra $\mathfrak{g}$ of $\mathfrak{g}(V)$ such that $\mathfrak{g} \subset \mathfrak{X}(V)$. Moreover, for that $\mathfrak{g}_\lambda$, $\mathfrak{g}^{(\lambda)}(\mathfrak{g})$ is contained in $\mathfrak{X}(V)$ for every $\lambda \in \Pi(\mathfrak{g})$.

Proof. Since $X \in \mathfrak{X}(V)$, 2.1 Lemma shows that $X$ can be written in the form $X = \sum \hat{\gamma}^\alpha / \gamma^\alpha$ by a suitable biholomorphic change of variables. Therefore, every $u \in \mathfrak{g}^{(\lambda)}(\mathfrak{g})$ is contained in $\mathfrak{X}(V)$, because $u$ is a polynomial vector field in $y_1, \ldots, y_n$.

2.C Isomorphisms of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$.

Let $V, V'$ be germs of varieties in $\mathbb{C}^n, \mathbb{C}^{n'}$ respectively. Suppose there is a bicontinuous isomorphism $\phi$ of $\mathfrak{g}(V)$ onto $\mathfrak{g}(V')$.

2.5 Lemma Let $\mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}(V)$. Then, so is $\phi(\mathfrak{g})$ of $\mathfrak{g}(V')$.

Proof. Set $\mathfrak{g}' = \phi(\mathfrak{g})$. Since $\phi : \mathfrak{g}(V) \rightarrow \mathfrak{g}(V')$ is continuous, for every $k'$ there is an integer $k = k(k')$ such that $\phi(\mathfrak{g}_k(V)) \subset \mathfrak{g}(k')(V')$. Thus, $\mathfrak{g}_k, \mathfrak{g}'$ is a nilpotent subalgebra of $\mathfrak{g}(V')/\mathfrak{g}(k')(V')$ and $\mathfrak{g}^{(0)}(\mathfrak{g}') \supset \phi(\mathfrak{g}^{(0)}(\mathfrak{g}))$. Thus, replacing $\phi$ by $\phi^{-1}$, we get the desired result.
Now, suppose that \( V \) and \( V' \) have expansive singularities at the origins respectively. By 2.4 Corollary, \( \mathcal{X}(V) \) and \( \mathcal{X}(V') \) contain Cartan subalgebras of \( \mathcal{G}(V) \) and \( \mathcal{G}(V') \) respectively.

2.6 Corollary Assume as above, let \( \mathcal{H} \) be a Cartan subalgebra of \( \mathcal{G}(V) \) contained in \( \mathcal{X}(V) \). Suppose there is a bicontinuous isomorphism \( \phi \) of \( \mathcal{G}(V) \) onto \( \mathcal{G}(V') \). Then, there is a bicontinuous isomorphism \( \psi \) of \( \mathcal{G}(V) \) onto \( \mathcal{G}(V') \) such that \( \psi(\mathcal{H}) \subseteq \mathcal{X}(V') \), that is, \( \mathcal{H}' \) is a Cartan subalgebra of \( \mathcal{G}(V') \) contained in \( \mathcal{X}(V') \).

Proof. By the above lemma, \( \phi(\mathcal{H}) \) is a Cartan subalgebra of \( \mathcal{G}(V') \). By 2.4 Corollary, there is a Cartan subalgebra \( \mathcal{H}' \) of \( \mathcal{G}(V') \) contained in \( \mathcal{X}(V') \). By Proposition A, there is \( g \in G \) such that \( \text{Ad}(g) \phi(\mathcal{H}) = \mathcal{H}' \). Note that \( \text{Ad}(g) : \mathcal{G}(V') \rightarrow \mathcal{G}(V') \) is a bicontinuous isomorphism. Thus, \( \psi = \text{Ad}(g) \phi \) is the desired one.

In the remainder of this section, we assume that there is a bicontinuous isomorphism \( \phi : \mathcal{G}(V) \rightarrow \mathcal{G}(V') \) such that \( \phi(\mathcal{H}) = \mathcal{H}' \) where \( \mathcal{H}, \mathcal{H}' \) are Cartan subalgebras of \( \mathcal{G}(V), \mathcal{G}(V') \) respectively such that \( \mathcal{H} \subseteq \mathcal{X}(V) \) and \( \mathcal{H}' \subseteq \mathcal{X}(V') \). By 2.3-4 Corollaries, there is a local coordinate system \( (y_1, \ldots, y_n) \), related biholomorphically to the original one such that every \( \mathcal{G}^{(\lambda)}(\mathcal{H}) \) is a finite dimensional space of polynomial vector fields in \( y_1, \ldots, y_n \). We choose such a local coordinate system \( (x_1, \ldots, x_n) \) for \( \mathcal{G}(V') \). Let

