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J-integral in two dimensional fracture mechanics

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1. Introduction The principal objective of this paper is the mathematical interpretation of two dimensional fracture mechanics (cf. Okamura[10], Rice[12], Sih[13]). The fundamental concepts of fracture mechanics are stress intensity factors, energy release rate and J-integral. In two pioneering papers published in the early 1920's, A. A. Griffith studied measurements of crack extension forces for the problem of a crack of length $2a$ in a plate under tension σ as in Fig. 1. The basic idea behind his theory is that a crack will begin to propagate if the elastic energy released by its growth is greater than the energy required to create the fractured surface (see [13], Historical remarks). This concept of Griffith's energy balance has been generalized (see, e.g., Palaniswamy and Knauss [11]) by means of the concept of energy release rate which is defined as the variational derivative of potential energies with respect to crack growth. Irwin[4] in applying Griffith's concept to solve fracture problems recognized the importance of the intensity of the local stress field. He proposed three modes of crack extension which are identified with their respective stress intensity factors K_1, K_2 and K_3 (see [13]). In two dimensional case it has been shown by Rice[12] that the energy

release rate is expressed as a path-independent integral, which is called J-integral in fracture mechanics. In three dimensional case it has been proved more precisely by Ohtsuka!9! that analogous representations of the energy release rates hold for a linear (non-homogeneous) elastic body containing a smooth crack which means a 2-dimensional oriented smooth manifold with boundary. Perhaps these representations will be valid for nonlinear elastic body, but nonlinearity gives rise to mathematical difficulty in the calculation of the energy release rate.

In this paper we shall connect these concepts from mathematical viewpoint. We consider only an isotropic plate containing a crack lying on the line $x_2 = 0$. In the next section we shall explain the classical energy release rate and J-integral in such elastic plate and show that the result of Rice!12! indicates the crack extension force is attributed to singularity at a crack tip. In section 3 we shall analyze singularity at a crack tip and show that the stress intensity factors are coefficients of singular terms. Furthermore it will be proved that J-integral acts only on this singularity.

Fig. 1

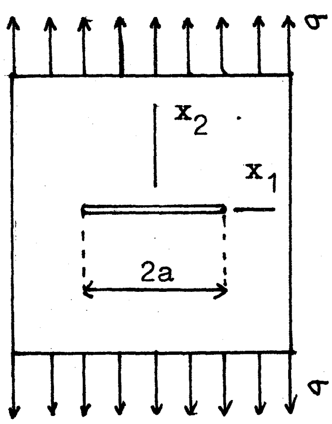
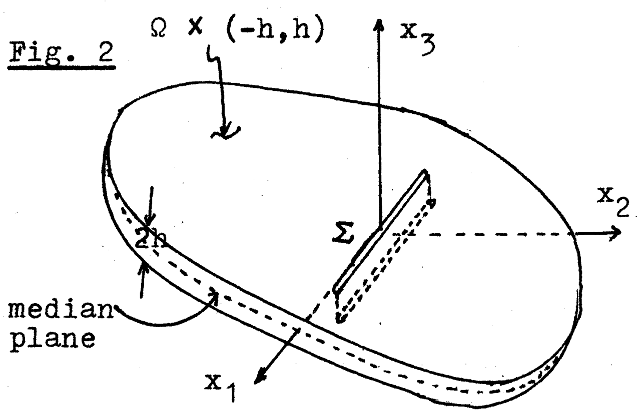


Fig. 2



2 Energy release rate and J-integral Let G be a simply connected and bounded domain of R^2 containing the origin and let the boundary S of G be smooth. Let Σ be the segment

$$\{x \in R^2;\ -a \leqq x_1 \leqq a,\ x_2 = 0\},\ \text{with some}\ \ a > 0,$$

such that $\Sigma \subset G$. Throughout this paper we denote $\Omega = G - \Sigma$.

Let us now consider an elastic plate containing the crack Σ, that is:

For a positive number h being "small", we denote by $\Omega \times (-h,h)$ the set occupied by the interior of this elastic plate in its non-deformed state (see Fig. 2), and call $[u,\varepsilon,\sigma]$, in this order, the displacement vector, the linearized strain tensor and the stress tensor. We consider here the case when $[u,\varepsilon,\sigma]$ is a plane stress state which is approximately achieved in a thin lamina deformed under the action of forces lying in its median plane. Then $[u,\varepsilon,\sigma]$ depends only on two Cartesian coordinates x_1 and x_2, and the stress components σ_{31}, σ_{32}, σ_{33} are equal to zero. We admit possible discontinuities of $[u,\varepsilon,\sigma]$ across Σ. In addition we assume the following; $[u,\varepsilon,\sigma]$ is a linear isotropic state on Ω, the elastic plate cannot move along S_0 ($\subset S$), the surface force F is given on $S_1 = S - S_0$, the body force f is given in Ω, and the stress is free on Σ (see Fig. 3). Here S_0 is a subset of S which is measurable with respect to the line element dS of S and has positive measure.

