Nonlinear Stefan Problems in One-Space Dimension
(An Approach by the Theory of Subdifferential Operators)

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Introduction.

In the physical processes we find many phenomena with change of state, such as melting of ice, recrystallization of metals, evaporation, condensation and flow in porous media. Stefan problems in mathematics represent such physical models and have been studied by many authors (e.g., [3, 4, 6 - 20, 25 - 28, 30, 32, 33, 35] and their references).

The present paper is devoted to the study of Stefan problems for nonlinear parabolic differential equations of the forms

\[(\#) \quad u_t - \beta(u)_{xx} = f, \quad (**) \quad u_t - (|u_x|^{p-2}u_x)_x = f\]

in one-space dimension, where \(\beta: R \to R\) is a given function with \(\beta(0) = 0\) which is strictly increasing and \(\beta\)-Lipschitz continuous, and \(p\) is a number satisfying \(2 \leq p < \infty\). Equations (\#) and (**) are special cases of the following general form

\[(***) \quad u_t - (|\beta(u)_x|^{p-2}\beta(u)_x)_x = f, \]

so we shall deal with equation (***) in what follows.

Given a non-negative number \(\lambda_0\), functions \(g\) on \([0, T]\) (\(T\) is a fixed positive number), \(u_0\) on \([0, \lambda_0]\) and \(f\) on \([0, T] \times [0, \infty)\), our problem (one phase Stefan problem) is to find a non-negative function \(x = \lambda(t)\) on \([0, T]\) and a function \(u = u(t, x)\) on \([0, T] \times [0, \infty)\) such that
\( (E) \quad u_t - (|\beta(u)_x|^{p-2}\beta(u)_x)_x = f \quad \text{for } 0 < t < T, \quad \lambda(t) > 0, \quad 0 < x < \lambda(t) \)

subject to

\( (C1) \quad \lambda(0) = \lambda_0 \text{ and if } \lambda_0 > 0, \text{ then } u(0, x) = u_0(x) \text{ for } 0 < x < \lambda_0, \)

\( (C2) \quad \left\{ \begin{array}{l} |\beta(u)_x(t, 0^+)|^{p-2}\beta(u)_x(t, 0^+) = g(t) \quad \text{for } 0 < t < T, \\ \beta(u)(t, \lambda(t)) = 0 \quad \text{for } 0 < t < T, \end{array} \right. \)

\( (C3) \quad \frac{d\lambda(t)}{dt} = -|\beta(u)_x(t, \lambda(t)-)|^{p-2}\beta(u)_x(t, \lambda(t)-) \quad \text{for } 0 < t < T, \)

where \( \beta(u)_x(t, x^+) \) (resp. \( \beta(u)_x(t, x^-) \)) stands for the right (resp. left) hand partial derivative of \( \beta(u)(t, x) \) at \( x \) with respect to \( x \). The unknown boundary \( x = \lambda(t) \) is called the free boundary.

To this kind of problems there are such approaches as listed below:

(a) difference method (cf. [32, 35]),
(b) approach by the theory of nonlinear semigroups (cf. [3, 7, 25]),
(c) reduction to variational inequalities (cf. [8, 9, 11, 16 - 18, 28]),
(d) reduction to nonlinear integral equations (cf. [12, 15, 26, 32]),

etc. In this paper we employ (d) with (c). A class of nonlinear Stefan problems was treated earlier by Kyner [26], in which he established an existence and uniqueness theorem by employing (d) and by making use of a strong maximum principle of Nirenberg [29] for parabolic equations with variable coefficients. But his method is not directly available to our case, in particular to the case that \( p > 2 \) or \( \beta \) is not smooth. Our approach to problem \{(E), (C1) - (C3)\}, which is different from that of Kyner in some points of view, is based upon recent results on the existence, uniqueness and stability of solutions to nonlinear evolution equations involving subdifferential operators of time-dependent convex functions on Hilbert spaces (cf. [1, 2, 21 - 23, 31, 34, 36]).
Notations. For a (real) Banach space $V$ we denote by $|\cdot|_V$ the norm in $V$, by $V^*$ its dual and by $(\cdot, \cdot)_V$ the duality pairing between $V^*$ and $V$; in case $V$ is a Hilbert space and is identified with its dual space, we mean by $(\cdot, \cdot)_V$ the inner product in $V$.

By an operator $A$ from a Banach space $V$ into another Banach space $W$ we mean that to each $v$ in $V$, $A$ assigns a subset $Av$ of $W$, namely $A$ is a multi-valued mapping from $V$ into $W$; in particular, if $Av$ consists of at most one element of $W$ for every $v$ in $V$, then $A$ is called singlevalued. For an operator $A: V \to W$ the set $D(A) = \{v \in V; Av \neq \emptyset\}$ is called the domain.

Let $\phi$ be a lower semi-continuous convex function on a Hilbert space $H$ with values in $(-\infty, \infty]$ such that $\phi \not< \infty$ on $H$. Then the set $D(\phi) = \{z \in H; \phi(z) < \infty\}$ is called the effective domain of $\phi$, and the subdifferential $\partial \phi$ of $\phi$ is an operator from $H$ into itself defined as follows: $z^* \in \partial \phi(z)$ if and only if $z \in D(\phi)$, $z^* \in H$ and

$$(z^*, y - z)_H \leq \phi(y) - \phi(z), \quad \forall \ y \in H.$$ 

For fundamental properties of $\partial \phi$ we refer to a book of Brézis [5].

1. Formulation as a quasi-variational problem.

Let $2 \leq p < \infty$ and $0 < T < \infty$ be numbers which are fixed, and set for simplicity

$$H = L^2(0, \infty), \quad X = W^{1,p}(0, \infty).$$

Let $\beta: R \to R$ be a function with $\beta(0) = 0$ and assume that $\beta$ is strictly increasing and bi-Lipschitz continuous on $R$, i.e.,

$$\frac{1}{c_{\beta}}|r - r_1|^2 \leq (\beta(r) - \beta(r_1))(r - r_1) \leq c_{\beta}|r - r_1|^2, \quad \forall \ r, r_1 \in R$$

with a positive constant $c_{\beta}$.
Given a non-negative continuous function \( \lambda: [0, T] \rightarrow \mathbb{R} \) and a continuous function \( g: [0, T] \rightarrow \mathbb{R} \), we define for each \( t \) in \( [0, T] \)

\[
K_\lambda(t) = \{ z \in X; z(x) = 0, \quad \forall \ x \geq \lambda(t) \}
\]

and

\[
\phi_{\lambda, g}^t(z) = \begin{cases} 
\frac{1}{p} \int_0^\infty |z_x|^p \, dx + g(t)z(0) & \text{if } z \in K_\lambda(t), \\
\infty & \text{otherwise.}
\end{cases}
\]

Clearly \( \phi_{\lambda, g}^t \) is a lower semi-continuous convex function on \( H \) with \( D(\phi_{\lambda, g}^t) = K_\lambda(t) \). We then consider the nonlinear evolution equation

\[
u'(t) + \partial \phi_{\lambda, g}^t(Bu(t)) \ni f(t) \quad \text{for } 0 < t < T,
\]

where the unknown \( u \) is an \( H \)-valued function on \( [0, T] \), \( u'(t) = (d/dt)u(t) \) and \( B \) is the singlevalued operator from \( H = D(B) \) into itself defined by

\[
[Bz](x) = \beta(z(x)) \quad \text{for } z \in H \text{ and } x \in [0, \infty).
\]

By our assumption we see that \( B \) is Lipschitz continuous on \( H \) with \( c_\beta \) as a Lipschitz constant as well as \( B^{-1} \) with \( 1/c_\beta \) as a Lipschitz constant.

**Definition 1.1.** Let \( \lambda, g \) be as above, \( u_0 \) be in \( H \) and \( f \) in \( L^2(0, T; H) \). Then we mean by VP\((\lambda, g, u_0, f)\) the Cauchy problem for (1.2) to find a function \( u \) in \( C([0, T]; H) \) such that

(A1) \( u \) is in \( W^{1,2}(0, T; H) \) and \( u(0) = u_0 \);

(A2) the function \( t \mapsto \phi_{\lambda, g}^t(Bu(t)) \) is bounded on \( [0, T] \);

(A3) \( u'(t) + \partial \phi_{\lambda, g}^t(Bu(t)) \ni f(t) \) for a.e. \( t \) in \( [0, T] \).

