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On seminormal rings (general survey)

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Introduction

The notion of seminormal rings comes from two sources. One is the works of Endô and Bass-Murthy about Picard groups under polynomial ring extensions : $\mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(A[X])$ is not an isomorphism in general and is an isomorphism for a normal ring.[1)] Thus a problem is to characterize the rings for which $\mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(A[X])$ are isomorphisms. The answer is given by seminormal rings. The other source is the work of Andreotti-Bombieri about weakly normal varieties : Given an algebraic variety X, we seek the largest variety X^* which is birationally equivalent and homeomorphic to X. If X is normal, then $X^* = X$ by the Zariski's Main Theorem. They constructed X^*, the weak normalization of X and called X is weakly normal if $X^* = X$. Weak normality coincides with seminormality if the ground field has characteristic zero. After these works, Traverso defined seminormal rings and proved their fundamental theorems. Since then, many authors have studied seminormal rings and obtained many interesting results. In this paper, we shall report on some results known about seminormal rings.

Firstly, we begin with the following definition :

<u>Definition</u> (cf. C.Traverso [30]) Let $A \subset B$ be an integral extension of commutative rings. Put

$$A_B^+ = \left\{ b \in B \mid b/1 \in A_p + \mathrm{Rad}(B_p) \quad (\forall\, p \in \mathrm{Spec}(A)) \right\},$$

where $Rad(B_p)$ is the Jacobson radical of B_p. Then A_B^+ is the largest ring C between A and B satisfying the following properties:

(*) For every $p \in Spec(A)$, there is a unique prime ideal q of C over p and q satisfies $k(q) = k(p)$.

When $A \subset B$ is a general extension of rings, we define A_B^+ by A_D^+, where D is the integral closure of A in B. If $B = Q(A)$, the total quotient ring of A, then we write A^+ instead of A_B^+. We call A_B^+ (resp. A^+) the <u>seminormalization</u> of A in B (resp. the seminormalization of A). If $A_B^+ = A$ (resp. $A^+ = A$), then we say that A is <u>seminormal</u> in B (resp. A is seminormal).

We shall give other characterizations of A_B^+. Let $A \subset B$ be an extension of rings. A ring C between A and B which is integral over A and satisfies the above condition (*) is said to be <u>strongly integral</u> over A. An element $b \in B$ is called strongly integral over A if $C = A[b]$ is strongly integral over A. (For example, if b^n, $b^{n+1} \in A$ for some $n > 0$, then b is strongly integral over A.) Then,

(1) A_B^+ is the largest ring between A and B which is strongly integral over A,

(2) A_B^+ is the smallest ring between A and B which is seminormal in B,

and (3) $A_B^+ = \{ b \in B \mid b \text{ is strongly integral over } A \}$.

Other characterizations of A_B^+ can be given in some special cases :

(1) (Weak normalization) (cf. [1], [21], [22]) For an integral extension $A \subset B$ of commutative rings, we put

$A_B^* = \{b \in B \mid \forall q \in \mathrm{Spec}(A),\ (b/1)^{p^n} \in A_q + \mathrm{Rad}(B_q) \text{ for some } n\}$, where p is the characteristic exponent of k(q) and call this ring the weak normalization of A in B. A_B^* is the largest ring between A and B which is radicial over A. If all residue field extensions $k(p) \subset k(q)$ $(q \in \mathrm{Spec}(B),\ p = q \cap A)$ are separable (e.g., A contains a field of characteristic zero), A_B^* coincides with A_B^+. Manaresi showed that $A_B^* = \{b \in B \mid b \otimes 1 - 1 \otimes b = 0 \text{ in } (B \otimes_A B)_{red}\}$.[2)]