\( \mathcal{H}(V; y_1, \ldots, y_n) \) (resp. \( \mathcal{H}(V'; z_1, \ldots, z_n) \)) be the totality of \( u \in \mathcal{G}(V) \) (resp. \( \mathcal{G}(V') \)) such that \( u \) can be expressed as a polynomial vector field in \( y_1, \ldots, y_n \) (resp. \( z_1, \ldots, z_n \)) \( \mathcal{H}(V; y_1, \ldots, y_n) \) and \( \mathcal{H}(V'; z_1, \ldots, z_n) \) are Lie subalgebras of \( \mathcal{X}(V), \mathcal{X}(V') \) respectively. Since \( \mathcal{G}^{(\lambda)}(\mathcal{H}) \subseteq \mathcal{H}(V; y_1, \ldots, y_n) \) for every \( \lambda \in \Pi(\mathcal{H}) \), we get the
following:

2.7 Corollary Notations and assumptions being as above, the above isomorphism \( \Phi : \mathcal{O}(V) \rightarrow \mathcal{O}(V') \) induces an isomorphism of \( \mathcal{O}(V; y_1, \ldots, y_n) \) onto \( \mathcal{O}(V'; z_1, \ldots, z_n) \).

Proof. Note that \( \Phi(\mathcal{O}(\lambda)(\ell)) = \mathcal{O}(\lambda)(\ell') \), because \( \mathcal{O}(\lambda)(\ell) \) is an eigenspace of \( \text{ad}(\ell) \). Every \( u \in \mathcal{O}(V; y_1, \ldots, y_n) \) can be written in the form \( u = \sum_{\lambda \in \mathfrak{g}} u_{\lambda} \), but the summation in this case is a finite sum. Since \( \Phi(u) = \sum_{\lambda \in \mathfrak{g}} \Phi(u_{\lambda}) \) and \( \Phi(u_{\lambda}) \in \mathcal{O}(\lambda)(\ell') \), we see that \( \Phi(u) \in \mathcal{O}(V'; z_1, \ldots, z_n) \). Replacing \( \Phi \) by \( \Phi^{-1} \), we get the desired result.

Let \( \mathcal{C}[y_1, \ldots, y_n] \) be the ring of all polynomials in \( y_1, \ldots, y_n \). Then, since \( \mathcal{O}(V) \) is an \( \mathcal{O} \)-module, \( \mathcal{O}(V; y_1, \ldots, y_n) \) is a \( \mathcal{C}[y_1, \ldots, y_n] \)-module.
3 Theorem of Pursell-Shanks' type

In this chapter, we consider two Lie algebras \( \mathfrak{g}(V; y_1, \ldots, y_n) \) and \( \mathfrak{g}(V'; z_1, \ldots, z_n') \) of polynomial vector fields such that they are \( \mathbb{C}[y_1, \ldots, y_n] \) and \( \mathbb{C}[z_1, \ldots, z_n] \)-module respectively and that there is an isomorphism \( \phi \) of \( \mathfrak{g}(V; y_1, \ldots, y_n) \) onto \( \mathfrak{g}(V'; z_1, \ldots, z_n') \). The goal is as follows:

Theorem II Notations and assumptions being as above, there is a biholomorphic mapping \( \varphi \) of \( \mathbb{C}^n \) onto \( \mathbb{C}^{n'} \) such that \( d\varphi(\mathfrak{g}(V; y_1, \ldots, y_n)) = \mathfrak{g}(V'; z_1, \ldots, z_n') \). Moreover, \( \varphi(V) = V' \) as germs of varieties.

Note at first that Theorem II implies Theorem I in the introduction, for 2.6-7 Corollaries show that an isomorphism between \( \mathfrak{g}(V) \) and \( \mathfrak{g}(V') \) induces an isomorphism between \( \mathfrak{g}(V; y_1, \ldots, y_n) \) and \( \mathfrak{g}(V'; z_1, \ldots, z_n') \).

3.A Characterization of maximal subalgebras

Let \( \mathfrak{g} \) be a subalgebra of \( \mathfrak{g}(V; y_1, \ldots, y_n) \). We denote by \( \mathfrak{g}^{(\infty)} \) the ideal consisting of all \( u \in \mathfrak{g} \) such that \( \text{ad}(v_1) \cdots \text{ad}(v_k)u \in \mathfrak{g} \) for every \( k \geq 0 \) and any \( v_1, \ldots, v_k \in \mathfrak{g}(V; y_1, \ldots, y_n) \). Let \( V_\mathfrak{g} \) be the set of all points \( q \in \mathbb{C}^n \) such that \( \mathfrak{g}(V; y_1, \ldots, y_n) \) does not span \( n \)-dimensional vector space at \( q \), that is, \( \dim \mathfrak{g}(V; y_1, \ldots, y_n)(q) < n \).

For a point \( p \in \mathbb{C}^n \), let \( \mathfrak{g}_p \) be the isotropy subalgebra of \( \mathfrak{g}(V; y_1, \ldots, y_n) \) at \( p \), i.e. \( \mathfrak{g}_p = \{ u \in \mathfrak{g}(V; y_1, \ldots, y_n) : u(p) = 0 \} \).

3.1 Lemma For a point \( p \in \mathbb{C}^n - V_\mathfrak{g} \), \( \mathfrak{g}_p \) is a maximal, finite codimensional subalgebra such that \( \mathfrak{g}_p^{(\infty)} = \{0\} \).

