

広田氏の Bilinear Equations について

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ここでは主に Korteweg-de Vries (KdV) 方程式, modified Korteweg-de Vries (MKdV) 方程式及び Nonlinear Schrödinger (NLS) 方程式を考える。

$\Delta(p_1, \dots, p_r)$ で差積: $\prod_{1 \leq i < j \leq r} (p_i - p_j)$ を表わす。このとき

$$\Delta(p_1^2, \dots, p_r^2) = \prod_{i < j} (p_i^2 - p_j^2) \quad \text{である。}$$

[KdV, MKdV]

$t = (t_0, t_1, t_2, \dots)$ において, $t_0, t_1, t_2, \dots$ の weight を各々 $1, 3, 5, \dots$ と定める。 $x = (x_0, x_1, x_2, \dots)$ についても同様とする。

$p_1, \dots, p_r$ の函数 $F(p_1, \dots, p_r)$ の totally even odd part を次のように定義する:

$$(1) \quad \text{totally even part of } F(p_1, \dots, p_r) \stackrel{\text{def.}}{=} \frac{1}{2^r} \sum_{\epsilon_1, \dots, \epsilon_r = \pm 1} F(\epsilon_1 p_1, \dots, \epsilon_r p_r).$$

$$(2) \quad \text{totally odd part of } F(p_1, \dots, p_r) \stackrel{\text{def.}}{=} \frac{1}{2^r} \sum_{\epsilon_1, \dots, \epsilon_r = \pm 1} (-1)^{\#} F(\epsilon_1 p_1, \dots, \epsilon_r p_r),$$

$$\# \stackrel{\text{def.}}{=} \# \{ j \mid \epsilon_j = -1, 1 \leq j \leq r \}.$$

$$(3) \quad \Delta(p_1^2, \dots, p_r^2) \Phi^{\pm}(t; p_1, \dots, p_r)$$

$$\stackrel{\text{def.}}{=} \text{totally even odd part of } \Delta(p_1, \dots, p_r) e^{\frac{t_0}{2}(p_1^2 + \dots + p_r^2) + \frac{t_1}{2}(p_1^3 + \dots + p_r^3) + \frac{t_2}{2}(p_1^5 + \dots + p_r^5) + \dots}$$

$$(4) \quad 2^r \Delta(p^1, \dots, p^r) \cdot (\Phi^-(t; p_1, \dots, p_r))^2 \stackrel{\text{def}}{=} \sum_{\substack{l_1 > \dots > l_r > 0 \\ \text{even}}} a_{l_1, \dots, l_r}(t) \cdot \det \begin{bmatrix} p_1^{l_1} & \dots & p_r^{l_1} \\ \vdots & & \vdots \\ p_1^{l_r} & \dots & p_r^{l_r} \end{bmatrix}.$$

$$(5) \quad 2^r \Delta(p^1, \dots, p^r) \cdot \Phi^+(t; p_1, \dots, p_r) \Phi^-(t; p_1, \dots, p_r) \stackrel{\text{def}}{=} \sum_{\substack{l_1 > \dots > l_r > 0 \\ \text{odd}}} a_{l_1, \dots, l_r}(t) \cdot \det \begin{bmatrix} p_1^{l_1} & \dots & p_r^{l_1} \\ \vdots & & \vdots \\ p_1^{l_r} & \dots & p_r^{l_r} \end{bmatrix}.$$

$r=0$ のとき $a_{l_1, \dots, l_r}(t) = 1$ とする。

主定理 u を KdV 方程式 $u_{x_1} = (u_{x_0 x_0} + 3u^2)_{x_0}$ の解, v を MKdV 方程式 $v_{x_1} = (v_{x_0 x_0} - 2v^3)_{x_0}$ の解とする (下の添字は微分を表わす). $u = (2 \log f)_{x_0 x_0}$, $v = (\log \frac{f}{g})_{x_0}$ となる f, g を適当に選ぶとは

$$(6) \quad \frac{f(x+t)f(x-t)}{f(x)^2} = \sum_{r=0}^{\infty} \sum_{\substack{l_1 > \dots > l_r > 0 \\ \text{even}}} a_{l_1, \dots, l_r}(t) \cdot K_{l_1, \dots, l_r}[u],$$

$$(7) \quad \frac{f(x+t)g(x-t)}{f(x)g(x)} = \sum_{r=0}^{\infty} \sum_{\substack{l_1 > \dots > l_r > 0 \\ \text{odd}}} a_{l_1, \dots, l_r}(t) \cdot K_{l_1, \dots, l_r}[v]$$