(2) (Weakly normal algebraic varieties) (cf. [19]) Let k be an algebraically closed field of characteristic zero and $(X, \mathcal{O}_X)$ be an algebraic variety over k in the sense of J.-P.Serre (i.e., as a set, X is the set of closed points of reduced separated algebraic scheme over k). Define $\mathcal{O}_X^c$, the (coherent) sheaf of c-regular functions over X as follows : for an open set U of X, put

$$\Gamma(U, \mathcal{O}_X^c) = \{f : U \to k \mid f \text{ is continuous and } f|\mathrm{Reg}(U) \text{ is regular}\}$$
$$= \{g \in \Gamma(w^{-1}(U), \mathcal{O}_Y) \mid g \text{ is constant on each fiber of } w\},$$

where $w : Y \to X$ is the normalization of X. Then we can show that $\mathcal{O}_{X,x}^c = \mathcal{O}_{X,x}^+$ = the integral closure of $\mathcal{O}_{X,x}$ in $C_{X,x}$ (C_X = the sheaf of continuous k-valued functions over X) for every $x \in X$.

(3) (Weakly normal complex analytic space) (cf. [14], [32], [33]) Let $(X, \mathcal{O}_X)$ be a reduced complex analytic space. Define $\mathcal{O}_X^c$, the (coherent) sheaf of c-holomorphic (or continuous weakly holomorphic) functions over X as follows : for an open set U of X, put $\Gamma(U, \mathcal{O}_X^c) = \{f : U \to \mathbb{C} \mid f \text{ is continuous and } f|\mathrm{Reg}(U) \text{ is holomorphic}\}$. Then $\mathcal{O}_{X,x}^c = \mathcal{O}_{X,x}^+$ for every $x \in X$.

Let $A \subset B \subset C$ be extensions of rings. Then,

(1) $A^+_B = A^+_C \cap B$.

(2) $A^+_C \subset B^+_C$. If B is an integral domain, then $A^+ \subset B^+$.

(3) If A is seminormal in C, then A is seminormal in B.

(4) If A is seminormal in B and B is seminormal in C, then A is seminormal in C.

<u>Conductor theorems</u> (cf. [30]) For an extension $A \subset B$, we denote by $c(B/A)$ the conductor of B in A. Then,

(1) If $A \subset B$ is an integral extension and A is seminormal in B, then $c(B/A)$ is a radical ideal of B.

(2) If $A \subset B$ is a finite extension of noetherian rings and A is not seminormal in B, then $c(A^+_B/A)$ is not a radical ideal of A^+_B .

The following criterion is very useful to verify the seminormality of given rings.

<u>E.Hamann's criterion</u> (cf. [4], [12], [15]) Let $A \subset B$ be an extension of rings. Then, the following statements are equivalent :

(1) A is seminormal in B.

(2) If $b \in B$, $b^2, b^3 \in A$ (resp. $b^n, b^{n+1} \in A$ for some $n > 0$), then $b \in A$.

(3) $c(A[b]/A)$ is a radical ideal of $A[b]$ for every $b \in B$ which is integral over A.

Using this criterion, we can prove some fundamental facts about localization and graded rings.

<u>Localization</u> (1) Let $A \subset B$ be an extension of rings and S be a multiplicative subset of A. Then $(A_S)^+_{B_S} = (A^+_B)_S$. In particular, if A is seminormal in B, then A_S is seminormal in B_S.

(2) Let A be a ring such that $\dim Q(A) = 0$[3] (e.g., an integral domain or a reduced noetherian ring). Then $(A_S)^+ = (A^+)_S$ for every multiplicative subset S of A. Hence if A is seminormal, then A_S is seminormal. A is seminormal if and only if A_m is seminormal for every $m \in \mathrm{Max}(A)$.

(3) Let A be a noetherian ring such that $\dim Q(A) = 0$. Then A is seminormal if and only if A_p is seminormal for every $p \in \mathrm{Spec}(A)$ such that $\mathrm{depth}(A_p) = 1$. (If A satisfies the Serre's condition (S_2), then "$\mathrm{depth}(A_p) = 1$" can be replaced by "$\mathrm{ht}(p) = 1$".) In particular, let A be as above and I be an ideal of A such that $\mathrm{depth}_I(A) \geq 2$. Then A is seminormal if and only if A_p is seminormal for every $p \in D(I) = \mathrm{Spec}(A) - V(I)$.

