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<td><strong>Author(s)</strong></td>
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<td><strong>Citation</strong></td>
<td>数理解析研究所講究録 (1980), 375: 129-135</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1980-02</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/2433/104736">http://hdl.handle.net/2433/104736</a></td>
</tr>
<tr>
<td><strong>Type</strong></td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td><strong>Textversion</strong></td>
<td>publisher</td>
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Kyoto University
Relation between Kaup's and Mikhailov's Equations, their Exact Solutions and Stories How It Discovered

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We know that

Kaup's equation

\[ u_t + u_{5x} + 30(u_{3x}u + \frac{5}{2} u_{xx}u_x) + 180u^2u_x = 0 , \]

Sawada – Koter's equation

\[ u_t + u_{5x} + 15(u_{3x}u + u_{xx}u_x) + 45u^2u_x = 0 \]

and Lax's 5-th order K-dV equation

\[ u_t + u_{5x} + 10(u_{3x}u + 2u_{xx}u_x) + 30u^2u_x = 0 \]

are reduce, using the potentials

\[ u = \frac{1}{2} w_x , \quad u = w_x \quad \text{and} \quad u = \sqrt{3/2} w_x , \]

to

\[ w_t + w_{5x} + 15(w_{3x}w_x + \frac{3}{4} w^2_{xx}) + 15w^3 w_x = 0 , \]
\[ w_t + w_{5x} + 15w_{3x}w_x + 15w_x^3 = 0 \]

and

\[ w_t + w_{5x} + \sqrt{3/2} 10(w_{3x}w_x + \frac{1}{2}w_{xx}^2) + 15w_x^3 = 0. \]

Kaup found the inverse scattering transform for his equation (more than 2 years ago)

\[ \psi_{xxx} + 6u\psi_x + 3u_x\psi = \lambda\psi, \quad (\lambda: \text{eigenvalue}) \]

\[ \psi_t = 9\lambda\psi_{xx} - 3(u_{xx} + 12u^2)\psi_x + 3(u_{xxx} + 12\lambda u + 24u_x u)\psi. \]

I found that the form is transformed into the bilinear form (2 years ago)

\[ D_x^3 f' \cdot f + 3D_x g' \cdot f = 4\lambda f' f, \]

\[ D_x^2 f' \cdot f = g' f, \]

\[ D_t f' \cdot f = -\frac{3}{8} D_x^5 f' \cdot f + \frac{15}{8} D_x^3 g' \cdot f + \frac{15}{2} \lambda D_x^2 f' \cdot f, \]

through the transformation

\[ \psi = f'/f, \quad u = \frac{1}{4} \frac{D_x^2 f' \cdot f}{f^2}. \]

A. Ramani found at my request that Kaup's equation satisfies the resonance criterion that is the necessary condition for the equation to be of Painlevé-type (about 4 months ago).

Kaup found a one-soliton solution to it. (2-3 months ago)

\[ u = 2p^2 \frac{e^n + e^{-n} + 1}{[e^n + e^{-n} + 4]^2}, \]

\[ \eta = px + \Omega t + \text{const}, \quad \Omega + p^5 = 0. \]
Ramani pointed out that Kaup's one-soliton solution can be obtained using the bilinear form (about 2 months ago).

He found, for $\lambda = 0$, 

$$f = e^n + e^{-n} + h,$$

$$f' = e^n + e^{-n} - 2,$$

$$g' = 2p^2,$$

are the solutions to the bilinear form.

About a month ago, I found that Kaup's equation is transformed into the bilinear form

$$D_x(D_t + \frac{1}{16}D^5_x)f \cdot f + \frac{15}{16}D^2_xg \cdot f = 0,$$

$$D^h_xf \cdot f = gf,$$

through the transformation

$$u = \frac{1}{4} \frac{D^2_xf \cdot f}{f^2},$$

and found 2-soliton solution (Oct. 4, '79)

$$f = 1 + 4(e^{n_1} + e^{n_2}) + e^{2n_1} + 2(1 + a_{12})e^{n_1 + n_2} + e^{2n_2} + 4\beta_{12}(e^{n_1 + 2n_2} + e^{n_1 + n_2}) + \beta_{12}^2 e^{2n_1 + 2n_2},$$

$$g = 8(p_1 e^{n_1} + p_2 e^{n_2}) + 16\gamma_{12} e^{n_1 + n_2} + 3\beta_{12}(p_1 e^{n_1 + 2n_2} + p_2 e^{n_1 + n_2}),$$
where

\[
\alpha_{12} = (p_1 - p_2)^2 \left[ \frac{3}{(p_1 + p_2)^2} + \frac{4}{p_1^2 + p_1 p_2 + p_2^2} \right],
\]

\[
\beta_{12} = (\frac{p_1 - p_2}{p_1 + p_2})^2 \left( \frac{p_1^2 - p_1 p_2 + p_2^2}{p_1^2 + p_1 p_2 + p_2^2} \right),
\]

\[
\gamma_{12} = (p_1 - p_2)^2 \left( \frac{2 p_1^4 + 3 p_1^2 p_2^2 + 2 p_2^4}{p_1^2 + p_1 p_2 + p_2^2} \right),
\]

\[
\eta_i = p_i x + \Omega_i t + \eta_i^0, \quad \eta_i + \eta_i^0 = 0, \quad \text{for } i = 1, 2.
\]

In his letter dated Oct. 18, 1979, Ramani wrote me that Shabat and Mikhailov found "L,A" pair for the equation

\[
u_{xt} = e^u - e^{-2u}
\]

(Correspondence to him by Mark Ablowitz).