Such a function \( u \) is called a (strong) solution to VP\((\lambda, g, u_0, f)\).

**Proposition 1.1.** Let \( \lambda, g, u_0, f \) be as in Definition 1.1. Then a solution \( u \) to VP\((\lambda, g, u_0, f)\) is able to be characterized by the following system:
\[
\begin{align*}
\{ u \in W^{1,2}(0, T; H) \text{ with } u(0) &= u_0, \\
\beta(u) &\in L^\infty(0, T; X), \\
\beta(u)(t, \cdot) &\in K_\lambda(t) \text{ (hence } \beta(u)(t, \lambda(t)) = 0 \text{) for all } t \in [0, T], \\
\left( u_t(t, \cdot) - (|\beta(u)_x(t, \cdot)|^{p-2}\beta(u)_x(t, \cdot))_x \right)_x &= f(t, \cdot) \\
\text{in the distribution sense on } (0, \lambda(t)) \text{ for a.e. } t \in I_0, \\
|\beta(u)_x(t, 0+)|^p - |\beta(u)_x(t, 0+)|^p &= g(t) \text{ for a.e. } t \in I_0,
\end{align*}
\]

where \( I_0 = \{ t \in [0, T]; \lambda(t) > 0 \} \). Moreover, \( \beta(u)_x(t, \lambda(t)) \) exists for a.e. \( t \in I_0 \).

Proof. Obviously (1.3) follows from (A1) and (A2). As is easily seen, (A3) can be written in the following equivalent form:

\[
\begin{align*}
\begin{cases}
(u'(t) - f(t), z)_H + \int_0^\infty |\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x)z_x(x)dx \\
\quad + g(t)z(0) = 0, \quad \forall \ z \in K_\lambda(t), \quad \text{for a.e. } t \in [0, T].
\end{cases}
\end{align*}
\]

We see from (1.6) that (1.4) holds and hence \( (|\beta(u)_x(t, \cdot)|^{p-2}\beta(u)_x(t, \cdot))_x = u_t(t, \cdot) - f(t, \cdot) \in L^2(0, \lambda(t)) \) for a.e. \( t \in I_0 \). This implies that \( |\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x) \) is an absolutely continuous function of \( x \) on \( (0, \lambda(t)) \) and \( \beta(u)_x(t, 0+) \) exists for a.e. \( t \in I_0 \) as well as \( \beta(u)_x(t, \lambda(t)) \), so that by integration by parts we obtain (1.5) from (1.6). Similarly we can show the converse. Q.E.D.

Now we are going to give a quasi-variational formulation associated with our Stefan problem ((E), (C1) - (C3)).

Definition 1.2. Let \( \ell_0 \) be a non-negative number, \( u_0 \) be in \( H \), \( g \) in \( C([0, T]) \) and \( f \) in \( L^2(0, T; H) \). Then we mean by \( \text{QVP}(\ell_0, g, u_0, f) \) to find a couple \( (\ell, u) \) such that

\[
\begin{align*}
\ell &\in W^{1,2}(0, T) \text{ and } \ell \geq 0 \text{ on } [0, T];
\end{align*}
\]
(B2) \( u \) is a solution to VP(\( \ell \), g, \( u_o \), f);

(B3) \[
\ell(t) = \ell_o - \int_0^t g(r)dr + \int_0^t \int_0^{\ell_r} u_o(x)dx + \int_0^t \int_0^{\ell(r)} f(r, x)dxdr - \int_0^\infty u(t, x)dx \quad \text{for all } t \in [0, T].
\]

Proposition 1.2. Let \( \ell_o \), g, \( u_o \), f be as in Definition 1.2 and \{\ell, u\} be a solution to QVP(\( \ell_o \), g, \( u_o \), f). Assume that g is non-positive and \( u_o \), f are non-negative. Then (1.3), (1.4) and the following (1.5)', (1.7) hold:

\[
(1.5)' \quad |\beta(u)_x(t, 0^+)|^{p-2}\beta(u)_x(t, 0^+) = g(t) \quad \text{for a.e. } t \in [0, T],
\]

\[
(1.7) \quad \frac{d\ell(t)}{dt} = - |\beta(u)_x(t, \ell(t)-)|^{p-2}\beta(u)_x(t, \ell(t)-) \quad \text{for a.e. } t \in [0, T].
\]

Proof. By Proposition 1.1 we have (1.3), (1.4), (1.5) and \( \beta(u)_x(t, \ell(t)-) \) exists for a.e. \( t \in I_o (= \{ t \in [0, T]; \ell(t) > 0 \}) \). From (B3) with (1.5) it follows that

\[
\frac{d\ell(t)}{dt} = - g(t) + \int_0^{\ell(t)} f(t, x)dx - \int_0^\infty u(t, x)dx
\]

\[
= - g(t) - \int_0^{\ell(t)} (|\beta(u)_x(t, x)|^{p-2}\beta(u)_x(t, x))_x dx
\]

\[
= - |\beta(u)_x(t, \ell(t)-)|^{p-2}\beta(u)_x(t, \ell(t)-)
\]

for a.e. \( t \in I_o \). Also, if \( t \in [0, T] - I_o \), then \( u(t, x) = 0 \) for all \( x \geq 0 \) and (B3) implies \( g(t) = 0 \). Therefore

\[
|\beta(u)_x(t, 0^+)|^{p-2}\beta(u)_x(t, 0^+) = 0 = g(t) \quad \text{for all } t \in (0, T] - I_o
\]

and

\[
\frac{d\ell(t)}{dt} = 0 = - |\beta(u)_x(t, \ell(t)-)|^{p-2}\beta(u)_x(t, \ell(t)-) \quad \text{for a.e. } t \in [0, T] - I_o.
\]

Thus (1.5)' and (1.7) are satisfied. Q.E.D.
By the above proposition we see that QVP($\ell_o$, $g$, $u_o$, $f$) is a quasi-variational problem associated with {(E), (C1) - (C3)}. 

Our results on QVP($\ell_o$, $g$, $u_o$, $f$) are stated as follows.

Theorem 1.1. Let $\ell_o \geq 0$, $u_o \in H$ be non-negative, $g \in C([0, T])$ be non-positive and $f \in L^2(0, T; H)$ be non-negative. Then we have:

(a) If $\{\ell, u\}$ is a solution to QVP($\ell_o$, $g$, $u_o$, $f$), then $u$ is non-negative and $\ell$ is non-decreasing in $t$.

(b) In addition suppose that $u_o \in X$, $u_0 = 0$ on $[\ell_o, \infty)$ and $g \in W^{1, 1}(0, T)$. Then QVP($\ell_o$, $g$, $u_o$, $f$) has at least one solution.

The proof of this theorem will be given in sections 2 and 5. In order to demonstrate the existence of a solution to QVP($\ell_o$, $g$, $u_o$, $f$) we shall introduce a mapping $P$ from a certain compact convex subset $S$ of $C([0, T])$ into itself defined as follows:

$$[P\ell](t) = \ell_o - \int_0^t g(r)dr + \int_0^t u_0(x)dx + \int_0^t f(r, x)dxdr$$

$$- \int_0^\infty u^\ell(t, x)dx \quad \text{for each } \ell \in S \text{ and } t \in [0, T],$$

where $u^\ell$ is a solution to VP($\ell$, $g$, $u_o$, $f$). We shall show that there is an element $\ell$ of $S$ satisfying $P\ell = \ell$ by a fixed point theorem and that the couple $\{\ell, u^\ell\}$ is a solution to QVP($\ell_o$, $g$, $u_o$, $f$).

The problem of uniqueness for a solution to QVP($\ell_o$, $g$, $u_o$, $f$) remains open, but in the special case that $p = 2$ and $f \equiv 0$ we shall show

Theorem 1.2. If $p = 2$, then QVP($\ell_o$, $g$, $u_o$, $f$) has at most one solution for $\ell_o \geq 0$, $g \in C([0, T])$ non-positive, $u_o \in X$ non-negative with $u_o = 0$ on $[\ell_o, \infty)$ and $f \equiv 0$. 