<u>(S_2)-Mori rings</u> We say a noetherian ring A is a <u>Mori ring</u> if it is reduced and its integral closure A' is finite over A. If A is a one-dimensional semi-local Mori ring, then $A^+ = A + \mathrm{Rad}(A')$. Hence A is seminormal $\Longleftrightarrow \mathrm{Rad}(A') = \mathrm{Rad}(A) \Longleftrightarrow c(A'/A)$ is a radical ideal of A'. If A is a Mori ring which satisfies (S_2), then A is seminormal if and only if $c(A'/A)$ is a radical ideal of A'(cf. [2]).

<u>Graded rings</u> (1) For an extension $A \subset B$, we have $A[X]^+_{B[X]} = (A^+_B)[X]$. If A is an integral domain or a reduced noetherian ring, then $A[X]^+ = (A^+)[X]$. Therefore A[X] is seminormal if and only if A is seminormal.[4]

(2) For an extension $A \subset B$ of graded rings, A_B^+ is a graded subring of B. If A is a graded domain, then so is A^+. If $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is a seminormal graded domain, then so are its Veronesean subrings $A^{(d)} = \bigoplus_{n \in \mathbb{Z}} A_{nd}$.

(3) If (A, I) is a Zariski pair and its associated graded ring $G_I(A) = \bigoplus I^n/I^{n+1}$ is a seminormal domain, then A is also a seminormal domain.

(4) Let A be an integral domain or a reduced noetherian ring and I be an ideal of A. If A is seminormal and $G_I(A)$ is reduced (e.g., I is a radical ideal generated by an A-regular sequence), then the Rees algebra $R_I(A) = \bigoplus I^n$ is also seminormal.[5)] In particular, if (A, m) is a one-dimensional local Mori ring, then A is seminormal if and only if $R_m(A)$ is seminormal (cf. [10]).

(5) Let $A = \bigoplus_{n \geqslant 0} A_n$ be a graded noetherian ring such that $\dim Q(A) = 0$. Assume that A_0 is a field and $A = A_0[A_1]$ and put $V = \mathrm{Proj}(A)$ and $C = \mathrm{Spec}(A)$. Then C is seminormal if and only if V and $\mathcal{O}_{C,(0)}$ are seminormal. If V is seminormal and $\mathrm{depth}(\mathcal{O}_{C,(0)}) \geqslant 2$, then C is seminormal.

Gluings (cf. [26], [29], [30], [31]) Let $A \subset B$ be an integral extension and p be a prime ideal of A. Put $A_B^+(p) = \{ b \in B \mid b/1 \in A_p + \mathrm{Rad}(B_p) \}$ and call it the ring obtained from B by gluing over p. Then $A_B^+(p)$ is the largest ring C between A and B which satisfies the following property :

$(*)_p$ There is a unique prime ideal q of C over p and q satisfies $k(q) = k(p)$.

Since $A_B^+ \subset A_B^+(p)$, $A_B^+(p)$ is seminormal in B. If $A \subset B$ is a finite extension and $\{q_1, \ldots, q_r\}$ is the set of prime ideals of B over p, we can also define $A_B^+(p)$ by the following cartesian diagram :

$$\begin{array}{ccc} A_B^+(p) & \longrightarrow & B \\ \downarrow & & \downarrow \\ k(p) & \longrightarrow & \prod_{i=1}^{r} k(q_i) = (B \otimes_A k(p))_{red} \end{array}.$$

Using this notion of gluings, we can state the following structure theorem of seminormal rings :

Structure theorem (cf. [30]) Let $A \subset B$ be a finite extension of noetherian rings. Then A is seminormal in B if and only if

$\exists\ p_1, \ldots, p_n \in \operatorname{Spec}(A)$ such that $B = B_0 \supset B_1 \supset \ldots \supset B_n = A$, where $B_{i+1} = A^+_{B_i}(p_{i+1})$, namely, A is obtained from B by a succession of gluings. Hence a Mori ring is seminormal if and only if it is obtained from a normal ring by a succession of gluings.