Ramani was able to transform it to the third Painlevé equation, and write it in bilinear form

\[
D_x D_t \hat{f} \cdot \hat{g} = 2 \hat{f} (\hat{f} - \hat{g})
\]

\[
D_x D_t \hat{g} \cdot \hat{g} = -2 (\hat{f}^2 - \hat{g}^2),
\]

through the transformation

\[
u = \log(\hat{g} / \hat{f})
\]

He and I found two-soliton solution to it independently. I found it on Oct. 30, '79.

\[
\hat{f} = h^2,
\]
\[ h = 1 + e_1^n + e_2^n + \beta_{12} e_1^{n_1} e_2^{n_2}, \]
\[ \hat{g} = 1 - 4(e_1^n + e_2^n) + e_1^{2n_1} + \beta_{12} e_1^{n_1} e_2^{n_2} + e_2^{2n_2} \]
\[ - 4\beta_{12} (e_1^{n_1} e_2^{n_2} + e_1^{n_2} e_2^{n_1}) + \beta_{12} e_1^{2n_1} e_2^{2n_2}, \]

where \( n_i = p_i x + w_i x^i + n_i^0, \) \( p_i w_i = 3 \) for \( i = 1, 2, \)

\[ \beta_{12} = \left( \frac{p_1 - p_2}{p_1 + p_2} \right)^2 \left( \frac{p_1^2 - p_1 p_2 + p_2^2}{p_1^2 + p_1 p_2 + p_2^2} \right), \]

\[ b_{12} = 8 \frac{2p_1^4 - p_1^2 p_2^2 + 2p_2^4}{(p_1^2 + p_2^2)^2(p_1^2 + p_1 p_2 + p_2^4)}. \]

To our surprise, the functional form of \( \hat{g} \) is equal to that of \( f, \) the solution to Kaup's equation, namely

\[ f(n_1, n_2) = \hat{g}(n_1 + i\pi, n_2 + i\pi). \]

Furthermore, \( h \) is equal to two-soliton solution to Sawada–Kotera's equation

\[ \ast\) \quad D_x (D_t + D_x^5) h \cdot h = 0 \]

and to the equation

\[ D_x^2 (D_t D_x - 3) h \cdot h = 0, \]

which is the special case of

\[ D_x (D_t - D_x^2 D_t + D_x^3) f \cdot f = 0, \]
which is the bilinear form of the model equation for shallow water waves (R. Hirota and J. Satsuma, J. Phys. Soc. Japan. 40 (1976) 611),

\[ u_x - u_{xxx} - 3uu_t + 3u_x \int_{\pi}^x u_t \, dx' + u_x = 0, \]

where

\[ u = 2(\log f)_{xx}. \]

Suggested by these facts, I found that the solution of Mikhailov's equation is expressed with the solution of eq. *) (Nov. 14, 1979)

\[ u = \log(1 - S), \]

\[ S = 2(\log h)_x, \]

where \( u \) and \( S \) are the solutions to

\[ u_{xt} = e^u - e^{-2u}, \quad \text{Mikhailov's equation,} \]

\[ S_{xxx} + 3S_x S_t - 3S_x = 0 \]

or \( D_x^2(D_x^2 - 3)h \cdot h = 0 \), Shallow water wave eq., respectively.

Furthermore, the inverse scattering form for Kaup's equation (for \( \lambda = 0 \), namely

\[ D_x^3 f'' \cdot f + 3D_x g'' \cdot f = 0, \]

\[ D_x^2 f'' \cdot f = g' f, \]

\[ D_t f'' \cdot f = -\frac{3}{8} D_x^5 f'' \cdot f + \frac{15}{8} D_x^3 g'' \cdot f, \]

is satisfied by
\[ f' = h^2, \quad g' = -D_x^2 h \cdot h, \quad f = h^2 - D_x D_t h \cdot h, \]

provided that \( h \) satisfies

\[ D_x (D_t + D_x^5) h \cdot h = 0, \quad \text{Sawada - Kortewa's eq.} \]

Hence, \( N \)-Soliton solutions to Kaup's and Mikhailov's equation are expressed with \( h \):

\[ u = \frac{1}{2} (\log f)_{xx}, \quad f = h^2 - D_x D_t h \cdot h \]

and

\[ u = \log (1 - S_t), \quad S_t = 2(\log h)_{xt}, \]

respectively, where

\[ h = \sum_{\mu = 0, 1} \exp \left\{ \sum_{i=1}^{N} \mu_i \eta_i + \sum_{i>j}^{(N)} \beta_{ij} \mu_i \mu_j \right\} \]

\[ \exp (\beta_{ij}) = \frac{(p_i - p_j)^2(p_i^2 + p_i p_j + p_j^2)}{(p_i + p_j)^2(p_i^2 + p_i p_j + p_j^2)}, \]

\[ \eta_i = p_i x + \Omega_i t + w_i \tau + i \pi + \eta_i^0. \]

\[ \Omega_i + p_i^5 = 0, \quad \omega_i p_i = 3 \quad \text{for} \quad i = 1, 2, \ldots, N. \]