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2. A comparison theorem for VP(\(\ell\), \(g\), \(u_0\), \(f\)).

We show the following comparison theorem for solutions to VP(\(\ell\), \(g\), \(u_0\), \(f\)).

Theorem 2.1. Let \(\ell\) be a non-negative function in \(C([0, T])\), \(g\), \(\bar{g}\) be in \(C([0, T])\) with \(g \leq \bar{g}\) on \([0, T]\), \(u_0\), \(\bar{u}_0\) in \(H\) and \(f\), \(\bar{f}\) in \(L^2(0, T; H)\). Let \(u\) and \(\bar{u}\) be solutions to VP(\(\ell\), \(g\), \(u_0\), \(f\)) and VP(\(\ell\), \(\bar{g}\), \(\bar{u}_0\), \(\bar{f}\)), respectively. Then we have:

\[
|\langle \bar{u}(t) - u(t) \rangle^+|_{L^1(0, L)} \leq |\langle \bar{u}(s) - u(s) \rangle^+|_{L^1(0, L)} + \int_s^t |\langle \bar{f}(r) - f(r) \rangle^+|_{L^1(0, L)} \, dr
\]

(2.1)

for any \(0 \leq s \leq t \leq T\) and any positive number \(L \geq |\ell|_{C([0, T])}\), where \((\cdot)^+\) stands for the positive part of \((\cdot)\).

Proof. Take a sequence \(\{\sigma_n\}\) of smooth functions on \(R\) with non-negative bounded derivatives \(\sigma_n^{'}\) such that \(-1 \leq \sigma_n \leq 1\) on \(R\), \(\sigma_n(0) = 0\) and

\[
\sigma_n(r) \longrightarrow \sigma_0(r) = \begin{cases} 
1 & \text{if } r > 0, \\
0 & \text{if } r = 0 \\
-1 & \text{if } r < 0 
\end{cases} \quad (\text{as } n \to \infty)
\]

for each \(r \in R\). Since \(\sigma_n(\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot))^+) \in K_\lambda(t)\) for all \(t\) in \([0, T]\), we have by (1.6) in the proof of Proposition 1.1

\[
(u'(t) - f(t), \sigma_n(\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot))^+)_{H} + \int_0^\infty |\beta(u)x(t, x)|^{p-2}\beta(u)x(t, x)\frac{\partial}{\partial x}\sigma_n(\beta(\bar{u})(t, x) - \beta(u)(t, x))^+)dx + g(t)\sigma_n(\beta(\bar{u})(t, 0) - \beta(u)(t, 0))^+) = 0
\]

and

\[
(\bar{u}'(t) - \bar{f}(t), \sigma_n(\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot))^+)_{H} = 0
\]
\[ + \int_0^\infty |\beta(\bar{u})_x(t, x)|P^2 \beta(\bar{u})_x(t, x) \frac{\partial}{\partial x_n} \sigma_n([\beta(\bar{u})(t, x) - \beta(u)_x(t, x)]^+) dx \\
+ \bar{g}(t) \sigma_n([\beta(\bar{u})(t, 0) - \beta(u)(t, 0)]^+) = 0 \]

for a.e. \( t \in [0, T] \), from which we infer that

\[ \langle \bar{u}'(t) - u(t), \sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+) \rangle_H \leq \langle \bar{F}(t) - f(t), \sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+) \rangle_H \]

\[ - \int_0^\infty \{ |\beta(\bar{u})_x(t, x)|P^2 \beta(\bar{u})_x(t, x) - |\beta(u)_x(t, x)|P^2 \beta(u)_x(t, x) \} \times \]

\[ \times \frac{\partial}{\partial x_n} \sigma_n([\beta(\bar{u})(t, x) - \beta(u)(t, x)]^+) dx \\
- (\bar{g}(t) - g(t)) \sigma_n([\beta(\bar{u})(t, 0) - \beta(u)(t, 0)]^+) \]

\[ = \langle \bar{F}(t) - f(t), \sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+) \rangle_H \]

\[ - \int_{E(t)} \{ |\beta(\bar{u})_x(t, x)|P^2 \beta(\bar{u})_x(t, x) - |\beta(u)_x(t, x)|P^2 \beta(u)_x(t, x) \} \times \]

\[ \times (\beta(\bar{u})_x(t, x) - \beta(u)_x(t, x)) \sigma_n([\beta(\bar{u})(t, x) - \beta(u)(t, x)]^+) dx \\
- (\bar{g}(t) - g(t)) \sigma_n([\beta(\bar{u})(t, 0) - \beta(u)(t, 0)]^+) \]

\[ \leq \langle \bar{F}(t) - f(t), \sigma_n([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+) \rangle_H \]

for a.e. \( t \in [0, T] \), where \( E(t) = \{ x; \beta(\bar{u})(t, x) > \beta(u)(t, x) \} \). Letting \( n \to \infty \), we get for a.e. \( t \) in \( [0, T] \)

\[ (2.2) \quad \langle \bar{u}'(t) - u'(t), \sigma_o([\beta(\bar{u})(t, \cdot) - \beta(u)(t, \cdot)]^+) \rangle_H \leq |(\bar{F}(t) - f(t))^+|_{L^1(0, L)} \]

Since \( \sigma_o([\beta(\bar{u})(t, x) - \beta(u)(t, x)]^+) = \sigma_o([\bar{u}(t, x) - u(t, x)]^+) \) and

\[ \langle \bar{u}'(t) - u'(t), \sigma_o([\bar{u}(t, \cdot) - u(t, \cdot)]^+) \rangle_H = \frac{d}{dt} |(\bar{u}(t) - u(t))^+|_{L^1(0, L)} \]

it follows from (2.2) that

\[ \frac{d}{dt} |(\bar{u}(t) - u(t))^+|_{L^1(0, L)} \leq |(\bar{F}(t) - f(t))^+|_{L^1(0, L)} \]

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for a.e. $t \in [0, T]$. Integrating this inequality, we get (2.1). Q.E.D.

Remark 2.1. The technic adopted above is found in Bénilan [3] and Demlamlan [8].

The following corollaries are immediate consequences of Theorem 2.1.

Corollary 1. Let $l$, $g$, $u_0$ and $f$ be as in Theorem 2.1. Then $VP(l, g, u_0, f)$ has at most one solution.

Corollary 2. Let $l$, $g$, $u_0$ and $f$ be as in Theorem 2.1 and further assume that $g$ is non-positive and $u_0$, $f$ are non-negative. Then a solution to $VP(l, g, u_0, f)$ is non-negative.

Proof of (a) of Theorem 1.1: Let $(l, u)$ be any solution to $QVP(l_0, g, u_0, f)$. Then, according to Corollary 2 to Theorem 2.1, $u$ is non-negative as well as $\beta(u)$. Therefore

$$\beta(u)(t, l(t))- \leq 0 \quad \text{for a.e. } t \in [0, T]$$

and by (1.7) of Proposition 1.2

$$\frac{dl(t)}{dt} \geq 0 \quad \text{for a.e. } t \in [0, T],$$

which shows that $l$ is non-decreasing in $t$. Q.E.D.

3. Some lemmas.

Throughout this section we fix $l \in C([0, T])$ and $g \in W^{1,1}(0, T)$, and assume that $l$ is non-decreasing and non-negative. For simplicity by $\phi^t$ we denote the convex function $\phi^t_{\lambda, g}$ given by (1.1).

First of all we show a lemma on the $t$-dependence of $\phi^t$.

