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Series of Graphs Generated by Rational Machines
— A New Developmental System —

by H.Nishio, Department of Biophysics,
Faculty of Science, Kyoto University

Developmental process of a multi-cellular organism is considered as a series of graphs, whose nodes correspond to cells and edges to cellular interconnections. Thus far several authors have devised their developmental systems or graph generating systems. In this paper we introduce a new machinary, called rational machine, for generating series of finite directed labeled graphs. From the biological motivation and approach we took, each node of a generated graph is labeled with a string of symbols from a certain alphabet.

1. Definitions

Graph Γ on alphabet Σ is a directed graph whose nodes are labeled with strings on Σ and edges are defined by finite set of pairs of strings. That is, Γ is a finite subset of $\Sigma^* \times \Sigma^*$. The empty graph is the empty set of $\Sigma^* \times \Sigma^*$. Series of graphs $\mathbb{\Gamma}$ is a possibly infinite series $\mathbb{\Gamma} = \Gamma_0, \Gamma_1, \Gamma_2, \ldots$ of which Γ_i's are possibly empty graphs on Σ.

A rational machine (RM) on Σ is defined as

$$M = (Q, \Sigma, \nu, q_0, F)$$

where Q is the finite state set,

Σ is the finite set of symbols,

ν is a subset of $Q \times \Sigma_\varepsilon \times \Sigma_\varepsilon \times Q$ where $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$ and ε is the null string of Σ^*,

q_0 is the initial state of Q, and

F is a subset of Q called the accepting state set.

Thus a rational machine is a finite state transition system which can be nondeterministic and incomplete. Each state transition is labeled with a pair of symbols.

A <u>path</u> of length n from q_1 to q_{n+1} is defined as n consecutive transitions (q_1,x_1,y_1,q_2) (q_2,x_2,y_2,q_3) ... ... (q_n,x_n,y_n,q_{n+1}) where $(q_i,x_i,y_i,q_{i+1})\in\nu$ $(i=1,2,\ldots,n)$. The pair of strings $\alpha=x_1x_2\ldots x_n$ and $\beta=y_1y_2\ldots y_n$ is said to be defined by this path. α and β are called the domain and the range, respectively. When a path starts at the initial state q_0 and ends at an accepting state, then it is called an <u>accepted path</u>. The string pair defined by an accepted path is an accepted string pair.

Now denote by E_n the set of string pairs defined by all accepted paths of length n. Let D_n and R_n be the set of domains and one of ranges of string pairs of E_n, respectively. Then $N_n=D_n\cup R_n$ is a subset of Σ^* and $E_n\subset\Sigma^*\times\Sigma^*$ is a binary relation on N_n. In other words, $\Gamma_n=(N_n,E_n)$ is a finite directed graph whose nodes are N_n and edges are E_n. Γ_n is called the graph at time n.

In this way, a rational machine M <u>generates</u>, or defines uniquely, a series of graphs $\mathbb{\Gamma}=\Gamma_0,\Gamma_1,\Gamma_2,\ldots,\Gamma_n,\ldots$ where $\Gamma_0=(\{\varepsilon\},\{\varepsilon/\varepsilon\})$ by definition.

A series of graphs $\mathbb{\Gamma}$ is called <u>rational</u>, if there is a rational machine which generates it.

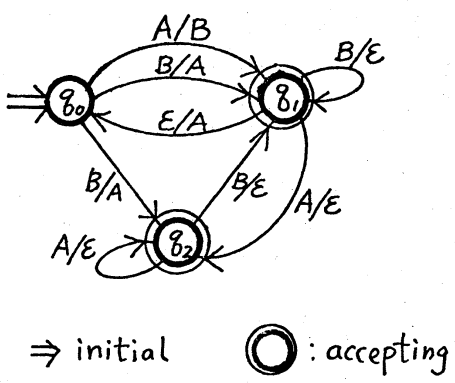


Fig. 1(a)

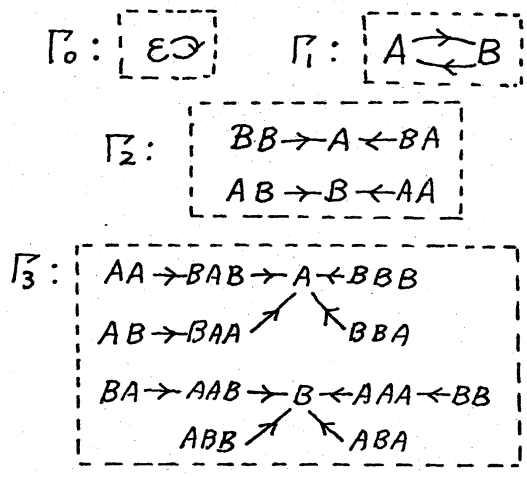


Fig.1 (b)

Example 1.1

Rational machine M illustrated in Fig.1(a), where a transition is denoted as $q \xrightarrow{X/Y} q'$, generates a series of graphs whose first four graphs are illustrated in Fig.1(b).

Remarks

(a) When we neglect the length of path in defining graphs generated by a RM and consider the set $R=\bigcup_{n=0}^{\infty} E_n$ of string pairs, we have a binary word relation R on Σ^*. We call R the relation defined by M. From our definition of RM, it is seen that the notion of rational relation thus defined is equivalent to that of "transduction" introduced by [Elgot and Mezei, 1965] and therefore that of "rational relation" defined by [Eilenberg, 1974].

(b) A natural generalization of our definition of RM is to define the transition relation ν as a finite subset of $Q \times \Sigma^* \times \Sigma^* \times Q$ rather than that of $Q \times \Sigma_\varepsilon \times \Sigma_\varepsilon \times Q$. This is nothing other than NDA of [Elgot and Mezei, 1965] as a machinary for defining word relations on Σ^*. As a machinary for defining series of graphs, this generalization might be significant, but we believe that our present definition is sufficient for studying essential points of this kind of developmental system.