In these circumstances the stress-displacement relations and stress equations of this plate are given by

$$
(2.1)\quad \begin{cases}
\varepsilon_{ij}(u) = (D_j u_i + D_i u_j)/2 & \text{(strain tensor)} \\
\sigma_{ij}(u) = a_{ijkl}\varepsilon_{kl}(u) & \text{(strain tensor; Hooke's law)} \\
a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) & \\
- D_j\sigma_{ij}(u) = f_i \quad \text{in} \quad \Omega & \text{(balance equation),}
\end{cases}
$$

in which $\lambda > 0$ and $\mu > 0$ denote Lamé's modulus and the shear modulus, respectively, and the following boundary conditions are given:

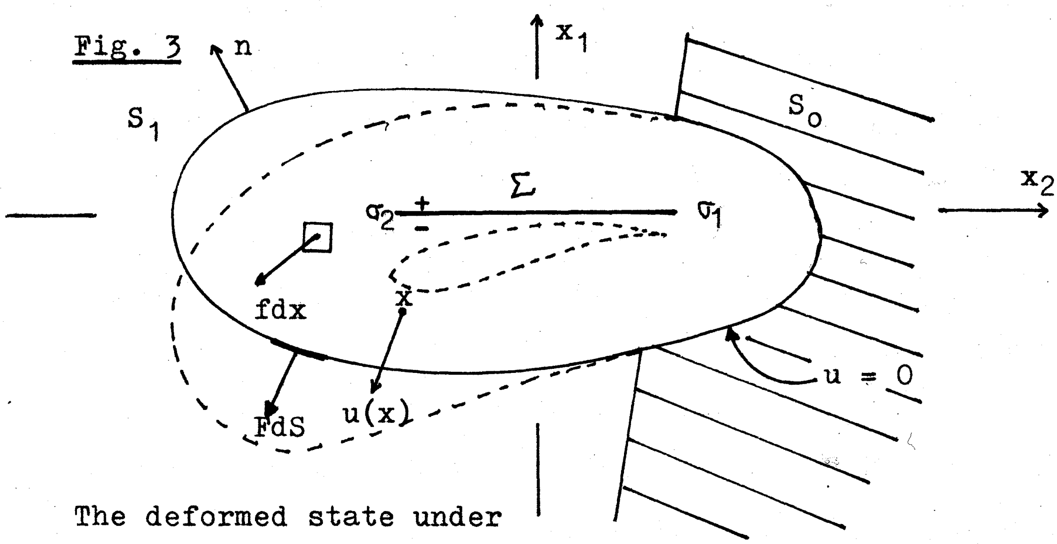
$$(2.2) \qquad u = 0 \quad \text{on} \quad S_o,$$

$$(2.3) \qquad \sigma_{ij}(u)n_j = F_i \quad \text{on} \quad S_1,$$

$$(2.4) \qquad \sigma_{i2}(u)^+ = \sigma_{i2}(u)^- = 0 \quad \text{on} \quad \Sigma \quad (i = 1,2),$$

where u_i, ε_{ij}, σ_{ij}, f_i and F_i are the components of u, ε, σ, f and F, respectively, n_j the components of the unit outward normal to S and δ_{ij} Kronecker's delta. Here $\delta_{i2}(u)^+$ $\delta_{i2}(u)^-$ denote the traces of $\delta_{i2}(u)$ on Σ_+ and Σ_- of Σ and we have chosen $(0,1)$ as the unit outward normal to Σ.

Fig. 3



The deformed state under a load (f,F) is shown by the dotted line.

Let Q be an open set of R^2. We denote by $H^m(Q)$ the usual Sobolev space of order m with the norm denoted by $|\cdot|_{m,Q}$. We shall use the notation

$$\mathbb{H}^m(Q) = \{H^m(Q)\}^2, \ \mathbb{L}^2(Q) = \{L^2(Q)\}^2 \ (= \mathbb{H}^0(Q)),$$

which are equipped with the usual product norm denoted by $\|\cdot\|_{m,Q}$.