Lemma 3.1. With $C = \left(\frac{1}{C} + 1\right) \frac{1}{p'} + 1$ and $p' = \frac{p}{p-1}$, we have
\[ \phi^s(z) \geq -C, \quad \forall \ s \in [0, T], \quad \forall \ z \in K_\lambda(s), \]

\[ \phi^t(z) - \phi^s(z) \leq C |g(t) - g(s)| (\phi^s(z) + C), \quad \forall \ s, t \in [0, T] \text{ with } s \leq t, \quad \forall \ z \in K_\lambda(s). \]

Proof. Let \( 0 \leq s \leq T \) and \( z \in K_\lambda(s) \). Then for any \( \delta > 0 \) we observe that

\[
|z(0)| \leq \int_0^{\lambda(s)} |z_x| \, dx \leq \int_0^{\lambda(s)} \left( \frac{\delta |z_x|^p}{p} + \frac{\delta^{1-p'} \lambda(s)}{p'} \right) \, dx
\]

\[
= \frac{\delta}{p} \int_0^{\lambda(s)} |z_x|^p \, dx + \frac{\delta^{1-p'} \lambda(s)}{p'}
\]

\[
= \delta \phi^s(z) - \delta g(s) z(0) + \frac{\delta^{1-p'} \lambda(s)}{p'},
\]

so that by taking \( \delta = (|g|_{C([0, T])} + 1)^{-1} \) we have

\[
\phi^s(z) = \frac{1}{p} \int_0^{\lambda(s)} |z_x|^p \, dx + g(s) z(0)
\]

\[
\geq \frac{1}{p} (1 - \delta |g|_{C([0, T])}) \int_0^{\lambda(s)} |z_x|^p \, dx - \frac{|g|_{C([0, T])} \delta^{1-p'} \lambda(T)}{p'}
\]

\[
\geq -C
\]

and for any \( t > s \)

\[
\phi^t(z) - \phi^s(z) = (g(t) - g(s)) z(0)
\]

\[
\leq \frac{\delta}{1 - |g|_{C([0, T])}} |g(t) - g(s)| (\phi^s(z) + \frac{\delta^{1-p'} \lambda(T)}{p'})
\]

\[
\leq C |g(t) - g(s)| (\phi^s(z) + C).
\]

Thus (3.1) and (3.2) hold. Q.E.D.

Next, we consider the regularization \( \phi^t_\lambda, \quad 0 < \lambda \leq 1, \) of \( \phi^t \) as is defined by

\[
\phi^t_\lambda(z) = \inf_{y \in H} \left\{ \frac{1}{2\lambda} |z - y|^2 + \phi^t(y) \right\}, \quad z \in H.
\]

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In what follows, without proof we shall use some of well-known properties of \( \phi_t^\lambda \) such as

1. \( \phi_t^\lambda \) is finite, continuous and convex on \( H \),
2. the subdifferential \( \partial \phi_t^\lambda : H = D(\partial \phi_t^\lambda) + H \) is singlevalued and Lipschitz continuous with \( 1/\lambda \) as a Lipschitz constant,
3. \( J^t_\lambda = (I + \lambda \partial \phi_t^\lambda)^{-1} \) is contractive on \( H \) and \( \partial \phi_t^\lambda = (I - J^t_\lambda)/(\lambda \mu) \),
4. \( J^\lambda_\mu z = J^\lambda_\mu (\lambda z + (I - \lambda J^t_\lambda z)) \) for \( \mu, \lambda > 0 \) and \( z \in H \),
5. \( \phi_t^\lambda(z) = |z - J^t_\lambda z|^2/(2\lambda) + \phi_t^\lambda(J^t_\lambda z) \) for \( z \in H \),

etc., for which we can refer to a book of Brézis [5].

**Lemma 3.2.** \( \phi_t^\lambda(z) \geq -C \) for any \( t \in [0, T] \), \( \lambda \in (0, 1] \) and \( z \in H \), where \( C \) is the same constant as in Lemma 3.1.

**Proof.** By (3.1) of Lemma 3.1,

\[
\phi_t^\lambda(z) = \frac{1}{2\lambda}|z - J^t_\lambda z|^2_H + \phi_t^\lambda(J^t_\lambda z) \geq \phi_t^\lambda(J^t_\lambda z) \geq -C. 
\]

Q.E.D.

**Lemma 3.3.** Let \( C \) be as in Lemma 3.1. Then we have:

1. \( |J^t_\lambda z|_H \leq |z|_H + 2\sqrt{C} \), \( \forall \ t \in [0, T] \), \( \forall \ \lambda \in (0, 1] \), \( \forall \ z \in H \);
2. \( |\partial \phi_t^\lambda(z)|_H \leq \frac{2}{\lambda}(|z|_H + \sqrt{C}) \), \( \forall \ t \in [0, T] \), \( \forall \ \lambda \in (0, 1] \), \( \forall \ z \in H \);
3. \( -C \leq \phi_t^\lambda(z) \leq \frac{2}{\lambda}(|z|_H + \sqrt{C})|z|_H \), \( |\phi_t^\lambda(z) - \phi_t^\lambda(z_1)| \leq \frac{2}{\lambda}(|z|_H + |z_1|_H + 2\sqrt{C})|z - z_1|_H \), \( \forall \ t \in [0, T] \), \( \forall \ \lambda \in (0, 1] \), \( \forall \ z, z_1 \in H \).

**Proof.** Using Lemma 3.2, we observe

\[
0 \geq \phi_t^\lambda(0) = \frac{1}{2}|J^t_\lambda 0|_H^2 + \phi_t^\lambda(J^t_\lambda 0) \geq \frac{1}{2}|J^t_\lambda 0|_H^2 - C,
\]

so

\[
|J^t_\lambda 0|_H \leq \sqrt{2C}, \quad \forall \ t \in [0, T].
\]

Making use of the relation \( J^t_\lambda z = J^t_\lambda (\lambda z + (1 - \lambda)J^t_\lambda z) \), \( z \in H \), we see that
\[ |J^t_{\lambda}0|_H \leq |J^t_{\lambda}0|_H + (1 - \lambda)|J^t_{\lambda}0|_H \leq 2|J^t_{\lambda}0|_H \]

and hence

\[(3.3) \quad |J^t_{\lambda}0|_H \leq 2\sqrt{c}, \quad \forall t \in [0, T], \quad \forall \lambda \in (0, 1].\]

Since \( J^t_\lambda \) is contractive on \( H \), we have by (3.3)

\[ |J^t_\lambda z|_H \leq |J^t_\lambda z - J^t_{\lambda}0|_H + |J^t_{\lambda}0|_H \leq |z|_H + |J^t_{\lambda}0|_H \leq |z|_H + 2\sqrt{c} \]

for any \( t \in [0, T], \lambda \in (0, 1] \) and \( z \in H \), which shows (1). Next from (1) it follows that

\[ |\phi^t_\lambda(z)|_H = \frac{1}{\lambda}|z - J^t_\lambda z|_H \leq \frac{2}{\lambda}(|z|_H + \sqrt{c}) \]

for any \( t \in [0, T], \lambda \in (0, 1] \) and \( z \in H \). Hence we have (ii). To show (iii) we note that

\[ \phi^t_\lambda(z) - \phi^t_\lambda(z_1) \leq (\phi^t_\lambda(z), z - z_1)_H \leq |\phi^t_\lambda(z)|_H|z - z_1|_H, \]

\[ \phi^t_\lambda(z) - \phi^t_\lambda(z_1) \geq (\phi^t_\lambda(z_1), z - z_1)_H \geq -|\phi^t_\lambda(z_1)|_H|z - z_1|_H \]

and

\[ \phi^t_\lambda(z) \leq |\phi^t_\lambda(z)|_H|z|_H \]

for any \( t \in [0, T], \lambda \in (0, 1] \) and \( z, z_1 \in H \). From these inequalities, (ii) and Lemma 3.2 we derive that

\[ |\phi^t_\lambda(z) - \phi^t_\lambda(z_1)| \leq \left( |\phi^t_\lambda(z)|_H + |\phi^t_\lambda(z_1)|_H \right)|z - z_1|_H \]

\[ \leq \frac{2}{\lambda}(|z|_H + |z_1|_H + 2\sqrt{c})|z - z_1|_H \]

and

\[ -C \leq \phi^t_\lambda(z) \leq \frac{2}{\lambda}(|z|_H + \sqrt{c})|z|_H \]

for any \( t \in [0, T], \lambda \in (0, 1] \) and \( z, z_1 \in H \). Thus we get (iii). Q.E.D.

The following lemma is due to Attouch-Bénilan-Damlamian-Picard [1]
(or Picard [31]).