2. Growth Functions

Let $\Gamma_n=(N_n,E_n)$ be the graph at time n of a certain graph series Γ. Now let

$$f(n)=\#(N_n) \quad \text{and} \quad h(n)=\#(E_n),$$

where #( ) is the number of elements of a set. f and h are called node and edge growth functions, respectively. Moreover we define the domain growth function $f_D(n)=\#(D_n)$ and the range growth function $f_R(n)=\#(R_n)$.

Then we have the following elementary propositions.

Proposition 2.1

$$\max(f_D(n),\ f_R(n)) \leq f(n) \leq f_D(n)+f_R(n)$$

Proof

Clear from $N_n = D_n \cup R_n$. ∎

Proposition 2.2

$$0 \leq f(n) \leq A \cdot C^n, \text{ where } A \text{ and } C \text{ are constants.}$$

Proof

By taking C as the maximum number of state transitions from every state, we have the upper bound. ∎

Proposition 2.3

$$(1/2)f \leq h \leq f_D \times f_R \leq f^2$$

Proof

This is a proposition which generally holds for a finite directed graph and is trivial. ∎

Remarks

There are RM's, for which equations $(1/2)f=h$ and $f^2=h$ holds respectively. In fact, RM illustrated in Fig.2(a) realizes the equation $(1/2)f=h$, with $f(n)=2^{n+1}$. The relation $f^2=h$, which means a complete graph, is realized by RM given in Fig.2(b). Moreover it is seen that there exist RM's for which equations $E_n = D_n \times R_n$ and a fortiori $h = f_D \times f_R$ hold.

Fig.2(a)

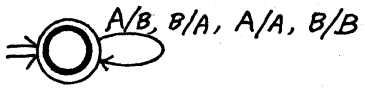


Fig.2(b)

Proposition 2.4

There is an algorithm which determines the order of growth of f_D and f_R for every rational machine. Especially it is decidable if a RM generates a graph series which grows indefinitely.

Owing to Propositions 2.1 and 2.4, we can estimate the growth order of f.

In order to prove the proposition, we need the definition of a new automaton.

A nondeterministic ε-automaton (ε-A) is an ordinary finite state nondeterministic incomplete automaton $A=(Q,\Sigma,\mu,q_0,F)$ which is allowed to have ε-transitions. That is, μ, the transition relation, is a subset of $Q\times\Sigma_\varepsilon\times Q$. As in the case of RM, a path of length n from q_1 to q_{n+1} is defined to be an n consecutive transitions (q_1,x_1,q_2) (q_2,x_2,q_3) $\ldots(q_n,x_n,q_{n+1})$ where $(q_i,x_i,q_{i+1})\in\mu$. The string $x_1x_2\ldots x_n$ is an accepted string in n steps, when there is a path of length n from q_0 to a state of F. Note that x_i can be ε. Let F_n be the set of accepted strings in n steps.

Then $\bigcup_{n=0}^{\infty} F_n$ is clearly a regular set.

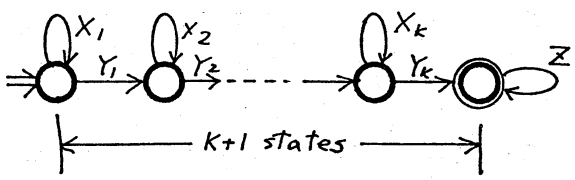
Now consider a ε-A $A_D=(Q,\Sigma,\mu,q_0,F)$ associated with a RM $M=(Q,\Sigma,\nu,q_0,F)$ which is defined as follows: $(q,x,q')\in\mu$ if and only if $(q,x,y,q')\in\nu$. A_D is called the <u>domain ε-automaton</u>. The range ε-automaton A_R is defined similarly. Clearly D_n and R_n are the sets of accepted strings in n steps by A_D and A_R, respectively. So, the domain $D=\bigcup D_n$ and the range $R=\bigcup R_n$ are regular sets.

<u>Proof of Proposition 2.4</u>

Since $f_D(n)=\#(D_n)$ and $f_R(n)=\#(R_n)$, it suffices to show an algorithm for determining the growth order of $\#(F_n)$ for a ε-automaton generally.

(i) If there is a state, which has at least two closed loops of transitions having different label strings, is reached from the initial state and can reach an accepted state, then the growth order is exponential.

(ii) If condition (i) fails and there is a one way chain of state transitions as illustrated in Fig.3, then the growth is of polynomial order.



$X_i \neq \varepsilon$

Fig. 3

(iii) If conditions (i) and (ii) fail, then the growth is of constant order. ∎

Corollary 2.1

If $\sup f(n)=\infty$, then $f(n) \geq C_1 n+C_2$, where C_1 and C_2 are constants.

This corollary means that if a graph series grows indefinitely, then the growth rate is faster than or equal to the linear order.

Remarks

(a) For ordinary finite state automaton, so having no ε-transition, it is known how to calculate the growth function $\#(F_n)$ precisely. In our case, the ε-transition causes the difficulty of evaluation of the growth function. Precise evaluation of $\#(F_n)$ is an open problem.

(b) For any k, the (k+1)-state ε-automaton illustrated in Fig.3 has the growth order of n^k.

(c) As to f and h, it is interesting to investigate behaviors of the ratio h/f. This can be a measure of complexity of graphs. It has not been known if there is an algorithm for deciding whether $\sup(h/f)=\infty$ or not.

(d) Besides growth functions f and h, growth rate of the number of edges emerging from nodes is also interesting. Define

$$d_n(x)=\#\left\{ y \mid (x,y) \in E_n \right\} \quad \text{for } x \in N_n$$

and

$$d(n)=\max_{x \in N_n} d_n(x)$$

d(n) is a function which indicates the maximum number of outgoing edges from nodes in graph Γ_n. It might be significant

to investigate the function d, especially sup d(n). There is a series of graphs where sup d(n) is not bounded, that is, the number of outgoing edges increases indefinitely.