Proof. Since \( p \in \mathbb{C}^n - V_\mathfrak{g} \), there are \( u_1, \ldots, u_n \in \mathfrak{g}(V; y_1, \ldots, y_n) \)
such that \( u_j(p) = \partial / \partial y_j \bigg|_p \) for \( 1 \leq j \leq n \). Consider

\[
(\text{ad}(u_1)_\ell, \ldots, \text{ad}(u_n)_\ell) (v)(p) = 0
\]

for any \( \ell_1, \ldots, \ell_n \), and we get easily that \( \mathfrak{g}_p^{(\omega)} = \{0\} \).

We next prove the maximality of \( \mathfrak{g}_p \). Let \( \mathfrak{g} \) be a subalgebra of \( \mathfrak{g}(V; y_1, \ldots, y_n) \) such that \( \mathfrak{g} \supseteq \mathfrak{g}_p \). There is then an element \( v \in \mathfrak{g} \) such that \( v(p) \neq 0 \). By a suitable linear change of variables, we may assume that \( v \) is written in the form

\[
v = g \partial / \partial y_1 + \sum_{j=2}^{n} h_j \partial / \partial y_j, \quad g(p) \neq 0, \quad h_j(p) = 0.
\]

Let \( (p_1, \ldots, p_n) \) be the coordinate of \( p \). Then, \( (y_1 - p_1)u_j \in \mathfrak{g}_p \) for \( 1 \leq j \leq n \). Therefore, \( [v, (y_1 - p_1)u_j] = v(y_1)u_j + (y_1 - p_1)[v, u_j] \in \mathfrak{g}_p \).

Since \( v(y_1)(p) = g(p) \neq 0 \), we have \( \mathfrak{g}_p(p) = \mathfrak{g}(V; y_1, \ldots, y_n)(p) \) and hence \( \mathfrak{g}_p = \mathfrak{g}(V; y_1, \ldots, y_n) \).

Let \( \mathcal{W}_\mathfrak{g} \) be the set of all points \( q \) such that \( \mathfrak{g}_q \) is a maximal subalgebra and \( \mathfrak{g}_q^{(\omega)} = \{0\} \). By the above lemma, \( \mathcal{W}_\mathfrak{g} \) contains \( \mathbb{C}^n - V_\mathfrak{g} \). The goal of this section is as follows:

3.2 Proposition Let \( \mathfrak{g} \) be a maximal, finite codimensional subalgebra of \( \mathfrak{g}(V; y_1, \ldots, y_n) \) such that \( \mathfrak{g}^{(\omega)} = \{0\} \). Then, there is a unique point \( p \in \mathcal{W}_\mathfrak{g} \) such that \( \mathfrak{g} = \mathfrak{g}_p \).

Let \( \mathfrak{g} \) be a subalgebra of \( \mathfrak{g}(V; y_1, \ldots, y_n) \), and let \( J = \{ f \in \mathbb{C}[y_1, \ldots, y_n] \mid f \mathfrak{g}(V; y_1, \ldots, y_n) \subseteq \mathfrak{g} \} \). Obviously, \( J \) is an ideal of \( \mathbb{C}[y_1, \ldots, y_n] \), for \( \mathfrak{g}(V; y_1, \ldots, y_n) \) is a \( \mathbb{C}[y_1, \ldots, y_n] \)-module.

3.3 Lemma Let \( \mathfrak{g} \) be a subalgebra of \( \mathfrak{g}(V; y_1, \ldots, y_n) \) such that \( \mathbb{C}[y_1, \ldots, y_n] \mathfrak{g} = \mathfrak{g}(V; y_1, \ldots, y_n) \). Then \( J \mathfrak{g}(V; y_1, \ldots, y_n) \) is an ideal of \( \mathfrak{g}(V; y_1, \ldots, y_n) \) contained in \( \mathfrak{g} \).
Proof. By definition \( J \mathcal{F}(V; y_1, \ldots, y_n) \subset \mathcal{G} \). Since \((uf)v = [u, fv] - f[u, v]\), we have \( \mathcal{G} J \subset J \), hence \( (\mathcal{G}[y_1, \ldots, y_n])_J J \subset J \). By the assumption, we get \( \mathcal{F}(V; y_1, \ldots, y_n) J \subset J \). Therefore, \( J \mathcal{F}(V; y_1, \ldots, y_n) \) is an ideal of \( \mathcal{F}(V; y_1, \ldots, y_n) \).

By the above lemma, we see also that \( J \mathcal{F}(V; y_1, \ldots, y_n) \subset \mathcal{G}^{(\infty)} \).

The next lemma is due to Amamiya [1]. The proof is seen also in [5], however we repeat the proof for the sake of selfcontainedness.

3.4 Lemma. Let \( \mathcal{G} \) be a finite codimensional subalgebra of \( \mathcal{F}(V; y_1, \ldots, y_n) \). Then, \( J \neq \{0\} \).

Proof. Set \( \mathcal{G}^{(1)} = \{ u \in \mathcal{G} : [u, \mathcal{F}(V; y_1, \ldots, y_n)] \subset \mathcal{G} \} \). Since \( \text{codim} \mathcal{G} < \infty \) and \( \text{ad}(u) \) for every \( u \in \mathcal{G} \) induces a linear mapping of \( \mathcal{F}(V; y_1, \ldots, y_n)/\mathcal{G} \) into itself, we see that \( \text{codim} \mathcal{G}^{(1)} < \infty \) and hence in particular \( \mathcal{G}^{(1)} \neq \{0\} \).

