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On Forming a Series-parallel Graph by Removing Nodes of a Planar Graph

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1. Introduction.

Let P_{SPNR} denote the problem of determining, for a graph $G=(V,E)$ and a nonnegative integer k, whether or not a series-parallel graph can be formed by removing k or fewer nodes of G, where V and E are the sets of nodes and of edges of G, respectively. The authors have already proved in [15] that P_{SPNR} is NP-complete. In this paper, we show that P_{SPNR} remains NP-complete even when the domain is restricted to planar graphs. From now on, let P_{PSPNR} denote P_{SPNR} with the constraint that G is planar.

Technical terms and notations not specified in this paper can be identified in [2],[7],[8] and [9].

In stead of a thorough description of the formal requirements of a proof of NP-completeness, we describe the two steps (i) and (ii) required in proving that a particular problem P_X is NP-complete.

(i) Prove that $P_X \in \mathcal{NP}$, the class of problems that can be solved in polynomial time by a nondeterministic Turing machine.

(ii) Prove that some known NP-complete problem $P_{X'}$ can be polynomially transformed into P_X ($P_{X'} \propto P_X$ for short), in such a way that any polynomial-time algorithm for solving P_X could be used to solve $P_{X'}$ in polynomial time.

We will omit verification of (i) from our proofs since there exists an algorithm that determines in polynomial time whether or not a given graph is series-parallel (For example, see [3] or [12]). Thus our proofs will focus on the transformation required by (ii).

2. Preliminaries.

Let $G=(V,E)$ and $X \subset V$. Let us denote

$$E(X)=\{(u,v) \in E: \{u,v\} \cap X \neq \phi\}$$

for $X=\{v_{i_1},\ldots, v_{i_x}\}$ ($x=|X|$). We call

$$G-X=(V-X, E-E(X))$$

the graph formed by removing nodes $v_{i_1},\ldots, v_{i_x}$ (or removing X) of G. Let us denote

$$\delta(G)=\mathrm{Max}\{\delta_G(v): v \in V\},$$

where $\delta_G(v)$ denotes the node degree of v in G. Let

$$P(u,v)=(V_{P(u,v)},E_{P(u,v)})$$

denote a path of length $|E_{P(u,v)}| \geq 2$, a subgraph of the graph G in consideration, connecting two nodes u and v of G. P(u,v) is said to be a disjoint path if and only if it satisfies the condition that no node in $V_{P(u,v)}-\{u,v\}$ is (and shall be) contained in any other path except P(u,v) (That is, any node of P(u,v) except u and v is of node degree 2 in any graph in consideration).

For two nodes u and v, let

$$L(u,v)=(V_{L(u,v)},E_{L(u,v)})$$

denote the graph which consists of $(\alpha+1)$ disjoint paths of length 2 each of which connects u and v, where

$$V_{L(u,v)}=\{b_i(u,v): i=0,\ldots,\alpha\} \cup \{u,v\}$$

and

$$E_{L(u,v)} = \{ (u,b_i(u,v)),\ (v,b_i(u,v)) : i=0,\ldots,\alpha \}.$$

L(u,v) is said to be a band of width α connecting u and v.

Remark 1.

Let L(u,v) be a band of width α connecting u and v. Then there exists at least one disjoint path even if α or fewer nodes of L(u,v) except u and v are removed. If we assume $\alpha \geq 2$, then $L(u,v)-\{u\}$ ($L(u,v)-\{v\}$) contains at least three blocks (maximal nonseparable components) each of which consists of a single edge having v (u) as a cutpoint.

A thinning is an operation that deletes one of two multiple edges e_1 and e_2. A shrinking is an operation that contracts one of two edges (u,v) and (v,w) satisfying that v is of node degree 2. A reduction is to repeat a thinning and/or a shrinking a number of times. A graph G is said to be series-parallel (s-p for short) if and only if there exists a reduction that leads G to a single edge. A well-known characterization of s-p graphs has been given by R. J. Duffin in [3].

Theorem 1 [3].

Let G be nonseparable. Then G is s-p if and only if G has no subgraph from which K_4 can be formed by contracting and/or deleting a number of edges.

If G has such a subgraph as mentioned in Theorem 1 then we say that G has a subgraph reducible to K_4 ($G \supset K_4$ for short).

We now turn our attention to the node cover problem. A node cover for G=(V,E) is a subset S of V such that any edge of G is incident upon some node of S. Given a graph G=(V,E) and a non-negative integer k, let P_{NC}, called the node cover problem, denote the problem of determining whether or not G has a node cover s of size $|S| \leq k$ (We can assume that G is a simple graph). It is well-known that P_{NC} is NP-complete (For example, see [9]).

In this paper we state all lemmas without proofs which are given in [16].