Similarly we can think of the function e(n), which means the number of incoming edges.

3. Generative Power of Rational Machines

In this section we discuss how RM's are powerful in generating series of graphs.

Proposition 3.1

Every finite series of graphs on Σ $\mathbb{\Gamma} = \Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_k$, such that each node of Γ_h is labeled with a string shorter than n+1, can be generated by a RM.

Proof

Consider Γ_i and an edge e=(v,w) in Γ_i. Let $v=x_1x_2\ldots x_k$ and $w=y_1y_2\ldots y_h$ where x_i and y_j are elements of Σ and $0 \leq k,h \leq i$. Then construct a RM $M_{i,e}$ which define an edge e as illustrated in Fig.4.

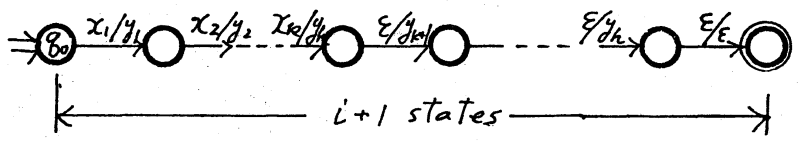


Fig. 4

By constructing RM's $M_{i,e}$ for every i and e and "taking union" of all $M_{i,e}$'s and considering all initial states as an identical initial state, we have a RM, which defines $\mathbb{\Gamma}$ as was wanted. As for "taking union", see Section 5. ∎

To minimize the number of states of RM which generates a finite graph series is an open problem.

Proposition 3.2

Every series of graphs, which is "isomorphic" to that of tree like structures generated by a bifurcating branching PDOL system, can be generated by a RM.

"Isomorphism" means here that two graphs are identical

besides labeling of nodes.

Remarks

PDOL system is the abbreviation of propagating deterministic interactionless Lindenmayer system. "Bifurcating" means that a symbol is rewritten by at most two symbols and "branching" does that the rewriting rule can have the type $a \rightarrow b(c)$ where (c) means a production of branch. For L system theory, see [Herman and Rozenberg, 1975] and [H.Nishio, 1978a].

Proof

Proof is described by showing an illustrative example. Generalization will be straightforward.

Consider the following PDOL system G as an example.

$G=(\{a,b,c,d,e\}, P, a)$

$P: \{ a \rightarrow bc,\ b \rightarrow b,\ c \rightarrow bd,\ d \rightarrow ed,\ e \rightarrow b(c) \}$

G generates a series of trees as illustrated in Fig.5.

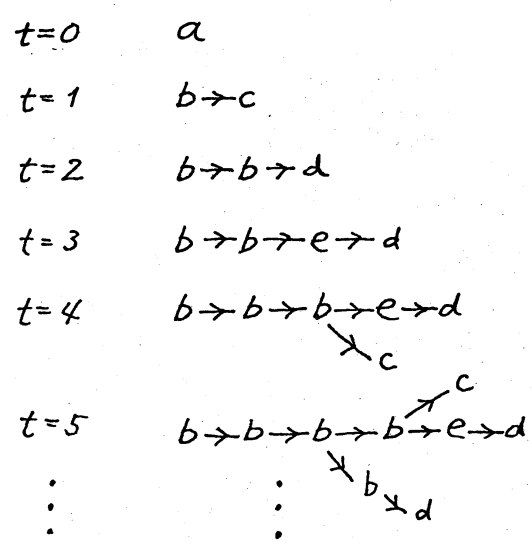


Fig. 5 Trees

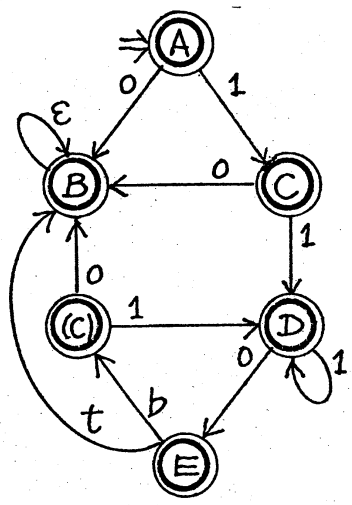


Fig. 6 S-Automaton M

In order to construct an equivalent RM, we use the alphabet $\Sigma=\{0,1,t,b\}$. From the rewriting rule P, we first construct the ε-automaton M called an <u>S-automaton</u>, as illustrated in Fig.6. S-automaton has been introduced with intention to define the series of sets of nodes N_n of graphs to be generated by RM. In constructing M, we consider that 0 corresponds to the left symbol and 1 the right symbol of each rewriting rule of the above P. For example, d is rewritten by ed. So the transition from state D to E has the label 0 and one from D to D the label 1 in Fig.6. e is rewritten by b and the branch (c). So, a transition from E to B has the label t, which means the "trunk" and one from E to (C) has the label b, which means the "branch". General construction of S-automaton from a given L system will be seen from this example.

From S-automaton M, we next construct a RM $M'=M\odot M$, which defines the complete connection among all nodes defined by M. Such a RM M' is called a <u>C-machine</u>. The construction is done by the semi-direct product method described in the proof of Proposition 1 of [Nishio, 1979b]. The relevant portion of C-machine M' for our example is illustrated in Fig.7.