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Lemma 3.4. Let \( v \in W^{1,1}(0, T; H) \) and \( 0 < \lambda < 1 \). Then \( \phi^t_\lambda(v(t)) \) is differentiable at a.e. \( t \in [0, T] \) and its derivative is integrable on \([0, T] \). Moreover, with the same constant \( C \) as in Lemma 3.1 we have

\[
\frac{d}{dt}\phi^t_\lambda(v(t)) - (\phi^t_\lambda(v(t)), v'(t))_H \leq C|g'(t)|(\phi^t_\lambda(v(t)) + C) 
\]

for a.e. \( t \in [0, T] \).

Proof. Let \( z \in H \). Then by Lemma 3.1 and (iii) of Lemma 3.3 we observe that for \( 0 < s < t < T \)

\[
\phi^t_\lambda(z) - \phi^s_\lambda(z) \leq \phi^t_\lambda(s^z_\lambda) - \phi^s_\lambda(s^z_\lambda) 
\]

\[
\leq C|g(t) - g(s)|(\phi^s_\lambda(z) + C) 
\]

\[
\leq c(z)\int_s^t|g'(r)|dr
\]

with

\[
c(z) = C\left\{ |z|_H + \sqrt{2C} |z|_H + C \right\}.
\]

From (3.6) we see that

\[
t \rightarrow \phi^t_\lambda(z) - c(z)\int_0^t|g'(r)|dr
\]

is non-increasing on \([0, T]\), so that \( \phi^t_\lambda(z) \) is differentiable at a.e. \( t \) in \([0, T]\), its derivative is integrable on \([0, T]\) and

\[
\phi^t_\lambda(z) - \phi^s_\lambda(z) \leq \int_s^t\frac{d}{dr}\phi^r_\lambda(z)dr 
\]

for any \( 0 < s < t < T \).

Moreover,

\[
\frac{d}{dr}\phi^r_\lambda(z) \leq C|g'(r)|(\phi^r_\lambda(z) + C) 
\]

for a.e. \( r \in [0, T] \).

Next, let \( v \in W^{1,1}(0, T; H) \). Using (3.7) and (3.8), we have
\[
\phi^t_\lambda(v(t)) - \phi^s_\lambda(v(s)) - (3\phi^t_\lambda(v(t)), v(t) - v(s))_H \\
\leq \phi^t_\lambda(v(s)) - \phi^s_\lambda(v(s)) \\
\leq \int_s^t \frac{d}{dr} \phi^r_\lambda(v(s)) dr \\
\leq C \int_s^t |g'(r)| (\phi^r_\lambda(v(s)) + C) dr
\]

for any \(0 \leq s < t \leq T\), from which (3.4) follows just as (3.7). Besides, dividing (3.9) by \(t - s\) and letting \(s \to t\), we obtain (3.5) with the help of (iii) of Lemma 3.3. Q.E.D.

Finally we show

Lemma 3.5. For each \(\lambda \in (0, 1]\), the operator \(v \mapsto \phi^t_\lambda(\cdot)(v(\cdot))\) is Lipschitz continuous on \(L^2(0, T; H)\) with \(1/\lambda\) as a Lipschitz constant.

Proof. As was seen in the proof of Lemma 3.4, \(t \mapsto \phi^t_\lambda(v(t))\) is measurable on \([0, T]\) for each \(v\) in \(W^{1,1}(0, T; H)\) and hence for each \(v\) in \(L^2(0, T; H)\). Now we consider a function \(\Phi^\lambda\) on \(L^2(0, T; H)\) which is defined by

\[
\Phi^\lambda(v) = \int_0^T \phi^r_\lambda(v(t)) dt, \quad \forall \ v \in L^2(0, T; H).
\]

Obviously \(\Phi^\lambda\) is finite, continuous and convex on \(L^2(0, T; H)\) (cf. Lemma 3.3). It is also easy to see that the subdifferential \(\partial \Phi^\lambda\): \(L^2(0, T; H) \to L^2(0, T; H)\) is a single-valued operator with \(D(\partial \Phi^\lambda) = L^2(0, T; H)\) and

\[
\partial \Phi^\lambda(v) = \partial \phi^t_\lambda(\cdot)(v(\cdot)), \quad \forall \ v \in L^2(0, T; H),
\]

so that \(\partial \phi^t_\lambda(v(t))\) is (strongly) measurable in \(t \in [0, T]\) for every \(v\) in \(L^2(0, T; H)\). Since \(\partial \phi^t_\lambda\) is Lipschitz continuous on \(H\) with \(1/\lambda\) as a Lipschitz constant for all \(t \in [0, T]\), so is \(\partial \Phi^\lambda\) on \(L^2(0, T; H)\) with the same Lipschitz constant. Q.E.D.
4. Existence and stability of solutions to \( VP(\lambda, g, u_0, f) \).

We first establish an existence theorem for \( VP(\lambda, g, u_0, f) \).

Theorem 4.1. Let \( \lambda \) be a non-negative and non-decreasing function in \( C([0, T]) \). Let \( u_0 \) be in \( K^1(0) \), \( g \) in \( W^{1,1}(0, T) \) and \( f \) in \( L^2(0, T; H) \). Then \( VP(\lambda, g, u_0, f) \) admits a (unique) solution.

Let \( \lambda, g, u_0 \) and \( f \) be in Theorem 4.1. Then in order to construct a solution to \( VP(\lambda, g, u_0, f) \), we consider an approximate problem of the following type:

\[
(u'_{\lambda}(t) + 3\phi_{\lambda, g}^t(Bu_{\lambda}(t)) \in f(t),
\]

\[
u_{\lambda}(0) = u_0,
\]

where \( 0 < \lambda \leq 1 \) and \( \phi_{\lambda, g}^t \) is the regularization of \( \phi_{\lambda, g}^t \). On account of Lemma 3.5, the operator \( \nu \rightarrow 3\phi_{\lambda, g}^t(B\nu(\cdot)) \) is Lipschitz continuous on \( L^2(0, T; H) \) with \( c_B/\lambda \) as a Lipschitz constant. Therefore for each \( \lambda \) in \( (0, 1] \), there is a unique function \( u_{\lambda} \) in \( W^{1,2}(0, T; H) \) satisfying (4.1) a.e. on \( [0, T] \) and \( u_{\lambda}(0) = u_0 \). We want to show that for a certain sequence \( \{\lambda_n\} \) with \( \lambda_n \rightarrow 0 \) (as \( n \rightarrow \infty \)) \( u_{\lambda_n} \) converges to a function in \( C([0, T]; H) \) and this limit is a solution to \( VP(\lambda, g, u_0, f) \).

For simplicity we denote \( \phi_{\lambda, g}^t \) by \( \phi^t \) again. To get estimations for \( \{u_{\lambda}; 0 < \lambda \leq 1\} \) which are independent of \( \lambda \), multiply (4.1) by \( (d/dt)Bu_{\lambda}(t) \).

We then have

\[
(u'_{\lambda}(t), \frac{d}{dt}Bu_{\lambda}(t))_H + (3\phi_{\lambda}^t(Bu_{\lambda}(t)), \frac{d}{dt}Bu_{\lambda}(t))_H = (f(t), \frac{d}{dt}Bu_{\lambda}(t))_H
\]

for a.e. \( t \in [0, T] \). We note here that

\[
(u_{\lambda}(t), \frac{d}{dt}Bu_{\lambda}(t))_H \geq \frac{1}{c_B}||\frac{d}{dt}Bu_{\lambda}(t)||^2_H,
\]

\[
(f(t), \frac{d}{dt}Bu_{\lambda}(t))_H \leq \frac{c_B}{4}||f(t)||^2_H + \frac{1}{4c_B}||\frac{d}{dt}Bu_{\lambda}(t)||^2_H
\]
and by Lemma 3.4

\[ (\phi^t_\lambda(Bu_\lambda(t)), \frac{d}{dt}Bu_\lambda(t))_H \geq \frac{d}{dt}\phi^t_\lambda(Bu_\lambda(t)) - C|g(t)|(|\phi^t_\lambda(Bu_\lambda(t))| + C) \]

for a.e. \( t \in [0, T] \), where \( C \) is the same constant as in Lemma 3.1. Therefore,

\[ (4.2) \quad \frac{3}{4c_B}|\frac{d}{dt}Bu_\lambda(t)|^2_H + \frac{d}{dt}\phi^t_\lambda(Bu_\lambda(t)) \leq k_1(t)\phi^t_\lambda(Bu_\lambda(t)) + k_2(t) \]

for a.e. \( t \in [0, T] \), where

\[ k_1(t) = C|g'(t)|, \quad k_2(t) = C^2|g'(t)| + c_3^2|f(t)|^2_H. \]

Applying Gronwall's inequality to (4.2), we have

\[ (4.3) \quad \phi^t_\lambda(Bu_\lambda(t)) \leq \phi^0(Bu_\lambda(t)) + \int_0^T k_2(r)dr \exp\left(\int_0^T k_1(r)dr\right) \leq C' \]

for all \( t \in [0, T] \) and

\[ (4.4) \quad \frac{3}{4c_B}|\frac{d}{dt}Bu_\lambda|^2_{L^2(0, T; H)} \leq C + \phi^0(Bu_\lambda(t)) + \int_0^T \{C'k_1(r) + k_2(r)\}dr. \]

From (4.3) and (4.4) we obtain the following lemma.