Lemma 1.

P_{C-QPNC}, the node cover problem for simple planar connected cubic graphs, is NP-complete.

3. Forming a series-parallel graph by removing nodes of a planar graph.

Throughout this section let k be nonnegative integer and $G=(V,E)$ be a cubic planar connected graph, where

$V=\{v_1,\ldots, v_n\}$ ($n=|V|$) and $E=\{e_1,\ldots e_m\}$ ($m=|E|$).

Represent G on a plane and fix this representation. Let

$F_G=\{f_0, f_1,\ldots, f_{r-1}\}$ (f_0 is the infinite face of G)

denote the set of all faces of G. We can assume that

$n \geq 4$, $m \geq 6$ and $r \geq 4$.

Beginning with the fixed representation of G we construct the planar graphs $G_i=(V_i,E_i)$ (i=1,2) and $G_1'=(V_1',E_1')$ by the following procedures (1), (2), (3) and (4).

(1) Construct the geometric dual $G^*=(V^*,E^*)$ of G, and then determine a spanning tree $T^*=(V_{T^*},E_{T^*})$ of G^*. There exists one to one correspondence between E and E^*. We denote this correspondence by $e^* \in E^*$ for $e \in E$. Let

$E_T=\{e \in E: e^* \in E_{T^*}\}$.

Then $|E_T|=r-1$.

(2) Let G_1 be the graph obtained by replacing each edge $e=(u,v) \in E_T$ with a disjoint path $P(u,v)=(V_{P(u,v)},E_{P(u,v)})$ of length 4. P(u,v) is called the T-path for e. Let us denote

$V_{P(u,v)} = \{v_0=u,\ v_1,\ v_2,\ v_3,\ v_4=v\}$, and

$E_{P(u,v)} = \{(v_i, v_{i+1}) : i=0,\dots,3\}$

(hereafter $V_{P(u,v)}$ and $E_{P(u,v)}$ are used in this sense). Let

$$A_1 = \bigcup_{(u,v)\in E_T} V_{P_a(u,v)},$$

where $V_{P_a(u,v)} = V_{P(u,v)} - \{u,v\}$.

Then $V_1 = V \oplus A_1$, where $\oplus$ denotes the disjoint union. Let F_{G_1} denote the set of all faces of G_1. Then there exists one to one correspondence between F_G and F_{G_1}, and we denote this correspondence by $f^{(1)} \in F_{G_1}$ for $f \in F_G$.

(3) For each edge $e=(u,v) \in E$, execute the following procedures (i), (ii) and (iii). Let f_i and $f_j \in F_G$ be the two faces (not necessarily distinct) that e touches. If e is a bridge of G then e* is a loop of G*. Since $e^* \notin E_{T^*}$, we have $e \notin E_T$. That is, if $e \in E_T$ then e is not a bridge of G.

(i) The case when $e \in E-E_T$.

(i-a) If e is not a bridge of G (then $f_i \neq f_j$), place two bands $L(u^i(e), v^i(e))$ and $L(u^j(e), v^j(e))$ of width $(k+r+2)$ in $f_i^{(1)}$ and $f_j^{(1)}$, respectively.

(i-b) If e is a bridge of G (then $f_i = f_j$), place two bands $L(u_a^i(e), v_a^i(e))$ and $L(u_b^i(e), v_b^i(e))$ of width $(k+r+2)$ in $f_i^{(1)}$, where we assume that one of them is located across e from the other.

(ii) The case when $e \in E_T$ (then $f_i = f_j$).

Let $V_{P(u,v)}$ be as before. Then place four bands $L(v_t^i(e), v_{t+1}^i(e))$ (i=0,1,2,3) of width $(k+r+2)$ in $f_i^{(1)}$, and similarly four bands $L(v_t^j(e), v_{t+1}^j(e))$ (j=0,1,2,3) of width (k+r+2) in $f_j^{(1)}$, where $v_0(e)=u^{\alpha}(e)$ and $v_4(e)=v^{\alpha}(e)$ for $\alpha=i,j$.

(iii) For each $u \in V$ and each pair of edges $e=(u,v)$ and $g=(u,w)$ ($e, g \in E$ and $v \neq w$), both of which are incident upon u, connect $u^i(e)$ and $u^i(g)$ in terms of a band of width (k+r+2) if and only if both e and g touch a face $f_i^{(1)} \in F_{G_1}$. If e (g) is a bridge of G, then we assume that $u^i(e)$ is set to either $u_a^i(e)$ or $u_b^i(e)$ (either $u_a^i(g)$ or $u_b^i(g)$).