The problem (2.1) - (2.4) is reformulated as follows:

(2.5) Find a displacement vector $u \in V(\Omega)$ under a load $(f,F) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(S_1)$ such that

$$\int_\Omega \sigma_{ij}(u)\varepsilon_{ij}(v)\, dx = \int_\Omega f\cdot v\, dx + \int_{S_1} F\cdot v\, dS$$

for all $v \in V(\Omega) = \{v \in \mathbb{H}^1(\Omega);\ v = 0 \ \text{ on } \ S_0\}$.

Then we have the following results (cf. [9]):

Theorem 2.1 (a) For each load $\mathcal{L} = (f,F) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(S_1)$, there exists a unique solution $u \in V(\Omega)$ of the problem (2.5). Furthermore Green's operator $T: \mathcal{L} \longrightarrow u$ is a bounded linear operator from $\mathbb{L}^2(\Omega) \times \mathbb{L}^2(S_1)$ into $V(\Omega)$.

(b) Let B be an arbitrary open neighborhood of Σ such that $\overline{B} \subset G$ and N an arbitrary open neighborhood of $\partial\Sigma$ such that $\overline{N} \subset B$. Then $\mathcal{L} \longrightarrow T(\mathcal{L})|_{(B-\overline{N})'}$ is a bounded linear operator from $\mathbb{L}^2(\Omega) \times \mathbb{L}^2(S_1)$ into $H^2((B-\overline{N})')$, where $(B-\overline{N})' = (B-\overline{N}) \cap \Omega$.

The crack extension process is considered to occur in a quasi-static manner. When we refer to time we use it as a parameter which indicates the sequence of events such as in crack progression (see [11]). Here we assume that the direction of crack propagation to be know a priori as follows: A newly created crack by extending the crack Σ in the length of time

$t\ (\geqq 0)$ is expressed by

$$\Sigma(t) = \{x \in R^2;\ -a - t \leqq x_1 \leqq a + t,\ x_2 = 0\}.$$

Here we assume that $\Sigma(t) \subset G$ for $t < a$. During crack extension let a load $\mathcal{L}$ be independent of t. Then the quasi-static problem we now consider is the following:

(2.6) For a given load $\mathcal{L} = (f,F) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(S_1)$, we seek displacement vectors $v(t) \in V(\Omega(t))$ such that

$$\int_{\Omega(t)} \sigma_{ij}(v(t))\,\varepsilon_{ij}(v)\,dx = \int_{\Omega(t)} f\cdot v\,dx + \int_{S_1} F\cdot v\,dS$$

for all $v \in V(\Omega(t))$; $0 \leqq t < a$, where $\Omega(t) = G - \Sigma(t)$ and

$$V(\Omega(t)) = \{v \in \mathbb{H}^1(\Omega(t));\ v = 0 \text{ on } S_o\}.$$

By virtue of Theorem 2.1 there exists a unique displacement vector $v(t)$ for each time t under a load $\mathcal{L}$.

The potential energy just prior to crack extension and same quantity after the crack has extended by the increment $\Sigma(t) - \Sigma$ are given as follows:

$$I(\mathcal{L};\Sigma) = \frac{1}{2}\int_{\Omega} \sigma_{ij}(u)\,\varepsilon_{ij}(u)\,dx - \int_{\Omega} f\cdot u\,dx - \int_{S_1} F\cdot u\,dS,$$

$$I(\mathcal{L};\Sigma(t)) = \frac{1}{2}\int_{\Omega(t)} \sigma_{ij}(v(t))\,\varepsilon_{ij}(v(t))\,dx - \int_{\Omega(t)} f\cdot v(t)\,dx - \int_{S_1} F\cdot v(t)\,dS,$$

where $u = v(0)$. Thus the potential energy released by the increment $\Sigma(t) - \Sigma$ is written in the form

$$I(\mathcal{L};\Sigma) - I(\mathcal{L};\Sigma(t)) = \frac{1}{2}\int_{\Omega(t)} \sigma_{ij}(u-v(t))\,\varepsilon_{ij}(u-v(t))\,dx.$$

Now consider the limit

$$G(\mathcal{L};\{\Sigma(t)\}) = \lim_{t \to 0} \frac{I(\mathcal{L};\Sigma) - I(\mathcal{L};\Sigma(t))}{|\Sigma(t) - \Sigma|}$$

where $|\Sigma(t) - \Sigma|$ is the length of the increment $\Sigma(t) - \Sigma$. If it exists, we call $G(\mathcal{L}; \{\Sigma(t)\})$ the (classical) energy release rate under the load $\mathcal{L}$. A. A. Griffith seems to have believed that the crack propagation in a brittle solid becomes possible when the energy release rate reaches a critical value depending on the material considered.

The following representation of the energy release rate is available (see [9]).

