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Inverse Scattering Transformation of the Benjamin-Ono Equation and Related Topics

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The purpose of this short article is to review recent development on nonlinear water wave equations mainly focusing attention on the various papers appeared around this year, 1979.

We have KdV equation

$$u_t+2uu_x+u_{xxx}=0, \tag{1}$$

which describes shallow water waves. As is well known, this nonlinear partial differential (=NPD) equation has been the most famous model equation that exibits soliton solutions and has been much investigated in the last decade beginning with the work by Zabusky and Kruskal in 1965.[1] Exact N-soliton solution and N-periodic wave solution are known. On the other hand, as a model equation for the propagation of waves in a deep stratified fluid, we have Benjamin-Ono(=BO) equation[2,3]

$$u_t+2uu_x+Hu_{xx}=0, \tag{2}$$

$$Hu(x)=\frac{P}{\pi}\int_{-\infty}^{\infty}\frac{u(z)}{z-x}\,dz, \qquad \text{(H: Hilbert transform)} \tag{3}$$

which is nonlinear integro-differential equation. The existence of the N-soliton solution to this equation has been shown by the pole expansion method.[4,5] Then explicit N-soliton solution is obtained by the bilinear method.[6] Exact N-periodic wave solution is also obtained.[7,8] The bilinear method can be outlined as follows. We consider dependent variable transformation

$$u(x,t)=i\partial_x \log(f'(x,t)/f(x,t)), \tag{4}$$

where f (f') is assumed to be finite or infinite product of $x-z_n$ ($x-z_n'$) with z_n (z_n') lying in the upper (lower) half complex pane. The Hilbert transform becomes very simple in the new variable[6]

$$H\left[i\partial_x \log(f'/f)\right]=-\partial_x \log(f'f). \tag{5}$$

This leads to the expression of BO equation in the very simple bilinear form[6]

$$(iD_t-D_x^{\,2})f'\cdot f=0. \tag{6}$$

Here bilinear operators D_t, D_x are defined by

$$D_t^{\,n}D_x^{\,m}a(x,t)\cdot b(x,t)\equiv(\partial_t-\partial_{t'})^n(\partial_x-\partial_{x'})^m a(x,t)b(x',t')\Big|_{x=x',\,t=t'}. \tag{7}$$

Equation (6) is rather surprizing because among all of the possible combinations of D_t and D_x, this is the simplest one other than the trivial one, $D_t \pm D_x$. Starting from eq. (6), the Bäcklund Transform ($\equiv$BT) in bilinear form can be obtained by the usual procedure of bilinear formalism leading to the expression[10)]

$$(iD_t-2i\lambda D_x-D_x^2-\mu)f\cdot g=0, \tag{8a}$$

$$(iD_t-2i\lambda D_x-D_x^2-\mu)f'\cdot g'=0, \tag{8b}$$

$$(D_x+i\lambda)f\cdot g'-i\nu f'g=0. \tag{8c}$$

Introducing "potential" by $u\equiv-\bar{u}_x$, $v\equiv-\bar{v}_x$, this can be transformed to BT in these variables

$$(\bar{u}+\bar{v})_x=2\lambda-2\nu\exp\{i(\bar{u}-\bar{v})\} -iH(\bar{u}-\bar{v})_x, \tag{9a}$$

$$(\bar{u}-\bar{v})_t=2\lambda(\bar{u}-\bar{v})_x-i(\bar{u}-\bar{v})_xH(\bar{u}-\bar{v})_x-i(\bar{u}+\bar{v})_{xx}. \tag{9b}$$

Inverse scattering transform can be also obtained by the usual procedure of putting $g=f\psi$, $g'=f'\psi'$ in bilinear BT, eqs. (8a-c), leading to

$$(i\partial_t-2i\lambda\partial_x+\partial_x^2+iu_x-(Hu_x)+\mu)\psi=0, \tag{10a}$$

$$(i\partial_t-2i\lambda\partial_x+\partial_x^2-iu_x-(Hu_x)+\mu)\psi'=0, \tag{10b}$$

$$(u+i\partial_x+\lambda)\psi'-\nu\psi=0. \tag{10c}$$

Existence of infinite number of conservation laws and Bianchi's exchange property of the present bilinear BT can be proved.[10)] Present system (BO eq.) is interesting because it is the first example of the nonlinear integro-differential equation which fits quite well into the "soliton" theory framework.

Recently, the fluid model equation which includes KdV and BO as the special limit is also studied. The equation is written as

$$u_t+2uu_x+Gu_{xx}=0, \tag{11}$$

$$Gu(x,t)=\frac{1}{2\delta}\,P\int_{-\infty}^{\infty}\left\{\coth\tfrac{1}{2}\delta^{-1}\pi(x'-x)-\mathrm{sgn}(x'-x)\right\}u(x',t)\,dx', \tag{12}$$

where δ is a depth parameter.[11)] Recently N-soliton solution has been obtained by bilinear method to this equation.[12,13)] Inverse scattering transform can be also obtained exactly in the same manner as the above BO case.[14)] Exact one- and two-periodic wave solution is obtained by the present author and Y. Matsuno.[15)]

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