Let G_1' denote the resulting graph. Let V_{C_i} denote the set of nodes placed in $f_i^{(1)} \in F_{G_1}$ in (i), (ii) and (iii) of (2), and let $C_i=(V_{C_i},E_{C_i})$ denote the subgraph of G_1' induced by V_{C_i}. C_i can be changed into a single circuit if we replace each band of C_i with a disjoint path of length 2. C_i is said to be the inner ring $IR(f_i)$ of f_i (or of $f_i^{(1)}$).

(4) Construct a maximum matching $M \subset E$ of G (It is well-known that there exists a polynomial-time algorithm to obtain a maximum matching of a planar graph G (see [4])). And execute the following procedures (i), (ii) and (iii) with respect to M.

(i) For each edge $e=(u,v) \in E_T$, let f_i and f_j ($f_i \neq f_j$) be two faces of G that e touches, where we can assume that one of them is a finite face. Then, for each $v_t \in V_{P_a(u,v)}$ (t=1,2,3), connect two pairs of nodes $\{ v_t, v_t^i(e) \}$ and $\{ v_t, v_t^j(e) \}$ by edges $(v_t, v_t^i(e))$ and $(v_t, v_t^j(e))$, respectively. A path of length 2 defined by each pair of these edges is called the c-path. Then we say that $IR(f_i)$ and $IR(f_j)$ are T-connected. e is said to be the c-edge of these two inner rings, and then these two inner rings are said to be the c-supporters of e.

(ii) For each edge $e=(u,v) \in M$, choose arbitrarily a face of G that e touches. Let, say, $f_i \in F_G$ denote this face. Let $g=(u,x)$ and $h=(v,y) \in E-M$ ($\{x,y\} \cap \{u,v\} = \emptyset$) denote the two edges touching f_i. Then connect four pairs of nodes $\{u,u^i(e)\}$, $\{u,u^i(g)\}$, $\{v,v^i(e)\}$ and $\{v,v^i(h)\}$ by edges $(u,u^i(e))$, $(u,u^i(g))$ $(v,v^i(e))$ and $(v,v^i(h))$, respectively. e is said to be the b-edge of $IR(f_i)$, and $IR(f_i)$ to be the supporter of e.

(iii) For each $v \in V-M(V)$ ($M(V)=\{u,v \in V: (u,v) \in M\}$), choose arbitrarily an edge of E incident upon v and choose again arbitrarily a face of G that e touches. Let, say, $f_i \in F_G$ denote this face. Let $g=(v,x)$ and $h=(v,y) \in E$ ($x \neq y$) denote the two edges touching f_i, both of which are incident upon v. Then connect two pairs of nodes $\{v,v^i(g)\}$ and $\{v,v^i(h)\}$ by edges $(v,v^i(g))$ and $(v,v^i(h))$, respectively. v is said to be the b-node of $IR(f_i)$, and $IR(f_i)$ be the supporter of v.

Let G_2 denote the graph constructed by applying the procedure (4) to G_1'. Since it is well-known that G*, T* and M can be obtained in polynomial time, G_2 can be constructed in polynomial time when G is given as input. Note that G_2 is a non-separable planar graph. Figure 1 shows an example of the graph transformation from G to G_2.

Let v be the node mentioned in (iii) of (4). Each $v \in V$ is connected to exactly one inner ring C_i in terms of a pair of edges (v,v') and (v,v'') ($v',v'' \in V_{C_i}$) both of which are introduced in either (ii) or (iii) of (4). Hence $\delta_{G_2}(v)=5$. Let $\Delta(v)$ denote the subgraph of G_2 called the triangle of C_i (or of v), where it is defined by nodes v, v' and v", edges (v,v') and (v,v"), and $E_{L(v',v")}$. A pair of edges (v,v') and (v,v") are called the Δ -edges of v. We say that v is connected to C_i in terms of $\Delta(v)$.

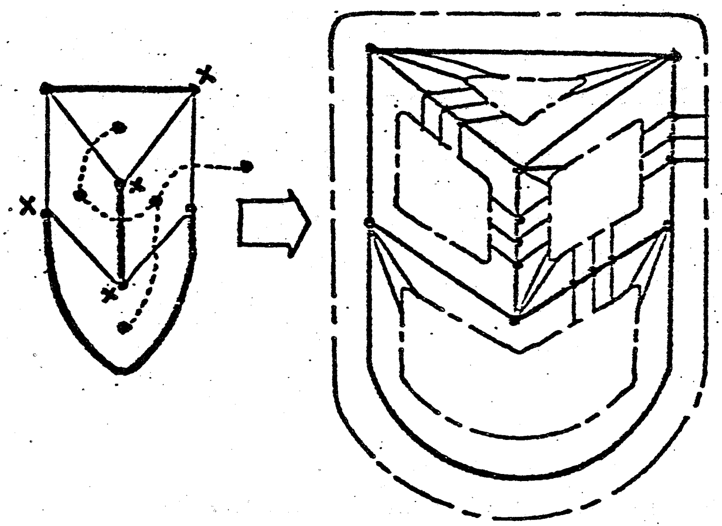

Figure 1. An example of the graph transformation from G to G_2 (In the figure ▬▬ and ——, respectively, denote an element of M and an inner ring, and both •——• and •---• edges).