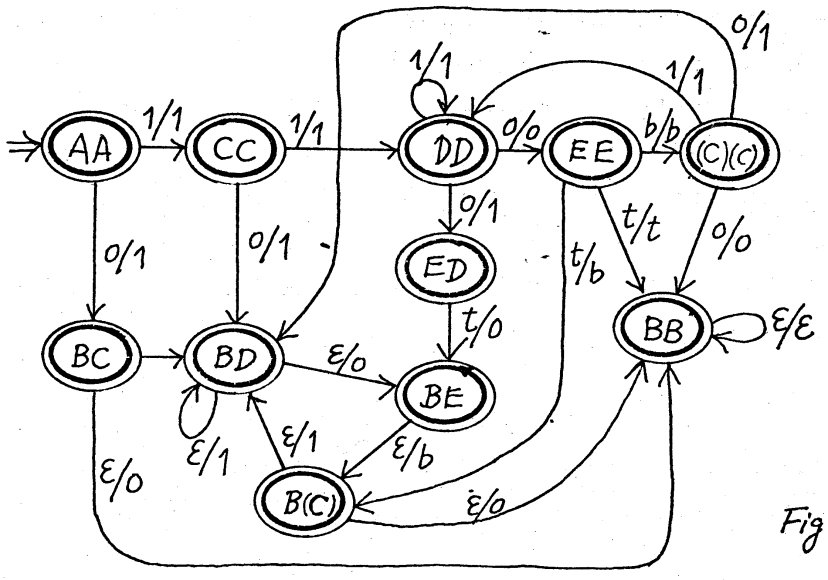


Fig.7 C-machine M'

Now we consider a rational machine M_{bf} which defines all possible connections among cells such that the left and right neighbors should be connected and the trunk-branch bifurcation should also be connected. See Fig.8. Note that M_{bf} is independent from the example under consideration and is universal.

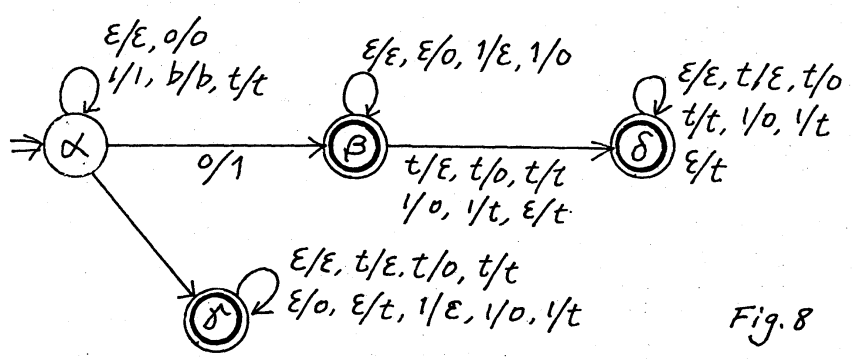


Fig.8 M_{bf}

Finally we take the "intersection" of series of graphs generated by C-machine M' and M_{bf}. For "intersection", see Section 5. Though intersection of two rational graph series is not rational in general, intersection of a C-machine with M_{bf} gives always a rational series. This comes from the special feature of M_{bf}.

For our example, the resultant rational series is generated by the machine M" illustrated in Fig.9.

It is seen from the above construction, that M" generates the graph series which is isomorphic to the series of trees generated by the L system G. Here "isomorphic" means that both graph series are identical besides labeling of nodes. In L system, nodes are labeled with symbols from $\{a,b,c,d,e\}$, but in RM-generated graphs, they are labeled with strings on the alphabet $\{0,1,t,b\}$. Fig.10 illustrates the graph Γ_η which is generated by M". ∎

Besides tree like graphs, which are essentially one-dimensional, RM's generally generate multi-dimenasional graphs. Since there is no adequate way of describing infinite series

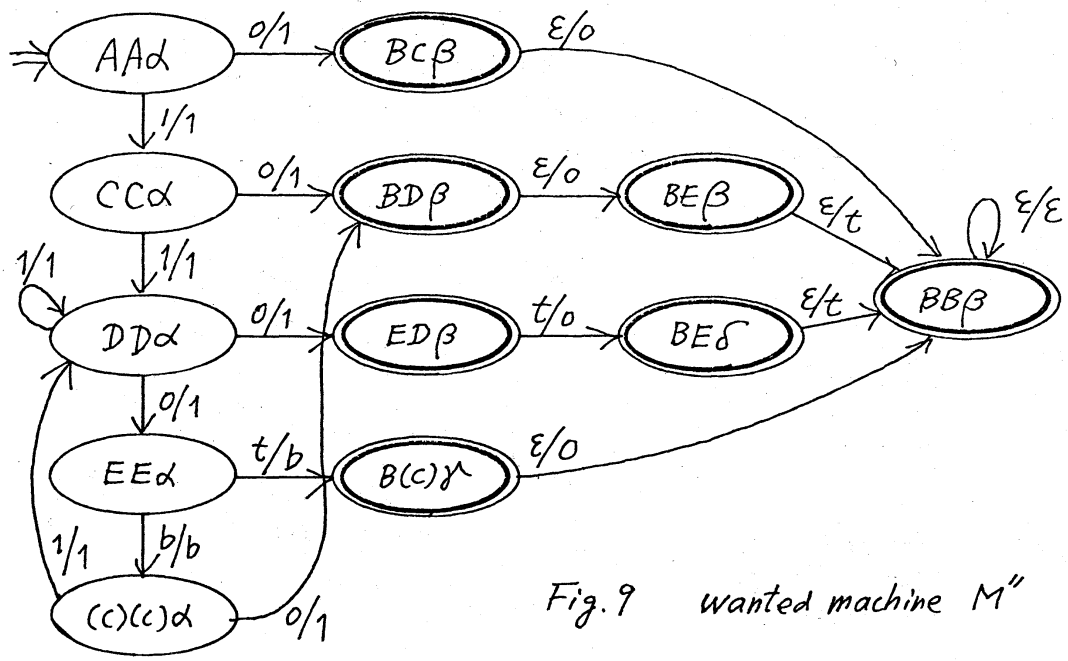


Fig.9 wanted machine M''