**Lemma 4.1.** There is a positive constant \( M \) such that

\[ |Bu_\lambda(\cdot)|_{W^{1,2}(0, T; H)} \leq M \text{ (hence } |u_\lambda|_{W^{1,2}(0, T; H)} \leq c_B M, \forall \lambda \in (0, 1]\]

and

\[ -C \leq \phi^t_\lambda(Bu_\lambda(t)) \leq M, \quad \forall t \in [0, T], \forall \lambda \in (0, 1], \]

where \( C \) is as in Lemma 3.1.

This lemma guarantees that \( \{Bu_\lambda(\cdot); 0 < \lambda \leq 1\} \) (hence \( \{u_\lambda; 0 < \lambda \leq 1\} \)) is relatively compact in \( C([0, T]; H) \), because the natural injection from \( X \) into \( H \) is compact. Now we can choose a sequence \( \lambda_n \) with \( \lambda_n \to 0 \) such that

\[ u_n = u_\lambda_n + u, \quad Bu_n(\cdot) + Bu(\cdot) \quad \text{in } C([0, T]; H) \]

and

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\[ u'_n + u' \quad \text{weakly in } L^2(0, T; H). \]

Evidently \( u \in W^{1,2}(0, T; H) \) and \( u(0) = u_0 \). Putting \( u_n^*(t) = \partial \phi^t_{\lambda_n}(Bu_n(t)) \), we see that \( u^*_n = f - u'_n + f - u' \) weakly in \( L^2(0, T; H) \). Also, since
\[
 u_n^*(t) = \frac{1}{\lambda_n}(Bu_n(t) - J^t_{\lambda_n}Bu_n(t)) \in \partial \phi^t_{J^t_{\lambda_n}Bu_n(t)},
\]
we get
\[
 J^t_{\lambda_n}Bu_n(t) \to Bu(\cdot) \quad \text{in } L^2(0, T; H),
\]
so the semi-continuity of the operator \( v \to \partial \phi^t(v(\cdot)) \) in \( L^2(0, T; H) \) gives
\[
f(t) - u'(t) \in \partial \phi^t(Bu(t)) \quad \text{for a.e. } t \in [0, T].
\]

Moreover
\[
\phi^t(Bu(t)) \leq \liminf_{n \to \infty} \phi^t_{J^t_{\lambda_n}Bu_n(t)} \leq \liminf_{n \to \infty} \phi^t_{\lambda_n}(Bu_n(t)) \leq M
\]
for all \( t \) in \( [0, T] \). Thus the function \( u \) is a solution to \( VP(\ell, g, u_0, f) \) and Theorem 4.1 was completely proved.

Next, we show a stability theorem for \( VP(\ell, g, u_0, f) \).

**Theorem 4.2.** Let \( \ell_0 \) and \( L \) be numbers such that \( 0 \leq \ell_0 \leq L \) and denote by \( \Lambda \) the set \( \{ \ell \in C([0, T]) : \ell \text{ is non-decreasing in } t, \ell(0) = \ell_0 \text{ and } \ell(T) \leq L \} \). Let \( g \in W^{1,1}(0, T), \) \( u_0 \in X \) with \( u_0 = 0 \) on \( [\ell_0, \infty) \) and \( f \in L^2(0, T; H) \). Then there is a positive constant \( K \) such that
\[
|u^\ell|_{W^{1,2}(0, T; H)} \leq K, \quad |\phi^t_{\ell, g}(Bu^\ell(t))| \leq K, \quad \forall \, t \in [0, T],
\]
for every \( \ell \in \Lambda \), where \( u^\ell \) is a unique solution to \( VP(\ell, g, u_0, f) \).

**Proof.** As is easily checked, the constants \( M \) in Lemma 4.1 and \( C \) in Lemma 3.1 can be taken so as to be bounded, as long as \( \ell \) varies in the set \( \Lambda \). From this fact the conclusion of the theorem follows.

Q.E.D.

This stability theorem plays an important role in solving
\[
QVP(\ell_0, g, u_0, f).
\]
5. Proof of (b) of Theorem 1.1.

Throughout this section, assume that \( \ell_0 \geq 0, u_0 \in X \) is non-negative with \( u_0 = 0 \) on \([\ell_0, \infty)\), \( g \in W^{1,1}(0, T) \) is non-positive and \( f \in L^2(0, T; H) \) is non-negative, and let \( \Lambda \) be the set \( \{ \ell \in C([0, T]); \ell \) is non-decreasing in \( t, \ell(0) = \ell_0 \) and \( \ell(T) \leq L \} \) with \( L \) satisfying

\[
L > \ell_0 + \int_0^{\ell_0} u_0(x) dx - \int_0^T g(r) dr + \sqrt{\ell(T)} |f|_{L^2(0, T; H)}.
\]

We now consider the operator \( P \) on \( \Lambda \) which is defined by (1.8).

Concerning this operator \( P \) we have

Lemma 5.1. \( P(\Lambda) \subset \Lambda \cap W^{1,2}(0, T) \).

Proof. Let \( \ell \in \Lambda \). Then \([P\ell](0) = \ell_0 \) and \( P\ell \in W^{1,2}(0, T) \), since a unique solution \( u^\ell \) to \( VP(\ell, g, u_0, f) \) belongs to \( W^{1,2}(0, T; H) \). By Proposition 1.1 we have

\[
\frac{d}{dt}[P\ell](t) = -g(t) + \int_0^{\ell(t)} f(t, x) dx - \int_0^{\ell(t)} u_t(t, x) dx
\]

\[
= -g(t) - \int_0^{\ell(t)} (|\beta(u^\ell)_x(t, x)|^{p-2}\beta(u^\ell)_x(t, x)) dx
\]

\[
= -|\beta(u^\ell)_x(t, \ell(t)-)|^{p-2}\beta(u^\ell)_x(t, \ell(t)-)
\]

for a.e. \( t \in I_0 \) (= \( \{ t \in [0, T]; \ell(t) > 0 \} \)). Also \( u^\ell \) is non-negative by Corollary 2 to Theorem 2.1 as well as \( \beta(u^\ell) \). Hence

\[
\beta(u^\ell)_x(t, \ell(t)-) \leq 0 \quad \text{for a.e. } t \in I_0,
\]

from which it follows that \( (d/dt)[P\ell](t) \geq 0 \) for a.e. \( t \in I_0 \). For a.e. \( t \) in \([0, T] - I_0 \) we have

\[
\frac{d}{dt}[P\ell](t) = -g(t) \geq 0,
\]

because \( u^\ell(t, x) = 0 \) for all \( x \geq 0 \) if \( t \in [0, T] - I_0 \). Therefore \( P\ell \) is non-
decreasing. By (5.1), \([P\mathcal{L}](T) \leq L\). Thus \(P \mathcal{L} \in \Lambda\). Q.E.D.