Let \( v \) be a non-trivial element in \( \mathcal{G}^{(1)} \), and let \( f \) be a polynomial such that \( vf \neq 0 \). Consider a sequence \( f v, f^2 v, f^3 v, \ldots \). Since \( \text{codim} \mathcal{G}^{(1)} < \infty \), there is a polynomial \( P(t) \) in \( t \) such that \( P(f)v \in \mathcal{G}^{(1)} \).

We next prove that if \( v \) and \( gv \) are contained in \( \mathcal{G}^{(1)} \), then \( (vg)^2 \in J \). For that purpose, let \( w \) be an arbitrary element of \( \mathcal{F}(V; y_1, \ldots, y_n) \). Then, we have

\[
[v, gw] = (vg)w + g[w, v] \in \mathcal{G}
\]

\[
[gv, w] = -(wg)v + g[w, v] \in \mathcal{G}
\]

Hence

\[(22) \quad (vg)w + (wg)v \in \mathcal{G}\]

for every \( w \in \mathcal{F}(V; y_1, \ldots, y_n) \). Replacing \( w \) by \( (wg)v \), we have

\((vg)(wg)v \in \mathcal{G}\). Replacing \( w \) in (22) by \( (vg)w \), we have also

\((vg)^2w + (vg)(wg)v \in \mathcal{G}\)

Hence \((vg)^2w \in \mathcal{G}\). Thus, \((vg)^2 \in J\).
Set \( g = P(f) \). Then, \( v, g v \in \mathfrak{g}^{(1)} \) and \( v g \neq 0 \) because of \( v f \neq 0 \). Thus, we get \( J \neq \{0\} \).

3.5 Corollary Let \( \mathfrak{g} \) be a maximal finite codimensional subalgebra of \( \mathfrak{g}(V; y_1, \ldots, y_n) \) such that \( \mathfrak{g}^{(\infty)} = \{0\} \). Then, \( \mathfrak{g} \) is a \( \mathbb{C}[y_1, \ldots, y_n] \)-module.

Proof. We have only to show that \( \mathbb{C}[y_1, \ldots, y_n] \mathfrak{g} \subseteq \mathfrak{g}(V; y_1, \ldots, y_n) \), because if so, the maximality of \( \mathfrak{g} \) shows that \( \mathbb{C}[y_1, \ldots, y_n] \mathfrak{g} = \mathfrak{g} \). Thus, assume that \( \mathbb{C}[y_1, \ldots, y_n] \mathfrak{g} = \mathfrak{g}(V; y_1, \ldots, y_n) \). Then by the above lemma, we get that \( \mathfrak{g}^{(\infty)} \cap J \mathfrak{g}(V; y_1, \ldots, y_n) \neq 0 \), contradicting the assumption.

Now, we have only to consider a maximal finite codimensional subalgebra \( \mathfrak{g} \) of \( \mathfrak{g}(V; y_1, \ldots, y_n) \) such that \( \mathfrak{g}^{(\infty)} = \{0\} \) and \( \mathfrak{g} \) is a \( \mathbb{C}[y_1, \ldots, y_n] \)-module. Let \( M_p = \{ f \in \mathbb{C}[y_1, \ldots, y_n] : f(p) = 0 \} \).

3.6 Lemma For a \( \mathbb{C}[y_1, \ldots, y_n] \)-submodule \( \mathfrak{g} \) of \( \mathfrak{g}(V; y_1, \ldots, y_n) \), if
\[
\mathfrak{g} + M_p \mathfrak{g}(V; y_1, \ldots, y_n) = \mathfrak{g}(V; y_1, \ldots, y_n)
\]
for every \( p \in \mathbb{C}^n \), then \( \mathfrak{g} = \mathfrak{g}(V; y_1, \ldots, y_n) \).

Proof. By Nakayama's lemma, we see that for each \( p \in \mathbb{C}^n \), there is \( f_p \in \mathbb{C}[y_1, \ldots, y_n] \) such that \( f_p(p) \neq 0 \) and \( f_p \mathfrak{g}(V; y_1, \ldots, y_n) = \mathfrak{g} \).

Since the ideal \( \mathfrak{J} \) generated by \( \{ f_p : p \in \mathbb{C}^n \} \) has no common zero, we see that \( \mathfrak{J} = \mathbb{C}[y_1, \ldots, y_n] \) and hence there are \( f_{p_1}, f_{p_2}, \ldots, f_p \), \( g_1, g_2, \ldots, g_t \in \mathbb{C}[y_1, \ldots, y_n] \) such that \( 1 = \sum_{j=1}^{t} g_j f_{p_j} \). Therefore,
\[
\mathfrak{g}(V; y_1, \ldots, y_n) = (\sum_{j=1}^{t} g_j) \mathfrak{g} \subseteq \mathfrak{g}.
\]

3.7 Corollary Let \( \mathfrak{g} \) be a maximal, finite codimensional subalgebra of \( \mathfrak{g}(V; y_1, \ldots, y_n) \) such that \( \mathfrak{g}^{(\infty)} = \{0\} \). Then, there exists uniquely a point \( p \in \mathfrak{g}(V; y_1, \ldots, y_n) \) such that \( \mathfrak{g} = \mathfrak{g}(V; y_1, \ldots, y_n) \).