Now, before going further, we shall give some examples of seminormal rings.

Example 1. Let d be an integer which is not square and put $A = \mathbb{Z}[\sqrt{d}\,]$. Then A is seminormal if and only if $d = \pm\prod p^{e_p}$ (p : prime and $e_p \in \mathbb{Z}$, $e_p \geqslant 0$), where $e_p \leqslant 2$ for all p and $e_2 \leqslant 1$. In particular, A is seminormal if d is a square-free integer. $\mathbb{Z}[\sqrt{-4}\,]$ is not seminormal. In fact, we can show that $\mathbb{Z}[\sqrt{-4}\,]^+ = \mathbb{Z}[i]$.

Example 2. (Abelian group rings, cf. Bass-Murthy [2]) Let G be a finite abelian group. Put $A = \mathbb{Z}[G]$. Then A is a one-dimensional Mori ring and A is seminormal if and only if Card(G) is square-free.

Example 3. (Semigroup rings, cf. Hochster-Roberts [16]) Let H be a commutative monoid written multiplicatively. For an integral domain A, A[H] is an integral domain if and only if H is cancellative and torsion-free (i.e., the quotient abelian group G of H is torsion-free). Then, the following statements are equivalent :

(1) If u, $v \in H$ and $u^3 = v^2$, then $u = w^2$, $v = w^3$ for some $w \in H$ (equivalently, if $w \in G$, w^p, $w^q \in H$, $(p, q) = 1$, then $w \in H$).

(2) K[H] is a seminormal domain for a field K.

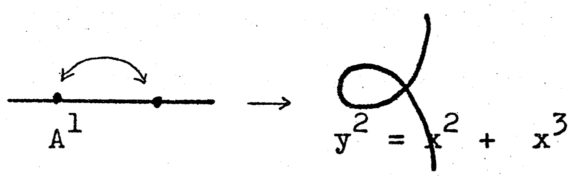
(3) A[H] is a seminormal domain for every seminormal domain A.

If these equivalent conditions are satisfied, the monoid H is said to be seminormal.

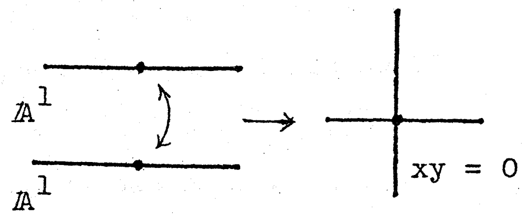
If H is a seminormal monoid such that $H \subset \mathbb{Z}$, then H is isomorphic to $\mathbb{N}$ or $\mathbb{Z}$, hence is normal. The submonoid of $\mathbb{N}^2$ generated by (2, 0), (1, 1) and (0, 1) is seminormal which is not normal (see Example 6 below).

Example 4. (N.Chiarli [8]) Let k be a field of characteristic zero and suppose that $V : Y^n = f(X_1,\dots,X_m)$ is an irreducible hypersurface in $\mathbb{A}_k^{m+1}$. Then V is seminormal if and only if either $n = 2$ and the multiplicity of every irreducible factor of f is at most two or $n \geqslant 3$ and f is square-free.

Example 5. (Algebraic curves) (1) $k[X, Y]/(Y^2 - X^2 - X^3) = k[T^2 - 1, T(T^2 - 1)]$ (char(k) $\neq 2$) is seminormal. This ring is obtained by gluing two points $x = \pm 1$ of $\mathbb{A}^1$.



(2) $k[X, Y]/(XY)$ is seminormal. This ring is obtained by gluing two affine line at $x = 0$.



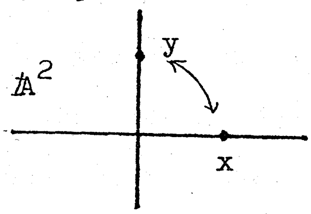
(3) $k[X, Y]/(Y^2 - X^3) = k[T^2, T^3]$ is not seminormal. In fact, T is strongly integral over $k[T^2, T^3]$.