Lemma 2.

G has a node cover of size less than or equal to k if and only if there exists a subset $X \subset V_2$ with $|X| \leq k+r-1$ such that G_2-X is an s-p graph.

Proof. Let $N \subset V$ be a node cover for G with $|N| \leq k$. Define $X_e \subset V_1$ for each $e=(u,v) \in E$ as follows:

$$X_e = \begin{cases} \phi & \text{when } e \notin E_T, \\ \{w\} & \text{when } e \in E_T \text{ and } |\{u,v\} \cap N|=1, \\ \{v_2\} & \text{when } e \in E_T \text{ and } \{u,v\} \subset N, \end{cases}$$

where if $\{u,v\} \cap N=\{u\}$ ($\{v\}$), then $w=v_3$ ($w=v_1$) for $V_{P(u,v)}= \{ v_0=u, v_1, v_2, v_3, v_4=v \}$. Let

$$X_1= \bigcup_{e \in E} X_e \quad (= \bigcup_{e \in E_T} X_e)$$

and

$X = N \cup X_1$.

Note that $N \cap X_1 = \phi$. Then we have $|X| \leq k+r-1$, since $|E_T| = r-1$. It is easy to see that $G_2 - X$ is an s-p graph (see Figure 2).

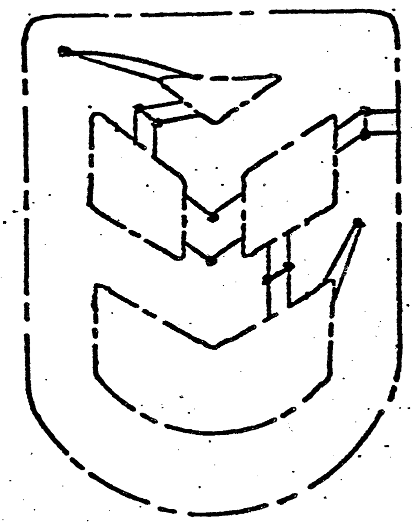

Figure 2. $G_2 - X$ for G_2 of Figure 1 (The node cover N that determines X consists of those nodes marked x in Figure 1).

Conversely, assume that an s-p graph can be formed by removing (k+r-1) or fewer nodes of G_2. Let $X \subset V_2$ be a minimal set satisfying that $G_2 - X$ is s-p and which maximizes, among all such sets, the number of nodes contained in V. By the construction of G_2 (in which any band of each inner ring is of width (k+r+2)), by Remark 1 and by the minimality of X, we can assume that

$X \subset V_1$.

Let

$N = X \cap V$

and let P(u,v) be the T-path for arbitrary edge $e=(u,v) \in E_T$. Then

$|X \cap \{v_1, v_2, v_3\}| \geq 1$

for $V_{P_a(u,v)} = \{v_1, v_2, v_3\} \subset V_{P(u,v)}$, since otherwise $G_2 - X \supset K_4$.

Therefore, since we have $|E_T| = r-1$,

$|N| \leq k$.

Let $G_3 = (V_3, E_3)$ denote $G_2 - X$ for simplicity.

Lemma 10.

G_3 satisfies the following (1) and (2).

(1) $\delta_{G_3}(v)=2$ for $\forall v \in V \cap V_3$.

(2) C_i and C_j are connected only in terms of exactly two c-paths (including the case when there exists an edge connecting these two c-paths) for any pair of distinct inner rings C_i and C_j that are T-connected in G_2.

By Lemma 10, we can prove that N is a node cover for G.

Q.E.D.

Lemma 2 and the fact that G_2 can be constructed in time polynomial in the size of input establish the main theorem.

Theorem 2.

P_{PSPNR} is NP-complete.

We will prove Lemma 10 by way of Lemma 3 through Lemma 9 in the rest of this section.

We have $\delta_{G_3}(v) \geq 2$ for $\forall v \in V_3$. And if $v \in V \cap V_3$ then there exist $\Delta(v)$ and the inner ring to which v is connected in terms of $\Delta(v)$.

By the construction of G_2, any two distinct inner rings C_i and C_j have no node in common. Thus G_3 has a path of length not less than 2 connecting C_i and C_j since $X \subset V_1$ and since G_3 is connected. Two blocks B and B' having a cutpoint u in common are said to be adjacent to each other at u.

Lemma 3.

Any block B of G_3 has at most two other blocks each of which is adjacent to B.