Theorem 2.2 For each load $\mathcal{L}$ the energy release rate is expressed as

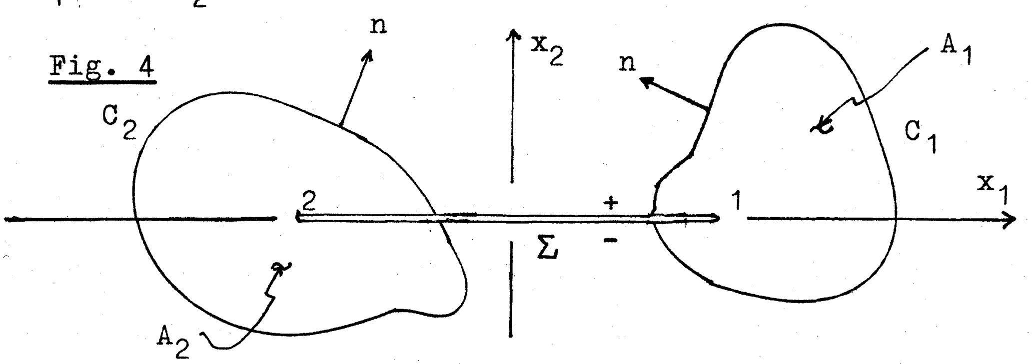
$$(2.7) \qquad G(\mathcal{L};\{\Sigma(t)\}) = \textstyle\sum_{k=1}^{2} J_k(u),$$

where $u = T(\mathcal{L})$ (see Theorem 2.1) and $J_k(u)$ is given by

$$(2.9) \qquad J_k(u) = (-1)^{k-1} \left\{ \int_{C_k} [Wn_1 - s\cdot(D_1u)]\, d\ell - \int_{A_k'} f\,(D_1u)\, dx \right\}.$$

As illustrated in Fig. 4, C_k is any closed path surrounding the crack tip σ_k, A_k the open set enclosed by C_k, $A_k' = A_k \cap \Omega$, $W = \frac{1}{2}\, \sigma_{ij}(u)\,\varepsilon_{ij}(u)$, s the traction vector defined by the outward normal n along C_k; $s_i = \sigma_{ij}(u)n_j$ and $d\ell$ the line element of C_1 (or C_2).

Fig. 4



$J_k(u)$ is independent of the choice of C_k. In the case when body force is zero, a path independent integral

$$J = \int_C \left[W n_1 - s\,(D_1 u)\right] d\ell$$

is called J-integral in fracture mechanics. Rice[12] has shown that the energy release rate is expressed as J-integral in the following case: the body force f is zero and $[u, \varepsilon, \sigma]$ is a homogeneous nonlinear elastic state, i.e., there exists the strain energy density W, depending only on ε, by which the stress-strain relation is given as follows; $\sigma_{ij} = \partial W/\partial \varepsilon_{ij}$.

J-integral will vanish if $u|_{N'} \in \mathbb{H}^2(N')$ for an arbitrary open neighborhood N of σ_1 (or σ_2), where $N' = N \cap \Omega$. Since the first component of the unit normal of Σ is zero, we obtain by the divergence theorem

$$\int_{C_1} W n_1\, d\ell = \int_{A_1'} D_1 W\, dx.$$

But $D_1 W = \sigma_{ij}(u)\varepsilon_{ij}(D_1 u)$, so that, integrating by parts, we have

$$\int_{A_1'} D_1 W\, dx = - \int_{A_1'} D_j \sigma_{ij}(u) D_1 u\, dx + \\ + \int_{C_1} \sigma_{ij}(u) n_j D_1 u_i\, d\ell.$$

Thus we have $J_1(u) = 0$.

This indicates the following interesting fact: If this elastic plate is regular at the crack tips $\{\sigma_1, \sigma_2\}$ for all loads, the crack will not progress by any loads, which is contradictory to our experience. Hence there exists a load such that the elastic plate under this load is singular at the crack tip. The cause of brittle fracture is such a singularity at the crack tip. Here we adopted the following definition.

Definition 2.3 Let β be a point of $\overline{\Omega}$ and let $u = T(\mathcal{L}) \in V(\Omega) \cap H^2_{loc}(\Omega)$. We call β a regular point of elastostatic field of this plate under the load $\mathcal{L}$ if there exists an open neighborhood V_β of β such that $u|_{(V_\beta \cap \Omega)} \in H^2(V_\beta \cap \Omega)$. We call β a singular point if β is not a regular point.