A plane algebraic curve is seminormal if and only if its singularities are at most nodes (= ordinary double points) (cf. [4], [28]). More generally, an algebraic curve is seminormal if and only if its singularities are ordinary n-fold points, where n is the dimension of the Zariski tangent space (cf. [3], [10]). E.D.Davis proved this theorem as follows : Let (A, m) be a one-dimensional local Mori ring. Then

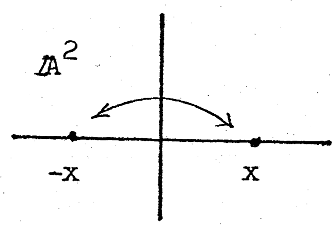
$$A \text{ is seminormal} \iff G_m(A) \text{ is reduced and } e(A) = \mathrm{emdim}(A)$$
$$\iff \mathrm{Proj}(G_m(A)) \text{ is reduced and } e(A) = \mathrm{emdim}(A),$$

where $e(A)$ is the multiplicity of A and $\mathrm{emdim}(A)$ is the embedding dimension $\dim_{A/m}(m/m^2)$ of A.

<u>Example 6</u>. (Algebraic surfaces) (1) $k[X, Y, Z]/(Y^3 + Z^2 - XYZ) = k[S + T, ST, S^2T]$ is seminormal. This ring is obtained by identifying x-axis with y-axis in an affine plane $\mathbb{A}^2$.

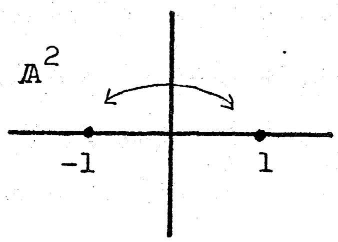


(2) $k[X, Y, Z]/(XZ^2 - Y^2) = k[S^2, ST, T]$ (Cayley's umbrella) is seminormal. This ring is obtained (in $\mathrm{char}(k) \neq 2$) by identifying $(x, 0)$ and $(-x, 0)$ in an affine plane $\mathbb{A}^2$.



On the contrary, $A = k[X, Y, Z]/(XZ^n - Y^n) = k[S^n, ST, T]$ $(n \geqslant 3)$ is not seminormal, since $(S^2T)^n = S^n(ST)^n \in A$, $(S^2T)^{n+1} = S^{2n}(ST)^2T^{n-1} \in A$ but $S^2T \notin A$.[6)]

(3) $k[XY, Y, X^2-1, X(X^2-1)]$ ($\mathrm{char}(k) \neq 2$) is seminormal. If we put $B = k[X, Y]$, $\mathfrak{m}_1 = (X-1, Y)$, $\mathfrak{m}_2 = (X+1, Y)$ and $m = \mathfrak{m}_1 \cap \mathfrak{m}_2$, then $A = k + m = A_B^+(m)$, i.e., A is obtained by gluing (1, 0) and (-1, 0) in $\mathbb{A}^2$. A has $\dim(A_m) = 2$, and $\mathrm{depth}(A_m) = 1$.



Characterizing seminormal surfaces by their singularities as in the case of algebraic curves seem to be unknown, though partial results have been obtained (cf. [6], [9]).

<u>Example 7</u>. (Ordinary singularities) Here we consider over $\mathbb{C}$, the complex number field. If X is a smooth projective variety of dimension n, the singularities of its generic projections in $\mathbb{P}^{n+1}$ are called <u>ordinary singularities</u>. Ordinary singularities of plane curves are precisely nodes. Hence these are seminormal by Example 5. Ordinary singularities of surfaces in $\mathbb{P}^3$ are of three types : $xy = 0$ (ordinary double curve),

$xyz = 0$ (ordinary triple point),

$x^2 - yz^2 = 0$ (pinch point),

which are easily shown to be seminormal. Thus a question arises : Are the ordinary singularities always seminormal ? (cf. [3]) This question has recently been solved affirmatively by S.Greco and C.Traverso using a classical result of A.Franchetta (cf. [14]).