Let

$K_\beta = (V_{K_\beta}, E_{K_\beta})$

denote the graph shown in Figure 3, where

$V_{K_\beta} = \{u, v, w_1, w_2\}$, and

$E_{K_\beta} = \{(u,w_i), (v,w_i): i=1,2\} \cup \{(w_1,w_2)\}$.

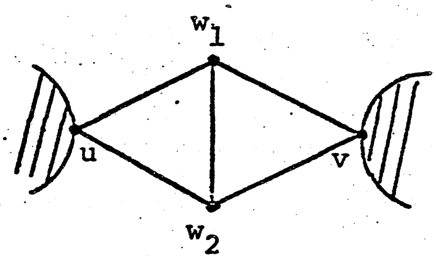

Figure 3. K_β .

Observe that a graph H is not s-p if K_β is a subgraph of H having both u and v as cutpoints of H (any reduction can not be applied to K_β since all of u, v, w_1 and w_2 are of node degree$\geq$3). Let

$\widetilde{K}_\beta = (V_{\widetilde{K}_\beta}, E_{\widetilde{K}_\beta})$

denote the graph obtained by replacing each edge $(w,w') \in E_{K_\beta}$ with a disjoint path of length not less than 1 (that is, $\widetilde{K}_\beta$ is a graph homeomorphic to K_β). We say that a graph H has a pair of K_β-cutpoints if and only if H has a subgraph isomorphic to $\widetilde{K}_\beta$ having both u and v as cutpoints of H.

Lemma 4 .

G_3 has no block which contains a pair of K_β-cutpoints.

Lemma 5 .

Assume that G_3 has a triangle $\Delta(u)$ ($u \in V$). If $\delta_{G_3}(u) \geq 3$ then u is a cutpoint of G_3.

Lemma 6.

Assume that G_3 has a triangle $\Delta(u)$ ($u \in V$). Let C_i be the inner ring to which u is connected in terms of $\Delta(u)$, $\delta_{G_3}(u) \geq 3$ and $B(C_i)$ be the block of G_3 containing C_i. Then $B(C_i)$ has no other cutpoint of G_3 except u.

Let $G_{A_1}=(V_{A_1},E_{A_1})$ ($V_{A_1}=A_1$) denote the subgraph of G_1 induced by A_1, and let $\overline{E}_1=E_1-E_{A_1}$. Then let

$$IR(G_2)=(V_{IR(G_2)},E_{IR(G_2)}),$$

called the inner ring tree, denote the subgraph of G_2 formed by deleting $\overline{E}_1$ from G_2, where $E_{IR(G_2)}=E_2-\overline{E}_1$. Let C and C' be a pair of inner rings that are T-connected in G_2. We say that C and C' are π -connected, $C\pi C'$ for short, if and only if they are connected only in terms of exactly two c-paths (including the case when there exists an edge connecting these two c-paths). For any two distinct inner rings C and C' of a block, a sequence of distinct inner rings $C_{i_0}=C$, $C_{i_1}, \ldots, C_{i_t}=C'$ ($t \geq 1$) of this block is called the π -connected sequence of C and C' if and only if $C_{i_j} \pi C_{i_{j+1}}$ for $j=0,\ldots, t-1$.

Let D denote the set of all cutpoints of G_3. Then, by the construction of G_2 and by the definition of X, we have

$$D \subset V_1=V \oplus A_1 .$$

Let us denote

$$D=D_V \oplus D_{A_1},$$

where $D_V=D \cap V$ and $D_{A_1}=D \cap A_1$. Then, by Lemmas 3, 5 and 6,

$$|D_V| \leq 2.$$

Let

$$X^{(1)}=X \cup D_V \quad \text{and} \quad G_3^{(1)}=G_2-X^{(1)} \ (=G_3-D_V).$$

Note that $G_3^{(1)}$ is not always connected. Let $B_1, B_2, \ldots$ denote the blocks of $G_3^{(1)}$, and let

$$\mathcal{B}^{(1)}=\{B_1, B_2, \ldots \} .$$

Hereafter, let us denote

$$G_3^{(i)}=(V_3^{(i)}, E_3^{(i)}) \text{ for } i=1,2 \text{ and } 3.$$

Lemma 7.

$G_3^{(1)}$ satisfies the following (1), (2) and (3).

(1) $\delta_{G_3^{(1)}}(v)=2$ for $^\forall v \in V \cap V_3^{(1)}$.

(2) $D_{A_1} \subset D^{(1)} \subset A_1$, where $D^{(1)}$ is the set of all cutpoints of $G_3^{(1)}$.

(3) $\delta_{G_3^{(1)}}(u) \geq 2$ for $^\forall u \in V_3^{(1)}$.

Lemma 8 .