が成り立つ。ここに $K_{l_1, \dots, l_r}[u]$ は $u, u_{x_0}, u_{x_0 x_0}, \dots$ を各々 weight $2, 3, 4, \dots$ と数えて、それらの weight $l_1 + \dots + l_r$ の斉重多項式、又 $K_{l_1, \dots, l_r}[v]$ は $v, v_{x_0}, v_{x_0 x_0}, \dots$ を各々 weight $1, 2, 3, \dots$ と数えて、それらの weight $l_1 + \dots + l_r$ の斉重多項式である (表2)。特に $r=0$ のとき $K_{l_1, \dots, l_r}[u] = K_{l_1, \dots, l_r}[v] = 1$ とする。

$$\mathbf{k} = (k_0, k_1, k_2, \dots) \quad \text{と} \quad (|\mathbf{k}| \stackrel{\text{def}}{=} k_0 + 3k_1 + 5k_2 + \dots, \quad D^{\mathbf{k}} = D_0^{k_0} D_1^{k_1} D_2^{k_2} \dots, \\ D_m = \frac{\partial}{\partial x_m} \quad \text{とする。} \quad a_{l_1, \dots, l_r}(t) \stackrel{\text{def}}{=} \sum_{|\mathbf{k}|=l_1+\dots+l_r} a_{l_1, \dots, l_r; \mathbf{k}} \frac{t^{\mathbf{k}}}{\mathbf{k}!} \quad \text{と書$$

くとき、主定理は次のように言いかえられる (左辺は広田氏[3])

の意味での微分とする)。

$$(6)_k \quad \frac{D^k f \cdot f}{f^2} = \sum_{r=0}^{\infty} \sum_{\substack{l_1 > \dots > l_r > 0 \\ \text{even} \\ l_1 + \dots + l_r = |k|}} a_{l_1, \dots, l_r; k} \cdot K_{l_1, \dots, l_r} [u],$$

$$(7)_k \quad \frac{D^k f \cdot g}{f \cdot g} = \sum_{r=0}^{\infty} \sum_{\substack{l_1 > \dots > l_r > 0 \\ \text{odd} \\ l_1 + \dots + l_r = |k|}} a_{l_1, \dots, l_r; k} \cdot K_{l_1, \dots, l_r} [v].$$

系 $P(D_x) = P(D_0, D_1, D_2, \dots)$ が $D_0, D_1, D_2, \dots$ と各々 weight $1, 3, 5, \dots$ と数之て、 $\forall$ k の weight n の斉重多項式であるとき ($\Leftrightarrow$ は必要十分を表わす),

$$(8) \quad [KdV] \quad P(D_x) f \cdot f = 0 \\ \Leftrightarrow P\left(\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_1}, \dots\right) a_{l_1, \dots, l_r}(t) = 0 \text{ for } \forall (l_1, \dots, l_r) \text{ s.t. } l_1 + \dots + l_r = n, \substack{l_1 > \dots > l_r > 0 \\ \text{even}}.$$

$$(9) \quad [MKdV] \quad P(D_x) f \cdot g = 0 \\ \Leftrightarrow P\left(\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_1}, \dots\right) a_{l_1, \dots, l_r}(t) = 0 \text{ for } \forall (l_1, \dots, l_r) \text{ s.t. } l_1 + \dots + l_r = n, \substack{l_1 > \dots > l_r > 0 \\ \text{odd}}.$$