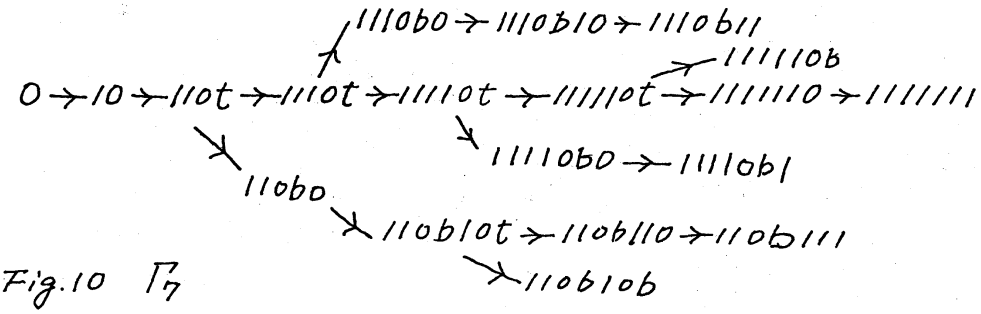


Fig.10 Γ_7

of multi-dimensional graphs, we show here some geometrically uniform examples in order to understand the generative capability of RM.

Example 3.1

$$\mathbb{\Gamma} = \Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$$

Γ_n is a square grid with (n+1)x(n+1) nodes as illustrated in Fig.11. So the growth functions $f(n)=(n+1)^2$ and $h(n)=2n^2+2n$. Therefore sup(h/f)=2. Nodes are labeled with strings

on the alphabet $\{A,B,C,D\}$ as in the figure. Edges are unidirectional. The rational machine which generates it is given in Fig.12. It is easy to alter the machine so that it may generate bidirectional graphs.

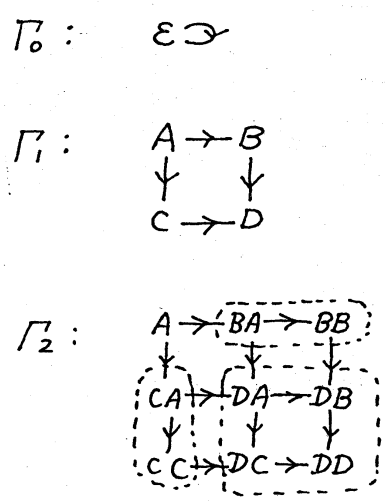


Fig.11 square grids

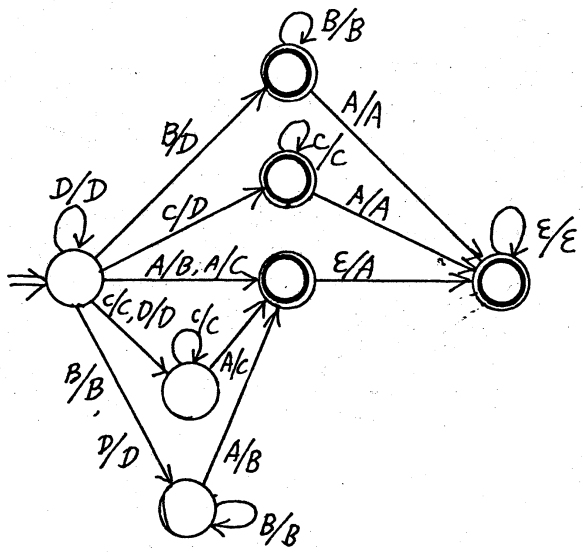


Fig.12

Example 3.2

Γ_n is also a square grid, but with $2^n \times 2^n$ nodes. $f(n)=2^{n+1}$ and $h(n)=2^{n+1}(2^n-1)$. This series is generated by the machine of Fig.13.

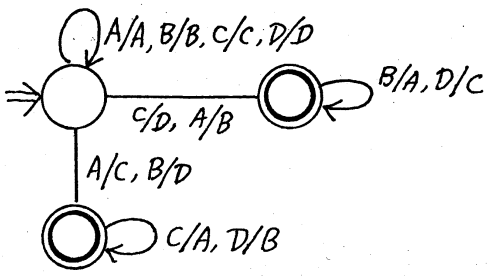


Fig.13

Example 3.3

Series of triangular graids as illustrated in Fig.14 is generated by the machine given in Fig.15.

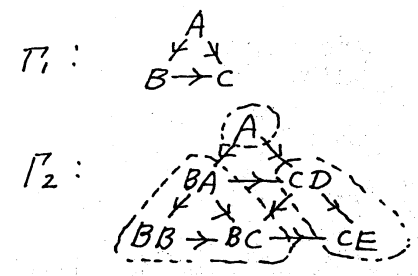


Fig.14 Triangular grids

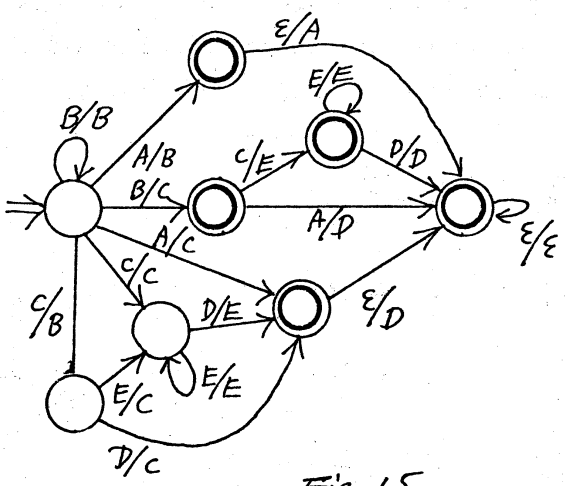


Fig.15

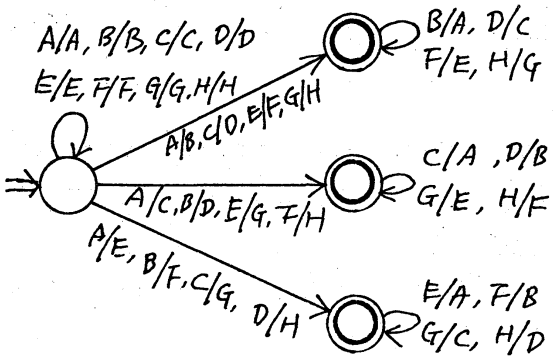


Fig.16 3-dim. Cube

Example 3.4

Three dimensional cube Γ_n with $2^n \times 2^n \times 2^n$ nodes. So $f(n+1) = 8f(n)$. We employ the alphabet $\Sigma = \{A,B,C,D,E,F,G,H\}$ which corresponds to 8 nodes which replace one node at each time. This series is generated by RM of Fig.16.