For any pair of distinct inner rings C and C' of any block $B_i \in \mathcal{B}^{(1)}$, there exists the π -connected sequence of C and C'.

Let

$$D^{(1)}= \{ a_1, \ldots, a_\theta \} \ (\theta = |D^{(1)}|),$$

where the order of elements of $D^{(1)}$, denoted by the subscript, can arbitrarily fixed. Let P(u,v) denote the T-path containing $a_1 \in D^{(1)}$. Then, clearly,

$$a_1 \in V_{P_a(u,v)}=\{v_1,v_2,v_3\} \subset V_{P(u,v)}=\{u=v_0,v_1,v_2,v_3,v=v_4\}.$$

We define $X_{a_1} \subset V_1$ for the following two cases. Note that, for each $a_i \in D^{(1)}$, there exists exactly one T-path containing a_i and that if $a_i \neq a_j$ ($a_i, a_j \in D^{(1)}$) then they are contained in distinct T-paths.

(i) The case when $a_1=v_1$ (respectively when $a_1=v_3$).

By the definition of $G_3^{(1)}$,

$$\{ v_0,v_2,v_3 \} \subset X^{(1)} \quad (\{v_1,v_2,v_4\} \subset X^{(1)}).$$

Then let

$$X_{a_1}=X^{(1)}-\{v_2\}.$$

(ii) The case when $a_1=v_2$.

By the definition of $G_3^{(1)}$,

$$\{v_1,v_3\} \subset X^{(1)}.$$

Note that if $v_0 \notin X^{(1)}$ (if $v_4 \notin X^{(1)}$) then $\delta_{G_3^{(1)}}(v_0)=2$ ($\delta_{G_3^{(1)}}(v_4)=2$) (that is, v_0 (v_4) is connected to some inner ring only in terms of $\Delta(v_0)$ ($\Delta(v_4)$).

Then let

$$X_{a_1}=\begin{cases} X^{(1)}-\{v_1\} & \text{when } v_0\in X^{(1)} \text{ and } v_4\notin X^{(1)}, \\ X^{(1)}-\{v_3\} & \text{when } v_0\notin X^{(1)} \text{ and } v_4\in X^{(1)}, \\ (X^{(1)}-\{v_1\})\cup\{v_0\} & \text{when } \{v_0,v_4\}\cap X^{(1)}=\phi, \\ X^{(1)}-\{v_1\} & \text{when } \{v_0,v_4\}\subset X^{(1)}. \end{cases}$$

Set

$$X_{a_0}=X^{(1)}$$

and, for each $j=2,\ldots,\theta$, define recursively

$$X_{a_j}\subset V_1,$$

similarily to the definition of X_{a_1}, by replacing $X_{a_{j-2}}$ and $X_{a_{j-1}}$ with $X_{a_{j-1}}$ and X_{a_j}, respectively. Clearly, for each $j=1,\ldots,\theta$, we have either

$$|X_{a_{j-1}}| > |X_{a_j}|$$

or

$$|X_{a_{j-1}}\cap V| < |X_{a_j}\cap V| \quad \text{with } |X_{a_{j-1}}| = |X_{a_j}|.$$

Set

$$X^{(2)}=X_{a_\theta} \quad \text{and} \quad G_3^{(2)}=G_2-X^{(2)}.$$

Then by the definition of $X^{(2)}$ and by Lemma 8 we have the next lemma.

Lemma 9.

$G_3^{(2)}$ satisfies the following (1), (2) and (3).

(1) Any maximal connected component of $G_3^{(2)}$ is a block of $G_3^{(2)}$.

(2) For any pair of distinct inner rings C and C' of any block of $G_3^{(2)}$, there exists the π-connected sequence of C and C'.

(3) $\delta_{G_3^{(2)}}(v)=2$ for $\forall v \in V \cap V_3^{(2)}$.

Lemma 9 shows that $G_3^{(2)}$ is a subgraph of the inner ring tree $IR(G_2)$.

For $w \in D_V$, let $B(w)$ denote the block of G_3 containing $\Delta(w)$, B be a block distinct from $B(w)$ and adjacent to $B(w)$ having w as a cutpoint. Let $(w,w^{(i_1)})$ and $(w,w^{(i_2)})$ be the two Δ-edges. Then we can show that any edge incident upon w and distinct from these two Δ-edges is contained in B. Therefore, $G_3^{(1)}$ consists of $|D_V|$ maximal connected components $B^{(1)}(w)=B(w)-\{w\}$ for $w \in D_V$ and another one C_Z which is determined by contracting $B(w)$ to w for $\forall w \in D_V$ in G_3. Hence, by Lemma 9, $G_3^{(2)}$ consists of ($|D_V|$ +1) blocks (or equivalently the same number of maximal connected components). That is, each $B^{(1)}(w)$ of $G_3^{(1)}$ is itself a block $B^{(2)}(w)$ of $G_3^{(2)}$ by Lemmas 6, 7 and 8, and C_Z of $G_3^{(1)}$ is changed into another block $B^{(2)}$ of $G_3^{(2)}$ in the process constructing $G_3^{(2)}$.