[NLS]

$t = (t_1, t_2, t_3, \dots)$ において t_1, t_2, t_3 と各々 weight $1, 2, 3, \dots$ と定める。

$x = (x_1, x_2, x_3, \dots)$ についても同様とする。

$$(10) \quad \Delta(p_1, \dots, p_{r+1}) \cdot \Phi(t; p_1, \dots, p_{r+1}) \\ \stackrel{\text{def}}{=} \frac{1}{r! (r+1)!} \sum^{(2r+1)!} \pm \Delta(p_1, \dots, p_r) \cdot \Delta(p_{r+1}, \dots, p_{r+1}) e^{\frac{t_1}{2}(-p_1^2 - \dots - p_r^2 + p_{r+1}^2 + \dots + p_{r+1}^2) + \frac{t_2}{2}(-p_1^2 - \dots - p_r^2 + p_{r+1}^2 + \dots + p_{r+1}^2) + \dots}$$

$$(11) \quad \Delta(p_1, \dots, p_r) \cdot \Phi(t; p_1, \dots, p_r) \\ \stackrel{\text{def}}{=} \frac{1}{r! r!} \sum^{(2r)!} \pm \Delta(p_1, \dots, p_r) \cdot \Delta(p_{r+1}, \dots, p_r) e^{\frac{t_1}{2}(-p_1^2 - \dots - p_r^2 + p_{r+1}^2 + \dots + p_r^2) + \frac{t_2}{2}(-p_1^2 - \dots - p_r^2 + p_{r+1}^2 + \dots + p_r^2) + \dots}$$

$$(12) \quad \Delta(p_1, \dots, p_{2r+1}) \cdot (\bar{\Gamma}(t; p_1, \dots, p_{2r+1}))^2 \stackrel{\text{def}}{=} \sum_{l_1 > \dots > l_{2r+1} \geq 0} a_{l_1, \dots, l_{2r+1}}(t) \cdot \det \begin{pmatrix} p_1^{l_1} & \dots & p_1^{l_{2r+1}} \\ \vdots & & \vdots \\ p_{2r+1}^{l_1} & \dots & p_{2r+1}^{l_{2r+1}} \end{pmatrix}.$$

$$(13) \quad \Delta(p_1, \dots, p_{2r}) \cdot (\bar{\Gamma}(t; p_1, \dots, p_{2r}))^2 \stackrel{\text{def}}{=} \sum_{l_1 > \dots > l_{2r} \geq 0} a_{l_1, \dots, l_{2r}}(t) \cdot \det \begin{pmatrix} p_1^{l_1} & \dots & p_1^{l_{2r}} \\ \vdots & & \vdots \\ p_{2r}^{l_1} & \dots & p_{2r}^{l_{2r}} \end{pmatrix}.$$

$r=0$ かつ $a_{l_1, \dots, l_{2r}}(t) = 1$ とする。

$k = (k_1, k_2, k_3, \dots)$ とし $|k| \stackrel{\text{def}}{=} k_1 + 2k_2 + 3k_3 + \dots$, $D^k = D_1^{k_1} D_2^{k_2} D_3^{k_3} \dots$,

$D_m = \frac{\partial}{\partial x_m}$ とする。 $a_{l_1, \dots, l_{2r+1}}(t) \stackrel{\text{def}}{=} \sum_{|k|=l_1+\dots+l_{2r+1}+r} a_{l_1, \dots, l_{2r+1}; k} \frac{t^k}{k!}$,

$a_{l_1, \dots, l_{2r}}(t) \stackrel{\text{def}}{=} \sum_{|k|=l_1+\dots+l_{2r}+r} a_{l_1, \dots, l_{2r}; k} \frac{t^k}{k!}$ と書く。

主定理 v と w (resp. u と v) は NLS 方程式 $v_{x_2} = (v^2 + w)x_1$,

$w_{x_2} = (v_{x_1}x_1 + 2vw)x_1$ (resp. $u_{x_2} = (u_{x_1} + 2uv)x_1$, $v_{x_2} = (-v_{x_1} + v^2 + u)x_1$) の