3 Stress intensity factors In this section we shall examine the behavior of a solution of the problem (2.5) near the crack tips. There are many attempts to obtain solutions for this crack problem (see, e.g., [13]). But these attempts are achieved by assuming some simplification of loading. Our interest is to decompose displacement vectors into singular terms and regular terms, under general loads $\mathcal{L} \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(S_1)$. To our knowledge such a calculation has not appeared in the literature. Our proof basically consists in applying the method by Merigot[7] in the analysis of single 2m-order elliptic equations in L^p-spaces. We note that Kondrat'ev's method[14] (in L^2-space) is not applicable, because the solution operator (which depends on a complex parameter ζ) for ordinary differential equations given by the biharmonic problem has poles at $\zeta = 0, \pm 1/2, \pm 1, \pm 3/2, \cdots$. In this paper we merely state our main results; the proofs will appear elsewhere.

Let us consider the following elastic plate problem without crack:

(3.1) Find a displacement vector $u_0 \in V(G)$ such that

$$\int_G \sigma_{ij}(u_0)\varepsilon_{ij}(\varphi)\, dx = \int_G f\cdot\varphi\, dx + \int_{S_1} F\cdot\varphi\, dS$$

for all $\varphi \in V(G)$. This problem is uniquely solvable and Green's operator $T_o : \mathcal{L} \to u_o$ is a bounded linear operator from $\mathbb{L}^2(\Omega) \times \mathbb{L}^2(S_1)$ into $V(G)$.

Next we consider the following elastic problem with crack:

(3.2) Find a displacement vector $u_I \in \mathbb{H}^1(\Omega)$ such that

$$\int_\Omega \sigma_{ij}(u_I)\varepsilon_{ij}(\psi)\,dx = -\int_{-a}^{a} \sigma_{i2}(T_o(\mathcal{L}))[\![\psi_i]\!]\,dx_1$$

for all $\psi \in \mathbb{H}^1(\Omega)$, where $[\![\psi_i]\!] = \psi_i^- - \psi_i^+$.

<u>Lemma 3.1</u> The problem (3.2) is solvable. Moreover there exists a function $U \in \mathbb{H}^2(\Omega) \cap C^\infty(\Omega)$, called Airy's stress function, such that

$$\sigma_{11}(u_I) = D_{22}U,\quad \sigma_{12}(u_I) = -D_{12}U \quad \text{and} \quad \sigma_{22}(u_I) = D_{11}U,$$

where $D_{ij} = D_i D_j$, and U satisfies:

$$\begin{cases} \Delta^2 U = 0 \quad \text{in} \quad \Omega, \\ (D_{11}U)^+ = (D_{11}U)^- = -\sigma_{22}(T_o(\mathcal{L})) \quad \text{on } \Sigma, \\ (D_{12}U)^+ = (D_{12}U)^- = \sigma_{12}(T_o(\mathcal{L})) \quad \text{on } \Sigma, \\ D^\alpha U = 0 \quad \text{on} \quad S \quad \text{for} \quad |\alpha| \leq 1. \end{cases}$$

The existence of Airy's stress function is not trivial, because Ω is not simply connected.

Let $\eta \in C_0^\infty(R^2)$ such that $\eta = 1$ near Σ and $\operatorname{supp}\eta \subset G$. We can easily construct the vector $u_{II} \in V(\Omega)$ such that

$$\int_\Omega \sigma_{ij}(u_{II})\varepsilon_{ij}(v)\,dx = \int_\Omega \sigma_{ij}((1 - \quad)u_I)\varepsilon_{ij}(v)\,dx$$

for all $v \in V(\Omega)$ and $u_{II} = 0$ near Σ. Then $u = T(\mathcal{L})$ is written in the form

$$u = u_o + \eta u_I + u_{II}.$$

Here we notice that u_o and u_{II} have not singular points in G. For ηu_I, the usual arguments using partition of unity together with Lemma 3.1 bring us to the following problem.

$$(3.3)\quad \begin{cases} \Delta^2 V = f \quad \text{in } \mathcal{O}, \\ D_{11}V(r,\pm\pi) = g_1^{\pm}(r), \\ D_{12}V(r,\pm\pi) = g_2^{\pm}(r), \\ D^{\alpha}V = 0 \quad \text{on } \partial\mathcal{O} \quad \text{for } |\alpha| \leqq 1, \end{cases}$$

where g_k^+, $g_k^- \in H^{1/2}(R_+)$ $(k = 1,2)$; $R_+ = (0,\infty)$, such that $g_k^+(r) = g_k^-(r)$ near $r = 0$ and $g_k^{\pm}(r) = 0$ for $r > 1$, and $f \in H^{-1}(\mathcal{O})$. Here we set

$$\mathcal{O} = \{x \in R^2;\ x = (r\cos\theta, r\sin\theta),\ 0 \leqq r \leqq 1,\ -\pi < \theta < \pi\}.$$

The problem

$$\begin{cases} \Delta^2 V_o = f \quad \text{in } B = \{|x| < 1\}, \\ D^{\alpha}V_o = 0 \quad \text{on } \partial B \quad \text{for } |\alpha| \leqq 1, \end{cases}$$

has a unique solution $V_o \in H^3(B)$.