Now let us assume that $|D_V| \geq 1$ (In the case when $D_V=\emptyset$, we omit the procedure to construct $X^{(3)}$ described in the following, and set $X^{(3)}=X^{(2)}$ and $G_3^{(3)}=G_3^{(2)}$). Let

$$\mathcal{B}^{(2)}=\{B^{(2)}(w) : w \in D_V\} \cup \{B^{(2)}\} = \{B_1, \ldots, B_{|D_V|+1}\}.$$

Let $B_i \in \mathcal{B}^{(2)}$ be arbitrarily chosen. By the definition of T* or of $IR(G_2)$, there exist some $B_j \in \mathcal{B}^{(2)} - \{B_i\}$, some inner ring C_i of B_i and some inner ring C_j of B_j such that C_i and C_j are T-connected in G_3. Then we say that B_j is a T-connected block of B_i and vice versa. Let $e=(u,v) \in E_T$ be the c-edge of C_i and C_j, and $P(u,v)$ be the corresponding T-path for e. Then

$$V_{P_a(u,v)} = \{v_1, v_2, v_3\} \subset X^{(2)},$$

where we set $u=v_0$ and $v=v_4$. Note that the following discussion does not depend on whether or not $D_V \cap \{u,v\}$ is empty. Define $X_{ij} \subset V_1$ as follows:

$$X_{ij} = \begin{cases} (X^{(2)} - \{v_1,v_2\}) \cup \{u\} & \text{when } \{u,v\} \cap X^{(2)} = \emptyset, \\ X^{(2)} - \{v_1,v_2\} & \text{when } \{u,v\} \cap X^{(2)} = \{u\}, \\ X^{(2)} - \{v_2,v_3\} & \text{when } \{u,v\} \cap X^{(2)} = \{v\}, \text{ and} \\ X^{(2)} - \{v_1,v_2\} & \text{when } \{u,v\} \subset X^{(2)}. \end{cases}$$

In the case when $|D_V| = 1$ define X_{ij} for $w \in D_V$ by setting $B_i = B^{(2)}(w)$. In the case when $|D_V| = 2$, first define X_{ij} for $w \in D_V$ by setting $B_i = B^{(2)}(w)$. Set $B_s = B^{(2)}(w')$ for $w' \in D_V - \{w\}$ and let $B_t \in \mathcal{B}^{(2)}$ be a T-connected block of B_s. Then define X_{st}, similarly to the definition of X_{ij}, by replacing X_{ij} and $X^{(2)}$ with X_{st} and X_{ij}, respectively. Now we set

$$X^{(3)} = \begin{cases} X_{ij} & \text{when } |D_V| = 1, \text{ and} \\ X_{st} & \text{when } |D_V| = 2, \end{cases}$$

and clarify the relation between X and $X^{(3)}$. First,

$$|X_{ij}| \le |X^{(2)}| - 1 \quad \text{and} \quad |X_{st}| \le |X_{ij}| - 1,$$

so that

$$|X^{(3)}| \le |X^{(2)}| - |D_V|,$$

where $|D_V| \le 2$. Secondly,

$$|X| = |X^{(1)}| - |D_V| \text{ and } |X^{(1)}| \geq |X^{(2)}| ,$$

so that

$$|X| \geq |X^{(3)}| .$$

We have

$$|X| = |X^{(3)}|$$

only if we have the following (i) and (ii):

(i) $|X^{(1)}| = |X^{(2)}|$.

(ii) $\{u,v\} \cap X^{(2)} = \phi$ and $\{u,v\} \cap X_{ij} = \phi$, respectively, for the c-edges $e=(u,v)$ in the definition of X_{ij} and of X_{st}.

If

$$|X^{(1)}| = |X^{(2)}|$$

then

$$|X^{(1)} \cap V| < |X^{(2)} \cap V|$$

as mentioned earlier. And (ii) implies that we have

$$|X_{ij} \cap V| = |X^{(2)} \cap V| +1 \text{ and } |X_{st} \cap V| = |X_{ij} \cap V| +1,$$

so that

$$|X^{(3)} \cap V| = |X^{(2)} \cap V| + |D_V| .$$

By the definition of $X^{(1)}$, we have

$$|X^{(1)} \cap V| = |X \cap V| + |D_V| .$$

Therefore, if

$$|X| = |X^{(3)}|$$

then

$$|X \cap V| < |X^{(3)} \cap V| .$$

Let

$$G_3^{(3)} = G_2 - X^{(3)} .$$

Then, clearly, we have the following statements (a) and (b).