Proof. By 3.5 Corollary, \( \mathfrak{g} \) is a \( \mathbb{C}[y_1, \ldots, y_n] \)-module, and hence
there is a point $p \in \mathbb{C}^n$ such that $\mathcal{J} + M_p\mathcal{O}(V; y_1, \ldots, y_n) \subseteq \mathcal{O}(V; y_1, \ldots, y_n)$. Thus, $\mathcal{J} \supseteq M_p\mathcal{O}(V; y_1, \ldots, y_n)$ by the maximality of $\mathcal{J}$.

It is easy to see that such a point is unique, because $M_p + M_q = \mathbb{C}[y_1, \ldots, y_n]$ if $p \neq q$.

If $\mathcal{O}(V; y_1, \ldots, y_n)(p) = \{0\}$, then $M_p\mathcal{O}(V; y_1, \ldots, y_n)$ is an ideal of $\mathcal{O}(V; y_1, \ldots, y_n)$, hence it must be contained in $\mathcal{J}(\infty)$. Thus, by the assumption, it must be $\{0\}$, contradicting the assumption. Therefore we get $\mathcal{O}(V; y_1, \ldots, y_n)(p) \neq \{0\}$. Now, there is $u \in \mathcal{O}(V; y_1, \ldots, y_n)$ such that $u(p) \neq 0$ and $f \in \mathbb{C}[y_1, \ldots, y_n]$ such that $f(p) = 0$ and $(uf)(p) \neq 0$. For every $v \in \mathcal{O}(V; y_1, \ldots, y_n)$, $fv$ is an element of $\mathcal{J}$. Therefore if $u$ were contained in $\mathcal{J}$, then $[u, fv] \in \mathcal{J}$. Thus, $(uf)v \in \mathcal{J}$. It follows that $(uf)(p)v \in (uf - (uf)(p))v + \mathcal{J} \subset \mathcal{J}$. Since $(uf)(p) \neq 0$, we get $v \in \mathcal{J}$, hence $\mathcal{J} = \mathcal{O}(V; y_1, \ldots, y_n)$, contradicting the assumption.

By the above argument, we see that $\mathcal{J} \subseteq \mathcal{O}_p$, and hence $\mathcal{J} = \mathcal{O}_p$ by the maximality of $\mathcal{J}$. Since $\mathcal{J}(\infty) = \{0\}$, we see $p \notin \mathcal{W}_\mathcal{O}$ by definition.

This completes the proof of 3.2 Proposition.

3.8 A diffeomorphism induced from $\Phi$.

Let $\mathcal{O}(V'; z_1, \ldots, z_n)$ be another Lie algebra of polynomial vector fields on $\mathbb{C}^n'$. Subsets $V'_\mathcal{O}$, $\mathcal{W}_\mathcal{O}$ are defined by the same way as in $\mathcal{O}(V; y_1, \ldots, y_n)$. Suppose there is an isomorphism $\Phi$ of $\mathcal{O}(V; y_1, \ldots, y_n)$ onto $\mathcal{O}(V'; z_1, \ldots, z_n)$. For a point $p \in \mathcal{W}_\mathcal{O}$, $\mathcal{O}_p$ is a maximal finite codimensional subalgebra such that $\mathcal{O}_p(\infty) = 0$. Then, $\Phi(\mathcal{O}_p)$ has the same property, hence there is a point $\Phi(p) \in \mathcal{W}_\mathcal{O}'$ such that $\Phi(\mathcal{O}_p) = \mathcal{O}'_{\Phi(p)}$, where $\mathcal{O}'_{\Phi(p)}$ is defined by the same manner as in $\mathcal{O}(V; y_1, \ldots, y_n)$. $\Phi : \mathcal{W}_\mathcal{O} \rightarrow \mathcal{W}_\mathcal{O}'$ is a bijective mapping. The goal of
this section is as follows:

3.8 Proposition Notations and assumptions being as above, assume further that \( \mathcal{O}(V; y_1, \ldots, y_n) \) (resp. \( \mathcal{O}(V'; z_1, \ldots, z_n') \)) contains a vector field \( X \) (resp. \( X' \)) such that \( X = \sum_{j=1}^{n} \beta_j y_j \partial / \partial y_j \) (resp. \( X' = \sum_{j=1}^{n'} \beta'_j z_j \partial / \partial z_j \)). Then \( \varphi \) can be extended to a holomorphic diffeomorphism of \( \mathbb{C}^n \) onto \( \mathbb{C}^{n'} \) such that \( \varphi (V) = V' \).

Note that the existence of \( X \) and \( X' \) are obtained by 2.1 Lemma.