Example 3.5

It will be easily seen that the same series of graphs is generated by different RM's if labeling of nodes is different. For example the square grids of Fig.17 are generated by the machine in Fig.18, which is different from one in Fig.12.

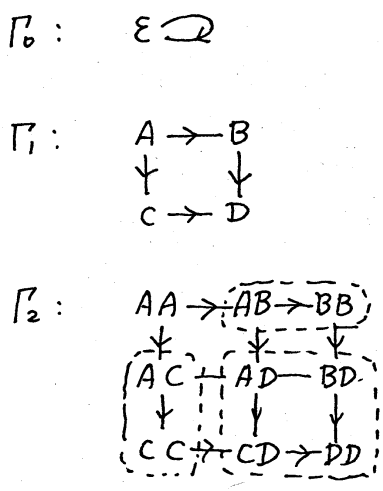


Fig.17 square grids (second)

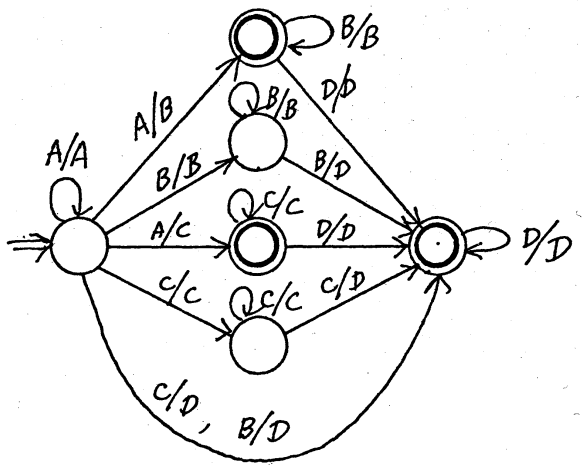


Fig. 18

Looking at examples given above and considering biological facts, we have here special classes of series of graphs.

(i) Graph series of division type

A series of graphs $\mathbb{\Gamma} = \Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \cdots$ is called of division type, if the following holds:

If $\Gamma_i \ni w$, then $\Gamma_{i-1} \ni w$ and $\Gamma_i \not\ni wa$ $(a \in \Sigma)$
or $\Gamma_{i-1} \ni u$ such that $ua=w$ $(a \in \Sigma)$ and $\Gamma_i \not\ni u$.

In a division type series, every node at a time is a concatenation of a node at the previous time with a symbol from Σ or a node at the previous time itself. Motivation for defining the division type comes from the fact in developmental biology that every cell is an outcome of a cell division and each cell can be represented by a string of symbols. See [Nishio, 1978b].

A daughter cell of a cell u is represented by ua where $a \in \Sigma$.

It is seen that all above examples of multi-dimensional graph series is of division type.

(ii) Graph series of conservation type

A series is called of conservation type, if $w \in \Gamma_i$ implies $wa \in \Gamma_{i+1}$ with $a \in \Sigma_\varepsilon$. In a conservation type series, every descendant of a node continues to exist in graphs. The above examples are also of conservation type.

4. Decision Problems

There are many interesting decision problems as to rational machines. We discuss some of them in this section.

Proposition 4.1

It is undecidable whether or not a pair of RM's generate the same series of graphs.

We employ the following lemmas.

Lemma 4.1 [Griffiths, 1968]

It is undecidable if a pair of ε-free nondeterministic gsm's define the same word relation.

Lemma 4.2

In the case of rational machines, such that the state transition ν is range-ε-free (i.e. $\nu \subset Q \times \Sigma_\varepsilon \times \Sigma \times Q$) or domain-$\varepsilon$-free ($\nu \subset Q \times \Sigma \times \Sigma_\varepsilon \times Q$), suppose that M and M' generate $\mathbb{\Gamma} = \Gamma_0, \Gamma_1, \Gamma_2, \ldots$ and $\mathbb{\Gamma}' = \Gamma_0', \Gamma_1', \Gamma_2', \ldots$, respectively. Then $\Gamma_i = \Gamma_i'$ (i=0,1,...) if and only if $\bigcup_{i \geq 0} \Gamma_i = \bigcup_{i \geq 0} \Gamma_i'$.

Proof

"Only if" part trivially holds for general rational machines. Since RM's are range-ε-free, relations $\bigcup \Gamma_i$ and $\bigcup \Gamma_i$ are partitioned into Γ_i and Γ_i' according to the length of range. So, "if" part holds. ∎

Proof of Proposition 4.1

Given a pair of ε-free nondeterministic gsm's M_1 and M_2,

construct rational machines M_1' and M_2', respectively, as follows: If M_i $(i=1,2)$ contains a state transition $q \xrightarrow{a/b_1 b_2 \cdots b_k} q'$ then define in M_i' the sequence of transitions

$$q \xrightarrow{a/b_1} q_1 \xrightarrow{\varepsilon/b_2} q_2 \xrightarrow{\varepsilon/b_3} \cdots \xrightarrow{\varepsilon/b_{k-1}} q_{k-1} \xrightarrow{\varepsilon/b_k} q'.$$

Then obviously such M_i's are range-ε-free. So, from Lemma 4.2 if it is decidable whether M_1' and M_2' generate the same series, it would also decidable thether M_1 and M_2 define the same relation. From Lemma 4.1 we have the proposition. ∎

Next we investigate the decision problem on the property of series: <u>Is it decidable if a given RM generates a series of division type (or conservation type) ?</u>

Though we have not reached the conclusion, some considerations are given here.