(a) $G_3^{(3)}$ is a nonseparable subgraph of the inner ring tree $IR(G_2)$ and satisfies the following (i) and (ii):

(i) $\delta_{G_3^{(3)}}(v)=2$ for $\forall v \in V_3^{(3)} \cap V$.

(ii) For any pair of distinct inner rings C and C' of G_2, there exists the π-connected sequence of C and C' in $G_3^{(3)}$.

(b) $|X| \geq |X^{(3)}|$, where if $|X| = |X^{(3)}|$ then $|X \cap V| < |X^{(3)} \cap V|$.

$G_3^{(3)}$ is essentially the same form as the G_2-X constructed in the first half of the proof of Lemma 2, and it is easy to see that $G_3^{(3)}$ is s-p. Now the discussion so far is summarized as follows.

The proof of Lemma 10. Assume that either (1) or (2) of the lemma is false. Then we can define $X^{(3)} \subset V_1$ which is distinct from X and which satisfies the above statements (a) and (b), contradicting our choice of X. Q.E.D.

4. Concluding remark.

The problems P_{PSPNR} and P_{SPNR} discussed in this paper and in [15], respectively, are not included in the class of problems with properties that are hereditary on induced subgraphs, while a large number of node-deletion NP-complete problems with hereditary properties have been presented by M. S. Krishnamoorthy and N. Deo [11], and by M. Yannakakis [17, 18].

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References.

[1] A. V. Aho, J. E. Hopcroft and J. D. Ullman, The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, Mass., 1974.

[2] C. Berge, Graphs and Hypergraphs, North-Holland, London, G. B., 1973.

[3] R. J. Duffin, Topology of series parallel networks, J. Math. and Appl., 10, 1965, 303-318.

[4] J. Edmonds, Paths, trees and flowers, Canad. J. Math., 17, 1965, 449-467.

[5] M. R. Garey, D. S. Johnson and L. J. Stockmeyer, Some simplified NP-complete graph problems, Theoret. Comput. Sci., 1, 1976, 237-267.

[6] M. R. Garey and D. S. Johnson, The rectilinear steiner tree problem is NP-complete, SIAM J. Appl. Math., 32, 4, 1977, 826-834.

[7] ————————, Computers and Intractability: A Guide to the Theory of NP-completeness, H. Freeman and Sons., San Francisco, 1978.

[8] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass., 1969.

[9] R. M. Karp, Reducibility among combinatorial problems, Complexity of Computer Computations, R. E. Miller and J. W. Thatcher, eds., Plenum Press, New York, 1972, 85-104.

[10] ——————, On the computational complexity of combinatorial problems, Networks, 5, 1975, 45-68.

[11] M. S. Krishnamoorthy and N. Deo, Node-deletion NP-complete

problems, SIAM J. Comput., 8, 4, 1979, 619-625.

[12] T. Nishizeki, K. Takamizawa and N. Saito, Algorithms for series-parallel graphs and D-charts, IECE of Japan Trans., J59-A, 3, 1976, 259-260 (in Japanese).

[13] T. Watanabe, T. Ae and A. Nakamura, On the node cover problem of planar graphs, Proc. of 1979 IEEE Int. Symp. on Circuits and Systems, Tokyo, Japan, 1979, 78-81.

[14] ——————————, On the contraction of nonplanar edges of a graph, Tech. Rep. No. C-1, Appl. Math. Dept., Faculty of Eng., Hiroshima Univ., Hiroshima, Japan, July 1979 (or Tech. Rep. IECE of Japan, AL79-34, July 1979 (in Japanese)).

[15] ——————————, On forming a series-parallel graph by edge-contraction, edge-deletion or node-removal, Tech. Rep. No. C-4, Appl. Math. Dept., Faculty of Eng., Hiroshima Univ., Hiroshima, Japan, September 1979 (or Tech. Rep. IECE of Japan, AL79-42, September 1979 (in Japanese)).

[16] ——————————, On forming a series-parallel graph by removing nodes of a planar graph, Tech. Rep. No. C-7, Appl. Math. Dept., Faculty of Eng., Hiroshima Univ., Hiroshima, Japan, February 1980 (or Tech. Rep. IECE of Japan, AL79-63, November 1979 (in Japanese)).

[17] M. Yannakakis, Node- and edge-deletion NP-complete problems, Proc. 10th Ann. ACM Symp. on Theory of Computing, 1978, 253-264.

[18] ——————————, The effect of a connectivity requirement on the complexity of maximum subgraph problems, J. Assoc. Comput. Mach., 26, 4, 1979, 618-630.