<u>The relation with Picard groups</u> (cf. [2], [4], [11], [12], [18], [34]) For a ring A, put $\mathrm{NPic}(A) = \mathrm{Coker}(\mathrm{Pic}(A) \longrightarrow \mathrm{Pic}(A[X]))$. Then, the following theorems hold :

Let $A \subset B$ be an integral extension of reduced rings and suppose that A is noetherian. Then A is seminormal in B if and only if NPic(A) $\longrightarrow$ NPic(B) is injective.

If A is an integral domain or a reduced noetherian ring, then A is seminormal if and only if Pic(A) $\longrightarrow$ Pic(A[X]) is an isomorphism, i.e., NPic(A) = 0.[7)]

An application to Serre's problem (cf. [4], [23]) Let A be a commutative ring. Consider the following property for A : For any set I, every finitely generated projective $A[X_i]_{i\in I}$-module is extended from A. (The set I can be assumed to be a finite set.) If A satisfies this property, we say that A is a Quillen ring (after the famous work of D.Quillen on Serre's problem). Next facts are known about Quillen rings :

(1) A is a Quillen ring if and only if A_{red} is so.

(2) If A is a Quillen ring, then A_S is also a Quillen ring for every multiplicative subset S of A (M.Roitman).

(3) A is a Quillen ring if and only if A_m is a Quillen ring for every $m \in Max(A)$.

(4) A regular ring of dimension at most two is a Quillen ring (Quillen-Suslin). It is conjectured that every regular ring is a Quillen ring. Some special cases of this conjecture are proved (H.Lindel-W.Lütkebohmert).

(5) A locally finite dimensional arithmetical ring is a Quillen ring (Lequain-Simis).

Using the method of Lequain-Simis (i.e., axiomatic treatment of Quillen rings) we can prove the following theorem :

(6) Let A be a one-dimensional integral domain whose integral closure is a Prüfer domain. Then A is a Quillen ring if and only

if A is seminormal. If A is a one-dimensional noetherian ring, then A is a Quillen ring if and only if A_{red} is seminormal.[8)]

Following the method of N.Mohan Kumar (cf. Inv. Math. 46 (1978), 225-236), we can show the next theorem about the number of generators of ideals of polynomial rings over Quillen rings (e.g., one-dimensional seminormal rings) : Let A be a noetherian Quillen ring of dimension d such that every finitely generated projective A-module is free (e.g., PID or two-dimensional semilocal regular domain or one-dimensional noetherian semilocal seminormal domain) and I be an ideal of $B = A[X_1,\dots,X_n]$ $(n>0)$. If $\mu(I/I^2) \geqslant \dim(B/I) + 2$ and $ht(I) \geqslant d + 1$, then $\mu(I) = \mu(I/I^2)$, where $\mu(M)$ denotes the minimal number of generators of a module M.

Other applications of seminormal rings include :

(1) The problem of invariance of coefficient rings of polynomial rings (E.Hamann [15]) : Let A be a seminormal domain. If $A \subset B$ and $A[X_0,\dots,X_n] = B[Y_1,\dots,Y_n]$, then $B = A[T]$ (X_i's, Y_j's and T are variables).

(2) Lüroth's problem for rings (S.Glaz, J.D.Sally and W.V. Vasconcelos, J. of Algebra 43 (1976), 699-708) : Let A be a seminormal domain. If $A \subsetneq B \subset A[X]$ and A[X] is faithfully flat over B, then $B \cong Symm_A(P)$ for some finitely generated projective A-module of rank one P. (If, moreover, A is semilocal, then $B \cong A[T]$.)

(3) The problem of efficient generation of maximal ideals in polynomial rings (E.D.Davis and A.Geramita) : They proved, in particular, the following theorem : Let A be a one-dimensional semilocal Mori domain. Then, $\mu(n) = \mu(n/n^2)$ for every maximal

ideal n of A[X] if and only if A is seminormal.