Setting $V_1 = V - V_o$, we can deduce from (3.3) that

$$\begin{cases} \left(\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}\right)^2 V_1 = 0 \quad \text{in } \mathcal{O}, \\ \dfrac{\partial^2}{\partial r^2} V_1(r,\pm\pi) = h_1^{\pm}(r), \\ \left(\dfrac{1}{r^2}\dfrac{\partial}{\partial\theta} + \dfrac{1}{r}\dfrac{\partial^2}{\partial r\partial\theta}\right) V_1(r,\pm\pi) = h_2^{\pm}(r), \end{cases}$$

where $h_1^{\pm}(r) = g_1^{\pm}(r) - D_{11}V_o(r,\pi)$, $h_2^{\pm}(r) = g_2^{\pm}(r) - D_{12}V_o(r,\pi)$. Here we notice that h_k^+, $h_k^- \in H^{1/2}(R_+)$ $(k = 1,2)$, $h_k^+(r) = h_k^-(r)$ near $r = 0$ and $h_k^{\pm}(r) = 0$ for $r > 1$.

Then we have:

Lemma 3.2 The problem (3.3) has a unique solution V, which is expressed in the form

$$V(x) = \frac{1}{3}(K_1/\sqrt{2\pi})\left\{3\cos(\theta/2) + \cos(3\theta/2)\right\} r^{3/2} + $$
$$+ (K_2/\sqrt{2\pi})\left\{- \sin(\theta/2) - \sin(3\theta/2)\right\} r^{3/2} + W(x),$$

where $W \in H^3(\Theta)$ and

$$K_j = -\sqrt{\frac{\pi}{2}}\int_0^\infty \left\{h_j^+(t) + h_j^-(t)\right\}t^{-1/2}\, dt.$$

The polar stress components is expressed by means of Airy's stress function $\widetilde{U}$ as follows:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\widetilde{U}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\widetilde{U}}{\partial\theta^2},\ \sigma_{\theta\theta} = \frac{\partial^2\widetilde{U}}{\partial r^2},\ \sigma_{r\theta} = \frac{1}{r}\frac{\partial\widetilde{U}}{\partial\theta} - \frac{\partial^2\widetilde{U}}{\partial r\partial\theta}\ .$$

Putting the above equations into transformations

$$\sigma_{11} = \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - \sigma_{r\theta}\sin 2\theta,$$
$$\sigma_{22} = \sigma_{rr}\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + \sigma_{r\theta}\sin 2\theta,$$
$$\sigma_{12} = (\sigma_{rr} - \sigma_{\theta\theta})\sin\theta\cos\theta + \sigma_{r\theta}(\cos^2\theta - \sin^2\theta),$$

we thus have

Theorem 3.3 Let u be a solution of the problem (2.5) under a load (f,F). We introduce local polar coordinates related to each of the crack tips $\{\sigma_1,\sigma_2\}$ (see Fig. 5); let r_j be the distance from x to σ_j and θ_j the angle between the vector $\overrightarrow{\sigma_j x}$ and the line $x_2 = 0$. Then there exist constants $K_{1,j}$ and $K_{2,j}$; $j = 1,2$ such that

$$(3.4)\qquad \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \sum_{j=1}^{2} \frac{K_{1,j}}{2\mu\sqrt{2\pi}} \begin{pmatrix} \cos\frac{\theta_j}{2}(\kappa-1+2\sin^2\frac{\theta_j}{2}) \\ \sin\frac{\theta_j}{2}(\kappa+1-2\cos^2\frac{\theta_j}{2}) \end{pmatrix}\zeta(r_j)r_j^{1/2} +$$

$$+\sum_{j=1}^{2}\frac{K_{2,j}}{2\mu\sqrt{2\pi}}\begin{Bmatrix}\sin\frac{\theta_j}{2}\left(\kappa+1+2\cos^2\frac{\theta_j}{2}\right)\\ -\cos\frac{\theta_j}{2}\left(\kappa-1-2\sin^2\frac{\theta_j}{2}\right)\end{Bmatrix}\zeta(r_j)r_j^{1/2}+\begin{Bmatrix}w_1(x)\\ w_2(x)\end{Bmatrix},$$

where $\kappa=\frac{3-\nu}{1+\nu}$ (ν; Poisson's ratio), $\zeta\in C_0^\infty(\overline{R_+})$ such that $\zeta(r)=1$ for $0\leqq r<a/2$ and $\zeta(r)=0$ for $r>a/2$, and $w_1, w_2\in \mathbb{H}^1(\Omega)$ are regular in G.