Let a given RM be $M=(Q,\Sigma,\nu,q_0,F)$. Denote its domain- and range-ε-automata by $A_D=(Q,\Sigma,\mu,q_0,F)$ and $A_R=(Q,\Sigma,\mu',q_0,F)$, respectively.

Here we need some definitions:

<u>$\exists$-equivalence</u>: Let A and A' be ε-automata with the same input alphabet. q' in A' is called $\exists$-equivalent to q in A if and only if there are a string s and a time t such that there is a path from q_0 to q with label s and length t in A and there is a path from q_0' to q' with the same label and length in A'.

<u>$\forall$-equivalence</u>: q' is called to be $\forall$-equivalent to q if and only if for <u>every</u> string s and t such that q defines s in t steps q' also defines s in t steps. Here that q defines s in t steps means that there is a path of length t from q_0 to q with label s.

<u>Proposition 4.2</u>

M generates a series of division type if and only if the following conditions hold:

(i) If $\exists p \in F$ in A_D (or in A_R) and $\exists q \in F$ such that $(q,x,p) \in \mu$, $x \in \Sigma$ (or $(q,y,p) \in \mu'$, $u \in \Sigma$), then every $\exists$-equivalent state <u>q</u> to q in A_D <u>and</u> A_R has not a transition enter-

ing an accepting state with label ε , i.e. $(\underline{q}, \varepsilon, \underline{p}) \notin \mu$ with $\underline{p} \in F$.

(ii) If $\exists p \in F$ in A_D (or A_R) and $\exists q \in F$ such that $(q, \varepsilon, p) \in \mu$ (or μ'), then for every $\exists$-equivalent state $\underline{q}$ in A_D and A_R, $(\underline{q}, x, \underline{p}) \notin \mu$ where $x \in \Sigma$ and $\underline{p} \in F$.

(iii) If $\exists p \in F$ in A_D (or A_R) and $\exists q \notin F$ such that $(q, x, p) \in \mu$ (or μ'), $x \in \Sigma$, then there is a $\forall$-equivalent state $\underline{q}$ to q in F of A_D <u>or</u> A_R and there is not a state in F which is "delayed $\forall$-equivalent" to p.

q' is "delayed $\forall$-equivalent" to q if for every string s and time t such that q defines s in t steps, q' defines s in <u>t+1</u> steps.

(iv) If $\exists p \in F$ in A_D (or A_R) and $\exists q \notin F$ such that $(q, \varepsilon, p) \in \mu$ (or μ'), then there is a $\forall$-equivalent state $\underline{q}$ to q in F of A_D or A_R.

<u>Proof</u>

Obvious from the definition of division type. ∎

Thus our decision problem has been reduced to those of $\exists$-equivalence, $\forall$-equivalence and delayed-$\forall$-equivalence. For example, $\exists$-equivalence problem: Given a pair of ε-automata having the same form of state transition, and a pair of states q and q', <u>is it decidable if q' is $\exists$-equivalent to q</u>? This problem has not been solved.

5. Operations on Series of Graphs

We discuss here the closure property of the <u>class of rational series</u> (C_R) under some operations defined below.

Let $\mathbb{\Gamma} = \Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$ and $\mathbb{\Gamma}' = \Gamma_0', \Gamma_1', \Gamma_2', \ldots$ be two series of graphs.

(i) Union of $\mathbb{\Gamma}$ and $\mathbb{\Gamma}'$ is the series

$$\mathbb{\Gamma} \cup \mathbb{\Gamma}' = \Gamma_0 \cup \Gamma_0', \Gamma_1 \cup \Gamma_1', \ldots, \Gamma_n \cup \Gamma_n', \ldots$$

where $\Gamma_n \cup \Gamma_n' = (N_n \cup N_n', E_n \cup E_n')$.

(ii) Intersection of $\mathbb{\Gamma}$ and $\mathbb{\Gamma}'$ is $\mathbb{\Gamma}\cap\mathbb{\Gamma}' = \Gamma_0\cap\Gamma_0', \Gamma_1\cap\Gamma_1', \ldots, \Gamma_n\cap\Gamma_n', \ldots$

where $\Gamma_n\cap\Gamma_n' = (N_n\cap N_n', E_n\cap E_n')$.

(iii) k-slow down of $\mathbb{\Gamma}$ is the series

$$\mathbb{\Gamma}_{(k)} = \overbrace{\Gamma_0, \Gamma_0, \ldots, \Gamma_0}^{k}, \overbrace{\Gamma_1, \Gamma_1, \ldots, \Gamma_1}^{k}, \ldots, \overbrace{\Gamma_n, \Gamma_n, \ldots, \Gamma_n}^{k}, \ldots$$

(iv) k-speed up of $\mathbb{\Gamma}$ is

$$\mathbb{\Gamma}^{(k)} = \Gamma_0, \Gamma_k, \Gamma_{2k}, \ldots, \Gamma_{ik}, \ldots$$

<u>Proposition 5.1</u>

C_R is closed under union of series.

<u>Proof</u>

By simply joining two RM's so that both initial states are identified, we obtain the RM which generates $\mathbb{\Gamma}\cup\mathbb{\Gamma}'$.

<u>Proposition 5.2</u>

C_R is not closed under intersection.

<u>Proof</u>

A counter example is given in Fig.19(a) and (b).