Let \( \mathcal{U}_\mathcal{O} \) be the totality of \( \mathbb{C} \)-valued functions \( f \) on \( \mathcal{U}_\mathcal{O} \) such that \( f u \) can be extended to an element of \( \mathcal{O}(V; y_1, \ldots, y_n) \) for every \( u \in \mathcal{U}_\mathcal{O} \). Remark that the extension of \( f u \) is unique, because \( \mathcal{U}_\mathcal{O} \) is dense in \( \mathbb{C}^n \). \( \mathcal{U}_\mathcal{O} \) is a ring and \( \mathcal{O}(V; y_1, \ldots, y_n) \) is an \( \mathcal{U}_\mathcal{O} \) module. For \( \mathcal{O}(V'; z_1, \ldots, z_n') \), we define \( \mathcal{U}_\mathcal{O}' \) by the same manner as above.

3.9 Lemma Notations and assumptions being as above, \( \varphi \) induces an isomorphism of \( \mathcal{U}_\mathcal{O} \) onto \( \mathcal{U}_\mathcal{O}' \).

Proof. Let \( f \in \mathcal{U}_\mathcal{O} \) and \( p \) an arbitrary point in \( \mathcal{U}_\mathcal{O} \). By definition, \( f(\mathcal{O}) \) can be extended to an element of \( \mathcal{O}(V'; z_1, \ldots, z_n') \), which will be denoted by the same notation. \( f(\mathcal{O}) - f(p) \in \mathcal{O}(\mathcal{O}') \), hence \( f^{-1}(f(u)) - f^{-1}(f(p)) \in \mathcal{O}(\mathcal{O}') \), that is, \( f^{-1}(f(u)) = f^{-1}(f(p)) \) if \( u = 0 \). Therefore, \( f^{-1}(f(u)) = f^{-1}(f(p)) \), that is, \( f^{-1}(f(u)) = f^{-1}(f(p)) \).

Since the left hand member is contained in \( \mathcal{O}(V; y_1, \ldots, y_n) \), we see \( \varphi f \in \mathcal{U}_\mathcal{O} \). It is easy to see that \( \varphi \) is a bi-holomorphic diffeomorphism of \( \mathbb{C}^n \) onto \( \mathbb{C}^{n'} \).

3.10 Lemma Under the same assumption as in the statement of 3.8

Proposition, we have \( \mathcal{U}_\mathcal{O} = C[y_1, \ldots, y_n] \). Hence \( \varphi \) is a bi-holomorphic diffeomorphism of \( \mathbb{C}^n \) onto \( \mathbb{C}^{n'} \).

Proof. Obviously \( \mathcal{U}_\mathcal{O} \supset C[y_1, \ldots, y_n] \). For any \( f \in \mathcal{O} \), \( fX \) is an element of \( \mathcal{O}(V; y_1, \ldots, y_n) \). Thus, \( f_1, \ldots, f_n \in C[y_1, \ldots, y_n] \). Hence
it is not hard to see \( f \in \mathcal{C}[y_1, \ldots, y_n] \).

3.11 Lemma \( \phi(c^n - V_{\phi}) = c^{n'} - V_{\phi'} \).

Proof. By the above lemma, we have \( n = n' \). Let \( p \) be a point of \( c^n - V_{\phi} \). Then \( \text{codim} \mu(\hat{\phi}^p) = n \), hence \( \text{codim} \mu(\hat{\phi}^p) = n \), because \( \mu(\hat{\phi}^p) = \mu(c^n - V_{\phi}) \). Therefore, we see \( \phi(c^n - V_{\phi}) = c^{n'} - V_{\phi'} \).

This completes the proof of 3.8 Proposition.

3.C Recapture of the germ.

Recall that \( V \) is a germ of variety with \( 0 \) as an expansive singularity. Hence there is \( X = \sum_{i = 1}^n \mu_i y_i \partial / \partial y_i \in \mathfrak{X}(V) \) such that \( \text{Re} \mu_i > 0 \) for \( 1 \leq i \leq n \). Since \( X \) is a linear vector field, \( \exp tX \) is a bi-holomorphic diffeomorphism of \( c^n \) onto itself. Remark that \( \exp tX) \ V = V \) as germs of varieties, for \( X \mathfrak{J}(V) \subseteq \mathfrak{J}(V) \) where \( \mathfrak{J}(V) \) is the ideal of \( V \) in \( \mathcal{O} \). Let \( \tilde{V} = \bigcup_{t \in \mathbb{R}} (\exp tX) \ V \). Though \( V \) is a germ of variety at \( 0 \), the expansive property of \( X \) yields that \( \tilde{V} \) is a closed subset of \( c^n \) such that \( (\exp tX) \tilde{V} = \tilde{V} \). Obviously, \( \tilde{V} = V \) as germs of varieties.