In fracture mechanics the constants $K_{1,j}$ and $K_{2,j}$ are called the stress intensity factors for mode 1 (opening) and mode 2 (inplane shearing), respectively, see Lemma 3.2 and Fig. 6.

Fig. 5

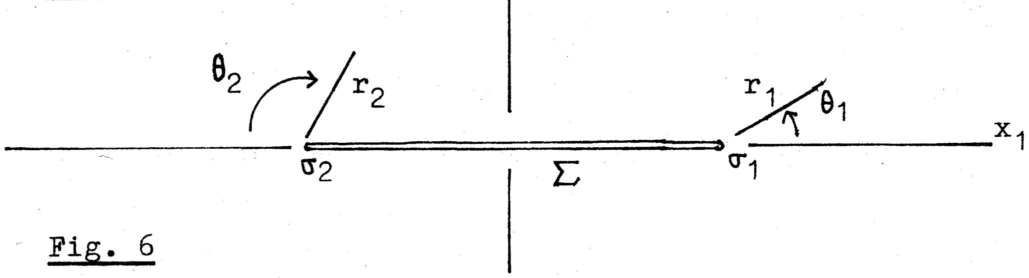
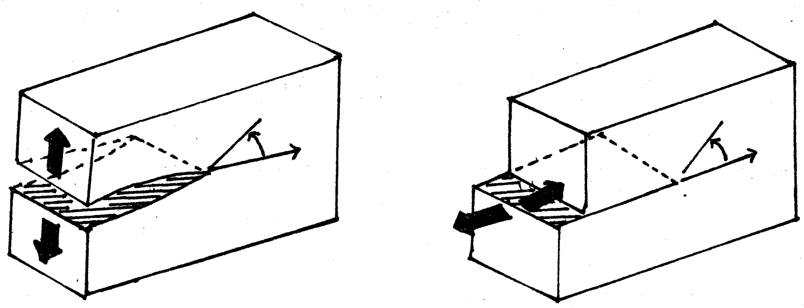


Fig. 6



Mode 1 Mode 2

Each mode represents deformation shown in above figure.

Mode 3 is obtained from antiplane strain state, but we study only plane stress state for the sake of simplicity.

Combining Theorem 2.2 with Theorem 3.3, we obtain the following.

Theorem 3.4 J-integral (2.8) acts only on the singular terms given in (3.4). Moreover

$$J_k(u) = \frac{1}{E}\left\{K_{1,k}^2 + K_{2,k}^2\right\},$$

where E is Young's modulus, and hence by (2.7)

$$G(\mathcal{L};\{\Sigma(t)\}) = \frac{1}{E}\sum_{k=1}^{2}\left\{K_{1,k}^2 + K_{2,k}^2\right\}.$$

It is important how these constants $K_{1,k}$ and $K_{2,k}$ depend on the given load and the crack, but, in general, we do not know it. For special cases we refer to Ishida[5], [12], [13].

Proof Substituting (3.4) into (2.8), we obtain

$$J_k(u) = J_k(z) + J_k(w) + M_k(z,w)$$

where z is the singular term of u, w the regular term of u, and

$$M_k(z,w) = \int_{C_k}\left[a_{ijkl}\sigma_{kl}(z)\varepsilon_{ij}(w)n_1 - \sigma_{ij}(u)n_jD_1w_i\right]d\ell - \\ - \int_{C_k}\sigma_{ij}(w)n_jD_1z_i\, d\ell.$$

Since w is regular at σ_k, $J_k(w) = 0$. Using the Schwarz inequality, we have

$$|M_k(z,w)| \leqq C\,\|z\|_{1,C_k}\|w\|_{1,C_k}.$$

Choosing $C_k = \{|x| = \rho\}$, we obtain the inequalities

$$\|z\|_{1,C_k} \leqq C\int_0^{2\pi}(\rho^{-1/2})^2\,\rho d\theta = 2\pi C,$$

$$\|w\|_{1,C_k} \leqq C\,\|w\|_{2,A_k'} \quad \text{(by the trace theorem)},$$

in which we can take for C the constant independent of ρ, so that we have

$$|M_k(z,w)| \leqq C \|w\|_{2,A_k'},$$

where the constant C is independent of ρ. Furthermore we obtain the estimate

$$\left| \int_{A_k'} f\cdot(D_1 z)\, dx \right| \leqq C \|f\|_{0,A_k'} \|z\|_{1,A_k'}.$$

Then, letting $\rho \longrightarrow 0$, it is clear that only the singular terms of (3.4) contribute. An explicit calculation based on these singular terms leads to

$$J_k(u) = \frac{1}{E}(K_{1,k}^2 + K_{2,k}^2).$$

Finally we shall state further results on J-integrals.