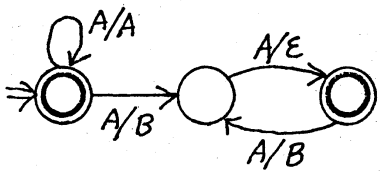


Fig. 19 (a) $\mathbb{\Gamma}$

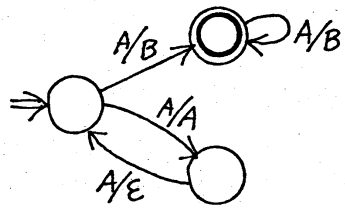


Fig. 19 (b) $\mathbb{\Gamma}'$

From the figures we have

$$\Gamma_n = \bigcup_{n=i+2j}\{(A^n, A^iB^j)\} \qquad (i\geq 0 \text{ and } j\geq 0)$$

$$\Gamma_n' = \bigcup_{n=2i+j}\{(A^n, A^iB^j)\} \qquad (i\geq 0 \text{ and } j\geq 1)$$

Therefore

$$\Gamma_n\cap\Gamma_n' = \{(A^{3i}, A^iB^i)\} \qquad (n=3i,\ i\geq 1)$$

and $\Gamma_m\cap\Gamma_m' = \emptyset \qquad (m\neq 3i)$

So, if there is a RM M which defines $\Gamma \cap \Gamma'$ then the range of the relation defined by M is equal to range $\bigcup_{n\geq 0}(\Gamma_n \cap \Gamma_n')$. But range $\bigcup(\Gamma_n \cap \Gamma_n') = \bigcup$ range$(\Gamma_n \cap \Gamma_n') = \bigcup_{i\geq 0}\{(A^iB^i)\}$. This is obviously not a regular set. From the comment in Section 2, page 5, we conclude that there is no such M.

Proposition 5.3

C_R is closed under k-slow down.

Proof

Suppose that $M=(Q,\Sigma,\nu,q_0,F)$ generates Γ. $M_{(k)}=(Q',\Sigma,\nu',q_0',F')$ which generates $\Gamma_{(k)}$ is constructed as follows. If there is a transition in M $q \xrightarrow{x/y} p$ and q is not an accepting state, then assume in $M_{(k)}$ the chain of transitions $q \xrightarrow{\varepsilon/\varepsilon} q_1 \xrightarrow{\varepsilon/\varepsilon} q_2 \xrightarrow{\varepsilon/\varepsilon} \cdots \xrightarrow{\varepsilon/\varepsilon} q_{k-1} \xrightarrow{x/y} p$ where $q,q_1,q_2,\ldots,q_{k-1}$ are not accepting states. If q is an accepting state, then they are also so.

Proposition 5.4

C_R is closed under k-speed up. Moreover if Γ is generated by an n state RM, then $\Gamma^{(k)}$ is so by a RM having not more than n states.

Proof

Let Σ' be the alphabet of $M^{(k)}$ which generates $\Gamma^{(k)}$ and $\Sigma' = \Sigma_\varepsilon^k = \overbrace{\Sigma_\varepsilon \times \Sigma_\varepsilon \times \cdots \times \Sigma_\varepsilon}^{k}$. Next we define the product of label pairs: Suppose in M a k consecutive sequence of transitions $q_0 \xrightarrow{x_1/y_1} q_1 \xrightarrow{x_2/y_2} \cdots \xrightarrow{x_k/y_k} q_k$ where x_i, $y_i \in \Sigma_\varepsilon$. Then the product is $x_1x_2\cdots x_k/y_1y_2\cdots y_k = \alpha/\beta$ where α and $\beta \in \Sigma'$. Now in M, follow k step transitions from q_0 to find that the path is $q_0,q_1,\ldots,q_k$. Then assume in $M^{(k)}$ a single transition $q_{(0)} \xrightarrow{\alpha/\beta} q_{(k)}$ where α/β is the k-product of label pairs of this path. When q_k is an accepting state, define $q_{(k)}$ to be so. Do this procedure for all paths of length k in M. Next follow another k step path from q_k to q_{2k} and define a new state and a transition as in the first procedure. Note that in this second trial, q_{2k} can coincide with q_0 or q_k. In this

case, identify $q_{(2k)}$ with $q_{(0)}$ or $q_{(k)}$, respectively. By continuing this procedure, which surely terminate in finite steps, we obtain $M^{(k)}$. The latter half of the proposition is clear from this construction.

6. Concluding Remarks

As to the newly introduced notion of generation of graph series by means of rational machines, there are many other interesting topics to be discussed. For example, we can define the notion of limit of graph series, especially in the case of conservation type. Generally the graph obtained as the limit of infinite series is an infinite graph. So, it might be interesting to investigate properties of such infinite graphs.

7. Acknowledgements

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8. References

[C.C.Elgot and J.E.Mezei, 1965] On Relations Defined by Generalized Finite Automata, IBM Journal, Jan. 1965, 47-68.

[S.Eilenberg, 1974] 'Automata, Languages and Machines, vol.A', Academic Press

[G.T.Herman and G.Rozenberg, 1975] 'Developmental Systems and Languages', North-Holland

[H.Nishio, 1978a] Cell Lineage System for Describing Growths of Filamentous Organisms, Information and Control, vol.37, 280-301.

[H.Nishio, 1978b] A Mathematical Theory of Developmental Processes Based Upon the Cell Lineage, Proc.International Symposium on Mathematical Topics in Biology, Sept.11-12, 1978,

Kyoto, 164-173.
[H.Nishio, 1979] Mathematical Considerations on Developmental Multi-cellular Systems, Memories of Symposium at Research Institute for Mathematical Sciences, Kyoto University, April 1979, vol.353, 152-164.
[T.V.Griffiths, 1968] The Unsolvability of the equivalence problem for Λ-free Nondeterministic Generalized Machine, J.A.C.M., vol.15, no.3, 409-413.