Lemma 5.2. \(P\) is continuous on \(\Lambda\) with respect to the topology of \(C([0, T]).\)

Proof. Suppose that \(\ell_n \in \Lambda\) and \(\ell_n \to \ell\) in \(C([0, T])\), and denote by \(u_n\) and \(u\) the solutions to \(VP(\ell_n, g, u_0, f)\) and \(VP(\ell, g, u_0, f)\), respectively. Then, on account of Theorem 4.2, there is a constant \(K\) such that

\[
|u_n|_{W^{1,2}(0, T; H)} \leq K, \quad |\Phi_{\ell_n, g}^{\tilde{t}}(Bu_n(t))| \leq K, \quad \forall n, \quad \forall t \in [0, T].
\]

(5.2)

We note here that for each \(n\) the following holds:

\[
\int_0^T (u'_n(t) - f(t), Bu_n(t) - w(t)) \ dt \leq \phi(w) - \phi(Bu_n),
\]

\[
\forall w \in L^p(0, T; X) \text{ with } w(t) \in \mathcal{K}_n(t) \text{ for a.e. } t \in [0, T],
\]

(5.3)

where

\[
\phi(w) = \frac{1}{p} \int_0^T \int_0^T |w_x(t, x)|^p \ dx \ dt + \int_0^T g(t)w(t, 0) \ dt.
\]

By (5.2), \(\{u_n\}\) is relatively compact in \(C([0, T]; H)\). We want to show that 
\(u_n \to u\) in \(C([0, T]; H)\). For this purpose, let \(\{u_{n_k}\}\) be any subsequence of \(\{u_n\}\) such that 
\(u_{n_k} \to \bar{u}\) (hence \(Bu_{n_k} \to \bar{B}u\)) in \(C([0, T]; H)\). Then we have

\[
Bu_{n_k}(t) \to \bar{B}u(t) \quad \text{weakly in } X \text{ for each } t \in [0, T],
\]

\[
Bu_{n_k} \to \bar{B}u \quad \text{weakly in } L^2(0, T; X)
\]

\[
u_{n_k}' \to \bar{u}' \quad \text{weakly in } L^2(0, T; H)
\]

and by the way

\[
\phi_{\ell, g}^{\tilde{t}}(\bar{B}u(t)) \leq K, \quad \forall t \in [0, T],
\]

\[
\bar{u} \in W^{1,2}(0, T; H), \quad \bar{u}(0) = u_0,
\]
\[(5.4) \quad \liminf_{k \to \infty} \Phi(B_{n_k}) \geq \Phi(B) \cdot \]

Now denote by \(Z\) the set
\[\{v \in L^P(0, T; X); v(t) \in K_{\lambda}(t) \text{ for a.e. } t \in [0, T]\}.\]

Let \(v\) be any function in \(Z\) and \(\epsilon\) be any positive number. Putting \(v_\epsilon(t, x) = v(t, x + \epsilon)\), we see that \(v_\epsilon(t) \in K_{\lambda}(t) \text{ for a.e. } t \in [0, T]\) and for all \(n\) sufficiently large. Hence, taking \(n = n_k\) with \(w = v_\epsilon\) and letting \(k \to \infty\) in (5.3), we obtain by (5.4)
\[\int_0^T (\tilde{u}'(t) - f(t), B\tilde{u}(t) - v_\epsilon(t))_H dt \leq \Phi(v_\epsilon) - \Phi(B\tilde{u}).\]

Furthermore, since \(v_\epsilon \to v\) in \(L^P(0, T; X)\) and \(\Phi(v_\epsilon) \to \Phi(v)\) as \(\epsilon \to 0\),
\[\int_0^T (\tilde{u}'(t) - f(t), B\tilde{u}(t) - v(t))_H dt \leq \Phi(v) - \Phi(B\tilde{u}).\]

This inequality holds for every \(v\) in \(Z\), which is equivalent to
\[f(t) - \tilde{u}'(t) \in \partial \Phi^{\infty}_{\lambda, \delta}(B\tilde{u}(t)) \text{ for a.e. } t \in [0, T]\]
(cf. Kornuochi [21; Proposition 1.1]). Thus \(\tilde{u}\) is a solution to \(\text{VP}(\lambda, \delta, u_0, f)\).

By the uniqueness of solution we have \(\tilde{u} = u\). Therefore it must be true that \(u_{n_k} \to u\) in \(C([0, T]; H)\), so that \(P \lambda_{n_k} \to P\lambda\) in \(C([0, T])\). Q.E.D.

Proof of (b) of Theorem 1.1: Consider the following subset \(S\) of \(\Lambda\):
\[S = \left\{ \lambda \in \Lambda; \left. \begin{array}{l} |\lambda(t) - \lambda(s)| \leq |t - s| ||g||_{C([0, T])} \\ + \sqrt{|t - s|L_2(L_2(0, T; H)} \right\}, \right.\]
\[\text{for all } s, t \in [0, T]\]

where \(K\) is the same constant as in Theorem 4.2. Obviously \(S\) is compact and convex in \(C([0, T])\) and \(P(A) \subseteq S\). Taking Lemmas 4.1 and 4.2 into account, we see that \(P\) is continuous on \(S\) with respect to the topology of \(C([0, T])\)
and $P(S) \subseteq S$. Hence, by a well-known fixed-point theorem there is $l \in S$ such that $Pl = l$ and it is easy to see that the couple $\{l, u^l\}$, $u^l$ being a unique solution to $VP(l, g, u_o, f)$, is a solution to $QVP(l_o, g, u_o, f)$.

Q.E.D.

6. A uniqueness theorem in a special case.

Throughout this section we assume that $p = 2$, $l_o \geq 0$, $g \in C([0, T])$ is non-positive and $u_o \in X (= W^{1, \infty}(0, \infty))$ is non-negative with $u_o = 0$ on $[l_o, \infty)$.

Let $\{l, u\}$ be an arbitrary solution to $QVP(l_o, g, u_o, 0)$. Then we know the following facts (cf. Proposition 1.2 and Theorem 1.1):

(6.1) $l$ is non-decreasing with $l(0) = l_o$ and $u$ is non-negative;

(6.2) $u_t(t, \cdot) - \beta(u)_{xx}(t, \cdot) = 0$ a.e. on $[0, l(t)]$ for a.e. $t \in [0, T]$;

(6.3) \begin{align*}
\begin{cases}
\begin{aligned}
u(0, x) &= u_o(x) \text{ for } 0 \leq x \leq l_o, \\
u(t, x) &= 0 \text{ for } x \geq l(t) \text{ and } \\
\beta(u)_x(t, 0+) &= g(t) \text{ for a.e. } t \in [0, T].
\end{aligned}
\end{cases}
\end{align*}

(6.4) $\frac{dl(t)}{dt} = -\beta(u)_x(t, l(t)-)$ for a.e. $t \in [0, T]$.

We define

$$v(t, x) = \int_0^t \beta(u)(r, x) dr \quad \text{for } x \geq 0, \ 0 \leq t \leq T$$

and note that

$$v_t(t, x) = \beta(u)(t, x) \geq 0, \quad v_x(t, x) = \int_0^t \beta(u)_x(r, x) dr.$$

Now, let $\eta$ be any function in $X$. Then we have by (6.1) - (6.4)

$$\int_0^\infty v_x(t, \cdot) \eta_x dx = \int_0^\infty \beta(u)_x(t, \cdot) \eta_x dx dr = \int_0^\infty \beta(u)(r, \cdot) \eta_x dx dr$$

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\[ \int_0^t \left( - \int_0^\beta(\alpha_x(x), \cdot) \eta \, dx - \beta(\alpha_x(x), 0) \eta(0) + \beta(\alpha_x(x), \alpha(\cdot)) \eta(\alpha(\cdot)) \right) \, dt \\
= \int_0^t \left( \int_0^\alpha(\cdot) u_t(x, \cdot) \eta \, dx - (\int_0^t g(\cdot) \eta(0) - \int_0^t \frac{d\alpha}{dt} \eta(\alpha(\cdot)) \right) \, dt \\
= \int_0^\infty u(t, \cdot) \eta \, dx + \int_0^\infty u_0 \eta \, dx - (\int_0^t g(\cdot) \eta(0) - \int_0^t \alpha(t) \eta) \, dx, \]

from which we get the following lemma.