4 Comments 1) Let us consider the following direction of crack extension (see Fig. 7):

(4.1) Let e_1 be the unit vector in the x_1-direction and let $\sigma_k(t)$; $k = 1,2$, the crack tips of newly created crack $\Sigma(t)$ by extending the crack Σ such that

(i) the map $t \longmapsto \sigma_k(t)$ is a smooth curve,

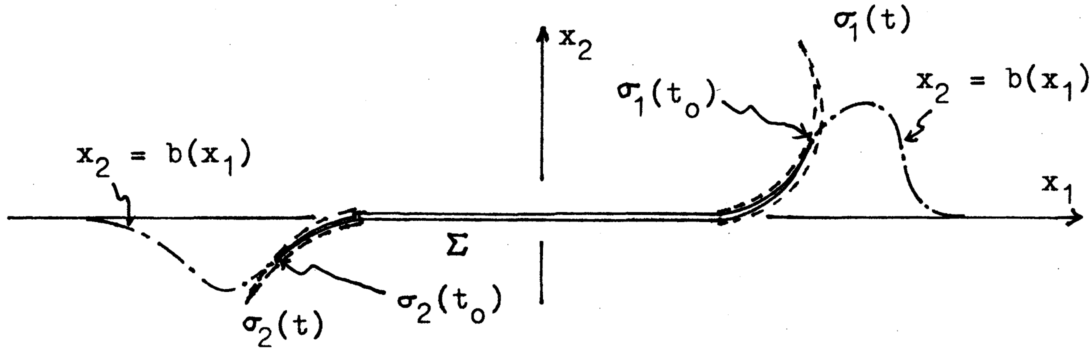
(ii) there exist positive number α, β such that

$$\sigma_1'(t)|_{t=0} = \alpha e_1,\ \sigma_2'(t)|_{t=0} = -\beta e_1.$$

Then, by implicit function theorem, there exist a positive number t_o and a smooth curve

$$\gamma = \left\{(x_1, b(x_1));\ x_1 \in R,\ b \in C_0^\infty(R)\right\}$$

such that $\Sigma(t_o) \subset \gamma$ (see Fig. 7).



Then we have (see [9]):

Theorem 4.1 We denote by $\tau(x)$ the vector field $x \to (1, b'(x_1))$ and by X_τ the differential operator $D_1 + b'(x_1)D_2$. Then the energy release rate $G(\mathcal{L};\{\Sigma(t)\})$ along the crack extension $\{\Sigma(t)\}$ given in (4.1) under the load $\mathcal{L}$ is independent of the given numbers α, β and

$$G(\mathcal{L};\{\Sigma(t)\}) = \sum_{k=1}^{2} J_k(u;\tau),$$

where $u = T(\mathcal{L})$ and $J_k(u;\tau)$ is given by

$$J_k(u;\tau) = (-1)^{k-1}\left\{\int_{C_k} [W(\tau\cdot n) - s\cdot(X_\tau u)]\, d\ell - \int_{A'_k} f\cdot(X_\tau u)\, dx - \int_{A'_k}[\sigma_{ij}(u)D_j\tau^h D_h u_i - W(\mathrm{div}\,\tau)]\, dx\right\},$$

where C_k, A'_k, W, s, n and $d\ell$ are the same symbols given in Theorem 2.2.

By virtue of Theorem 3.3 and 3.4, we can prove that

Corollary 4.2 For any crack extensions $\{\Sigma(t)\}$ as in (4.1), the energy release rates $G(\mathcal{L};\{\Sigma(t)\})$ take the same value

$$\frac{1}{E}\sum_{k=1}^{2}\left\{K_{k,1}^2 + K_{2,k}^2\right\}.$$

2) Next we shall consider general elastic bodies (or plates). When the elastic body does not contain the crack or another defect, i.e., elastic fields are regular (see Definition 2.3), J-integral gives rise to a coservation law for regular elasto-static fields appropriate to homogeneous but not necessarily isotropic solids. Günther[3], and Knowles and Sternberg[6] derive three types of surface (path) integrals, which is closely related to a famous work by Noether[8]. A generalization of these surface integrals are considered in [9]. By use of these surface integrals (conservation laws), physical interpretations of energy release rates are given in Eshelby[2], Budiansky and Rice[1].

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