Notes

1) This follows easily from the following commutative diagram for a noetherian normal domain A :

$$\begin{array}{ccccc} \mathrm{Pic}(A) & \xrightarrow{f} & \mathrm{Pic}(A[X]) & \xrightarrow{g} & \mathrm{Pic}(A[[X]]) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cl}(A) & \xrightarrow{p} & \mathrm{Cl}(A[X]) & \xrightarrow{q} & \mathrm{Cl}(A[[X]]), \end{array}$$

where p is an isomorphism, $q \circ p$ is injective and $g \circ f$ is an isomorphism.

2) This follows from the next facts : A ring homomorphism $A \to B$ is radicial $\Longleftrightarrow A_{red} \longrightarrow B_{red}$ is an epimorphism in the category of reduced rings $\Longleftrightarrow (B \otimes_A B)_{red} \longrightarrow B_{red}$ is an isomorphism. Hence, for an extension $A \subset B$, a ring C between A and B is radicial over $A \Longleftrightarrow C \subset \mathrm{Ker}(B \underset{j}{\overset{i}{\rightrightarrows}} (B \otimes_A B)_{red})$, where $i(b) = \overline{b \otimes 1}$ and $j(b) = \overline{1 \otimes b}$.

3) This condition ensures the commutativity of total quotient ring and localization (cf. J.Lipman , Proc. Amer. Math. Soc. 16 (1965), 1120-1122).

4) As for the power series extensions, we can show the following facts :

(1) Let $A \subset B$ be an integral extension. If A is seminormal in B, then A[[X]] is seminormal in B[[X]].

(2) If $A \subset B$ is a finite extension of noetherian rings, then $A[[X]]^+_{B[[X]]} = (A^+_B)[[X]]$. If A is a Mori ring, then $A[[X]]^+ = (A^+)[[X]]$.

(3) Let A be an integral domain or a reduced noetherian ring which is seminormal. Then, A[[X]] is also seminormal.

Concerning general base change, the following facts are known (cf. [14]) :

(1) Let $A \subset B$ be an integral extension and $A \longrightarrow A'$ be a reduced homomorphism of noetherian rings. If we put $B' = B \otimes_A A'$, then we have $A'^{+}_{B'} = (A^{+}_{B}) \otimes_A A'$.

(2) Let $A \longrightarrow B$ be a reduced homomorphism of noetherian rings with normal gereric fibers. Then we have $B^{+} = A^{+} \otimes_A B$.

(3) If A is a reduced excellent local ring, then A is seminormal if and only if $\hat{A}$, the completion of A, is seminormal.

5) By a theorem of J.Barshay (cf. J. of Algebra 25 (1973), 90 - 99) $R_I(A)$ is integrally closed in A[X] in this case.

6) Similarly $A = k[S^2, ST_1, \ldots, ST_{n-1}, T_1, \ldots, T_{n-1}]$ (Whitney's umbrella) is seminormal. If m is the maximal homogeneous ideal of A, then $\dim(A_m) = n$ and $\operatorname{depth}(A_m) = 2$. In general, if A is a seminormal domain in which 2 is a unit and I is an ideal of A such that $G_I(A)$ is reduced, then $A[It, t^2]$ is a seminormal domain. (The condition on 2 probably can be deletted.)

7) General situation is as follows :

(1) If A is seminormal and Min(A) is a finite set, then NPic(A) = 0 (cf. [34]).

(2) If A is reduced and NPic(A) = 0, then A is seminormal (Schanuel).

(3) Even if A is seminormal, NPic(A) = 0 does not hold in general (cf. [12]).

8) For the proof of this theorem, we use the next fact found by S.Itoh : Let A be an integral domain. Then, the integral closure of A is a Prüfer domain if and only if the canonical map Spec(A(X)) $\longrightarrow$ Spec(A) is bijective. (Concerning the ring A(X), see Nagata, Local rings, p.18.)

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