Lemma 6.1. Let \{\alpha, u\} be a solution to QVP(\alpha, g, u_0, 0) and v be as above. Also let \rho be the inverse of \beta. Then

\[ I(t; v, \eta) = \int_0^\infty \rho(v_t(t, x)) (v_t(t, x) - \eta(x)) \, dx + \int_0^\infty v_x(t, x) (v_{xt}(t, x) - \eta_x(x)) \, dx \\
- \int_0^\infty u_0(t, x) \eta \, dx + (\int_0^t g(\cdot) \eta(0) - \int_0^t \alpha(t) \eta) \, dx \leq 0 \]

for all \( t \in [0, T] \) and all \( \eta \in Y = \{ \eta \in X; \eta \text{ is non-negative and } \eta(x) = 0 \text{ for all sufficiently large } x \} \).

Proof. We set for simplicity

\[ J(t; \eta) = \int_0^\infty \rho(v_t(t, \cdot)) \eta \, dx + \int_0^\infty v_x(t, \cdot) \eta_x \, dx - \int_0^\infty u_0 \eta \, dx \\
+ (\int_0^t g(\cdot) \eta(0) + \int_0^\infty \eta \, dx \text{ for } \eta \in Y. \]

As was seen above, \( J(t; \eta) \geq 0 \) for all \( \eta \in Y \) and \( J(t; v_t(t, \cdot)) = 0 \).

Therefore \( I(t; v, \eta) = J(t; v_t(t, \cdot)) - J(t; \eta) \leq 0. \)

Q.E.D.

Proof of Theorem 1.2: Let \{\alpha, u\} and \{\bar{\alpha}, \bar{u}\} be two solutions to QVP(\alpha, g, u_0, 0). Then with the same notations as above it follows from
Lemma 6.1 that

\[ 0 \geq I(t; v, \vec{v}_t(t, \cdot)) + I(t; \vec{v}, v_t(t, \cdot)) \]

\[ = \int_0^\infty \{ \rho(v_t)(t, x) - \rho(\vec{v}_t)(t, x) \} (v_t(t, x) - \vec{v}_t(t, x)) \, dx \]

\[ + \int_0^\infty (v_x(t, x) - \vec{v}_x(t, x))(v_xt(t, x) - \vec{v}_xt(t, x)) \, dx \]

\[ \geq \frac{1}{2} \frac{d}{dt} |v_x(t, \cdot) - \vec{v}_x(t, \cdot)|^2_H \]

for a.e. \( t \in [0, T] \). This gives

\[ |v_x(t, \cdot) - \vec{v}_x(t, \cdot)|_H \leq |v_x(0, \cdot) - \vec{v}_x(0, \cdot)|_H = 0 \]

for all \( t \in [0, T] \), so \( v_t = \vec{v}_t \), i.e., \( u = \vec{u} \) as well as \( \lambda = \vec{\lambda} \). \textit{Q.E.D.}

**Remark 6.1.** The technics adopted above is found in Duvaut [11].

7. Some remarks.

A. Let \( l_o > 0 \) and \( g, u_o, f \) be as in (b) of Theorem 1.1 and let \( \{l, u\} \)
be a solution to QVF\( (l_o, g, u_o, f) \). Further assume that \( g \in W^{1,2}(0, T) \). Then
\( l \in W^{1,2}(0, T) \) and \( u \) is a unique solution to VP\( (l, g, u_o, f) \). It is not
difficult to verify that the family \( \{ \phi_{l, g}^t; 0 \leq t \leq T \} \) satisfies the following:

for each \( s, t \in [0, T] \) and \( z \in K_l(s) \) there is \( z_{\perp} \in K_l(t) \) such
that

\[
\begin{align*}
|z_{\perp} - z|_H & \leq C_o \{ |g(t) - g(s)| + |l(t) - l(s)| \} (|\phi_{l, g}^t(z)_{\perp}|^{1/2} + 1), \\
|\phi_{l, g}^t(z_{\perp}) - \phi_{l, g}^t(z)| & \leq C_o \{ |g(t) - g(s)| + |l(t) - l(s)| \} (|\phi_{l, g}^t(z)| + 1),
\end{align*}
\]

where \( C_o \) is a positive constant independent of \( s, t \) and \( z \).

In fact, if we take for \( z \) given in \( K_l(s) \)
\[ z_1(x) = z(\frac{2(s)}{z(t)})^x, \quad 0 \leq x < \infty, \]

then we obtain inequalities in (7.1) with a positive constant \( C_o \) independent of \( s, t \) and \( z \). Under (7.1) it can be shown (cf. Kermochi [24]) that the function \( t + \Phi_{x_i}^x(Bu(t)) \) is absolutely continuous on \([0, T]\). This implies that \( t \rightarrow [\beta(u)(t, \cdot)]_X \) is continuous on \([0, T]\), so that \( \beta(u) \in C([0, T]; X) \).

B. Let \( x_0 > 0, g \leq 0 \) be in \( W^{1,2}(0, T) \) and \( u_0, f \) be as in (b) of Theorem 1.1, and let \( h \) be in \( L^2(0, T) \). Then by QVP(\( l_o, g, u_o, f, h \)) we mean the problem to find a couple \( (\ell, u) \) satisfying (B1), (B2) of Definition 1.2 and the following (B3): instead of (B3):

(B3)' \[ \ell(t) = \ell_0 - \int_0^t g(r) dr + \int_0^t h(r) dr + \int_0^t u_0(x) dx \]

\[ + \int_0^t \int_0^r f(r, x) dxdx - \int_0^\infty u(t, x) dx, \quad \forall \ t \in [0, T]. \]

This integral equation (B3)' is corresponding to Stefan condition of the following type:

\[ \frac{d\ell(t)}{dt} - \beta(u)_x(t, \ell(t)-)P^{-2}\beta(u)_x(t, \ell(t)-) + h(t) \text{ for } 0 < t < T. \]

In this case we should notice that the free boundary \( x = \ell(t) \) is not necessarily non-decreasing in \( t \). However the same approach is possible to QVP(\( l_o, g, u_o, f, h \)).

C. Finally we consider the problem to find a couple \( (\ell, u) \) satisfying

(7.2) \[ u_t - \beta(u)_{xx} = f \quad \text{for } 0 < t < T, \ell(t) > 0, 0 < x < \ell(t), \]

subject to

(7.3) \[ \ell(0) = \ell_0 \text{ and } u(0, x) = u_0(x) \text{ for } 0 < x < \ell_0, \]

(7.4) \[ \beta(u)(t, 0) = g_0(t) \text{ for } 0 < t < T, \beta(u)(t, \ell(t)) = 0 \text{ for } 0 < t < T, \]

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and

\[ \frac{dl(t)}{dt} = -B(u)x(t, l(t)) \quad \text{for } 0 < t < T, \]

where \( l_0 > 0 \) is given as well as \( u_0 \geq 0 \) in \( W^{1,2}(0, \infty) \) with \( u_0 = 0 \) on \([l_0, \infty)\), 
\( f \geq 0 \) in \( L^2(0, T; H) \) and \( g_0 \geq 0 \) in \( W^{1,2}(0, T) \). By means of the family 
\( \{\Phi^t_{l_0, g_0} : 0 \leq t \leq T\} \) of convex functions on \( H \) given by

\[
\Phi^t_{l_0, g_0}(z) = \begin{cases} 
\frac{1}{2} \int_0^\infty |x|^2 dx & \text{if } z \in K_{l_0, g_0}(t), \\
\infty & \text{otherwise}
\end{cases}
\]

with

\[ K_{l_0, g_0}(t) = \{z \in W^{1,2}(0, \infty) ; z(0) = g_0(t), z(x) = 0 \text{ for } l(t) \leq x < \infty\}, \]

we can similarly give a quasi-variational formulation associated with system 
\((7.2) - (7.5))\), in which \((7.5)\) is transformed into the integral equation

\[ l(t)^2 = l_0^2 + 2 \int_0^t g_0(r) dr + 2 \int_0^l u_0(x) dx - 2 \int_0^\infty xu(t, x) dx \\
+ 2 \int_0^t l(r) \int_0^r x f(r, x) dx dr, \quad \forall t \in [0, T]. \]

Also in this case we can show the existence and uniqueness (in case \( f \equiv 0 \))
of a solution to this quasi-variational problem by modifying the arguments
developed in sections 2, 3, 4, 5 and 6.

References


[26] W. T. Kyner, An existence and uniqueness theorem for a nonlinear Stefan