In this section, we shall prove that \( V_{\phi} = \tilde{V} \), hence \( V_{\phi} = V \) as germs of varieties. Let \( \hat{\mathfrak{J}}(V) \) be the closure of \( \mathfrak{J}(V) \) in \( \hat{\mathcal{O}} \). Note that \( \mathfrak{J}(V) \) is also the closure of \( \mathfrak{X}(V) \) in \( \hat{\mathfrak{J}}(V) \). Hence \( \mathfrak{J}(V) \subseteq \hat{\mathfrak{J}}(V) \). Recall that \( \mathfrak{J}(V; y_1, \ldots, y_n) \) is given by using the eigenspace decomposition of \( \mathcal{J}(V) \) with respect to \( \text{ad}(X) \), that is, every \( u \in \mathfrak{J}(V) \) can be rearranged in the form \( u = \sum u_\lambda \), as in 1.6 Lemma, and \( \mathfrak{J}(V; y_1, \ldots, y_n) \) is generated by the \( u_\lambda \)'s. Similarly, we decompose \( \hat{\mathfrak{J}}(V) \) into eigenspaces of \( X \). Let \( f \) be an element of \( \hat{\mathfrak{J}}(V) \). Then, \( f \) can be rearranged in the form

\[
(23) \quad f = \sum f_\nu, \quad f_\nu = \sum_{(a, \lambda) = \nu} a_\lambda y^\nu.
\]

Then, \( f_\nu \) is a polynomial such that \( Xf_\nu = \nu f_\nu \). By the same proof
as in 1.6 Lemma, we see that $f_u \in \hat{\mathcal{J}}(V)$. We denote by $I_{\hat{\mathcal{J}}}$ the ideal of $\mathbb{C}[y_1, \ldots, y_n]$ generated by all $f_u$'s with $f \in \hat{\mathcal{J}}(V)$.

3.11 Lemma $I_{\hat{\mathcal{J}}} \subset \mathcal{J}(V)$.

Proof. Let $f \in \mathcal{J}(V)$. $f$ can be rearranged in the form $f = \sum_{\nu}^\infty f_{\nu}$, $f_{\nu} = \sum_{\lambda}^\infty a_{\lambda} y^{\nu}$. We may assume $0 < \nu_1 < \nu_2 < \cdots < \nu_k < \cdots$. First of all, we shall show $f_{\nu_k} \in \mathcal{J}(V)$. Note that $e^{\nu_1 t} (\exp - tx) f = \sum e^{-(\nu_i\nu_{i+1}) t} f_{\nu_{i+1}} \in \mathcal{J}(V)$ for $t > 0$. Suppose $f$ is defined on a neighborhood $N$ of $0$ in $\mathbb{C}^n$.

Then, $(\exp - tx)f$ is defined on $\exp tx)N$. Note that $\bigcup_{t > 0} (\exp tx)N = \mathbb{C}^n$ and $\bigcup_{t > 0} (\exp tx) (N \cap V) = \tilde{V}$. Since $e^{\nu t} (\exp - tx)f = 0$ on $(\exp tx) (N \cap V)$, taking $\lim_{t \to \infty}$ we see that $f_{\nu_k} = 0$ on $\tilde{V}$. Since $\tilde{V} = V$ as germs of varieties, we have $f_{\nu_k} \in \mathcal{J}(V)$. Repeating the same procedure to $f - f_{\nu_k}$, we have $f_{\nu_{k+1}} \in \mathcal{J}(V)$, and so on. Hence $f_{\nu_i} \in \mathcal{J}(V)$.

Let $f \in \hat{\mathcal{J}}(V)$. Then, there is a sequence $\{f^{(m)}_m\}$ in $\mathcal{J}(V)$ such that $\lim f^{(m)}_m = f$ in the topology of formal power series. For any eigenvalue $\nu$ of $X : \hat{\mathcal{O}} \to \hat{\mathcal{O}}$, we see $f^{(m)}_\nu \in \mathcal{J}(V)$, and $\lim_{m \to \infty} f^{(m)}_\nu = f_\nu$ as polynomials, because the degrees of $f^{(m)}_\nu$, $f_\nu$ are bounded from above by a number related only to $\nu_1, \ldots, \nu_n$ and $\nu$. Since $f^{(m)}_\nu |_V = 0$, we have $f_\nu |_V = 0$, hence $f_\nu \in \mathcal{J}(V)$. Recall that the $f_\nu$'s generate $I_{\hat{\mathcal{J}}}$. Thus, we see $I_{\hat{\mathcal{J}}} \subset \mathcal{J}(V)$.

3.12 Lemma Notations and assumptions being as above, a polynomial vector field $u$ with $u(0) = 0$ is contained in $\mathcal{O}(V; y_1, \ldots, y_n)$ if and only if $u I_{\hat{\mathcal{J}}} \subset I_{\hat{\mathcal{J}}}$.

Proof. For $u \in \mathcal{O}(V)$, $f \in \hat{\mathcal{J}}(V)$, let $u = f_\lambda u_\lambda$, $f = f_\nu f_\nu$ be the decompositions of eigenvectors with respect to $\text{ad}(X)$, $X$ respectively. Then, $u_\lambda \in \mathcal{O}(V; y_1, \ldots, y_n)$, $f_\nu \in I_{\hat{\mathcal{J}}}$. Since $X u_\lambda f_\nu = [X, u_\lambda] f_\nu + u_\lambda X f_\nu = (\lambda + \nu) u_\lambda f_\nu$, $u_\lambda f_\nu$ is also an eigenvector of $X$. Since $uf \in \hat{\mathcal{J}}(V)$, the $u_\lambda f_\nu$'s appear in the eigenspace decomposition of $uf$, and hence $u_\lambda f_\nu \in I_{\hat{\mathcal{J}}}$. Thus, we have $\mathcal{O}(V; y_1, \ldots, y_n) I_{\hat{\mathcal{J}}} \subset I_{\hat{\mathcal{J}}}$. 

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Conversely, if $uI_{\hat{\mathcal{F}}} \subset I_{\hat{\mathcal{F}}}$ for a polynomial vector field $u$ with $u(0) = 0$. Then, $u\hat{\mathcal{J}}(V) \subset \hat{\mathcal{J}}(V)$ by taking the closure in the formal power series. Note that $u\mathcal{J}(V) \subset \mathcal{O} \cap \hat{\mathcal{J}}(V)$. We next prove that $\mathcal{J}(V) = \mathcal{O} \cap \hat{\mathcal{J}}(V)$. For that purpose, we have only to show $\mathcal{J}(V) \supset \mathcal{O} \cap \hat{\mathcal{J}}(V)$, because the converse is trivial. Let $f \in \mathcal{O} \cap \hat{\mathcal{J}}(V)$, and $f = \sum f_\nu$ the eigenvector decomposition of $f$ with respect to $X$. Then, by 3.11 Lemma, we have $f_\nu \in I_{\hat{\mathcal{F}}} \subset \hat{\mathcal{J}}(V)$. Thus, $f_\nu = 0$ on $V$, hence $f = 0$ on $V$. This means $f \in \mathcal{J}(V)$. Thus, $uI_{\hat{\mathcal{F}}} \subset I_{\hat{\mathcal{F}}}$ yields $u \in \mathcal{X}(V) \subset \mathcal{O}(V)$. However $u$ is a polynomial vector field in $y_1, \ldots, y_n$, hence $u \in \mathcal{O}(V; y_1, \ldots, y_n)$.

3.13 Lemma $V_{\hat{\mathcal{F}}} = V_{I_{\hat{\mathcal{F}}}}$: the locus of zeros of $I_{\hat{\mathcal{F}}}$.

Proof. Let $p$ be a point in $\mathbb{C}^n - V_{\hat{\mathcal{F}}}$. By definition there are $u_1, \ldots, u_n \in \mathcal{O}(V; y_1, \ldots, y_n)$ such that $u_1(p), \ldots, u_n(p)$ are linearly independent. Assume for a while that $p \in V_{I_{\hat{\mathcal{F}}}}$. Since $u_1I_{\hat{\mathcal{F}}} \subset I_{\hat{\mathcal{F}}}$, we have

$$(u_{l_1}^1 u_{l_2}^2 \cdots u_{l_n}^n f)(p) = 0$$

for every $f \in I_{\hat{\mathcal{F}}}$ and any $l_1, l_2, \ldots, l_n$. Thus, $f = 0$, contradicting the fact $I_{\hat{\mathcal{F}}} \neq \{0\}$. Therefore, $V_{\hat{\mathcal{F}}} \supset V_{I_{\hat{\mathcal{F}}}}$.

Conversely, let $p \in \mathbb{C}^n - V_{I_{\hat{\mathcal{F}}}}$. There is then $g \in I_{\hat{\mathcal{F}}}$ such that $g(p) \neq 0$. By 3.12 Lemma, $g\partial/\partial y_1, \ldots, g\partial/\partial y_n \in \mathcal{O}(V; y_1, \ldots, y_n)$, which are linearly independent at $p$. Hence $p \in \mathbb{C}^n - V_{\hat{\mathcal{F}}}$. Thus, $V_{I_{\hat{\mathcal{F}}}} \supset V_{\hat{\mathcal{F}}}$.

3.14 Lemma $V_{I_{\hat{\mathcal{F}}}} = V$ as germs of varieties.

Proof. By 3.11 Lemma, we have $\mathcal{O}I_{\hat{\mathcal{F}}} \subset \mathcal{J}(V)$, hence $V_{I_{\hat{\mathcal{F}}}} \supset V$. Assume for a while that $V_{I_{\hat{\mathcal{F}}}} \supset V$. Then there is $f \in \mathcal{J}(V)$ such that $f \neq 0$ on $V$. Let $f = \sum f_\nu$ be the eigenvector decomposition of $f$. Then $f_\nu \in I_{\hat{\mathcal{F}}}$. Therefore $f_\nu = 0$ on $V$, hence $f = 0$ on $V$ contradicting the assumption. Thus, we get $V_{I_{\hat{\mathcal{F}}}} = V$ as germs of varieties, and hence $V_{I_{\hat{\mathcal{F}}}} = V$.  

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By the above result, we get that \( \varphi : \mathbb{C}^n \rightarrow \mathbb{C}^{n'} \) maps \( \tilde{V} \) onto \( \tilde{V'} \) and \( \varphi(V) = V' \) as germs. This implies that \( \varphi^* \mathcal{J}(V') = \mathcal{J}(V) \) and hence \( d\varphi \mathcal{X}(V) = \mathcal{X}(V') \). This completes the proof of Theorem 1 in the introduction.
References