解とする。 $u = (2 \log f)_{x_1 x_1}$, $v = (\log \frac{f}{g})_{x_1}$, $w = (\log f g)_{x_1 x_1}$ と

なる f, g を適当に選べば (このとき $u = v_{x_1} + w$ である),

$$(14)_R \quad \frac{D^k f \cdot g}{f g} = \sum_{r=0}^{\infty} \sum_{\substack{l_1 > \dots > l_{2r+1} \geq 0 \\ l_1 + \dots + l_{2r+1} = |k| - r}} a_{l_1, \dots, l_{2r+1}; k} \cdot K_{l_1, \dots, l_{2r+1}}[v, w],$$

$$(15)_R \quad \frac{D^k f \cdot f}{f^2} = \sum_{r=0}^{\infty} \sum_{\substack{l_1 > \dots > l_{2r} \geq 0 \\ l_1 + \dots + l_{2r} = |k| - r}} a_{l_1, \dots, l_{2r}; k} \cdot K_{l_1, \dots, l_{2r}}[u, v]$$

が成り立つ。ここに、 $K_{l_1, \dots, l_{2r+1}}[v, w]$ は $v, v_{x_1}, v_{x_1 x_1}, \dots$ を weight $1, 2, 3, \dots$,

$w, w_{x_1}, w_{x_1 x_1}, \dots$ を weight $2, 3, 4, \dots$ と数えて、それらの weight $l_1 + \dots + l_{2r+1}$

の斉重多項式、 $K_{l_1, \dots, l_{2r}}[u, v]$ は $u, u_{x_1}, u_{x_1 x_1}, \dots$, $v, v_{x_1}, v_{x_1 x_1}, \dots$

を weight $2, 3, 4, \dots, 1, 2, 3, \dots$ と数えて、それらの weight $l_1 + \dots + l_{2r} + r$ の

斉重多項式である(表4)。特に、 $r=0$ かつ $K_{l_1, \dots, l_{2r}}[u, v] = 1$

とする。

系 $P(D_x) = P(D_1, D_2, D_3, \dots)$ が $D_1, D_2, D_3, \dots$ を各々 weight $1, 2, 3, \dots$ と
数えて、それから weight n の齊重多項式であるとき、

$$(16) \quad P(D_x) f \cdot g = 0 \\ \Leftrightarrow P(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots) a_{i_1, \dots, i_{2r+1}}(t) = 0 \text{ for } \forall (i_1, \dots, i_{2r+1}) \text{ st. } i_1 + \dots + i_{2r+1} + r = n, i_1, \dots, i_{2r+1} \geq 0.$$

$$(17) \quad P(D_x) f \cdot f = 0 \\ \Leftrightarrow P(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots) a_{i_1, \dots, i_{2r}}(t) = 0 \text{ for } \forall (i_1, \dots, i_{2r}) \text{ st. } i_1 + \dots + i_{2r} + r = n, i_1, \dots, i_{2r} \geq 0.$$

$p(n) = n$ を自然数の和に分ける分け方の数,

$p_{\text{odd}}^{\text{even}}(n) = n$ を 偶数 奇数 の和に分ける分け方の数,

$q(n) = n$ を相異なる自然数の和に分ける分け方の数,

$q_{\text{odd}}^{\text{even}}(n) = n$ を相異なる 偶数 奇数 の和に分ける分け方の数

とおく。これらの母函数は次で与えられる。

$$(18) \quad \sum_{n=0}^{\infty} p(n) x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots},$$

$$(19) \quad \sum_{n=0}^{\infty} p_{\text{odd}}^{\text{even}}(n) x^n = \frac{1}{(1-x^2)(1-x^4)(1-x^6)\dots},$$

$$(17) \quad \sum_{n=0}^{\infty} p_{\text{odd}}^{\text{odd}}(n) x^n = \frac{1}{(1-x)(1-x^3)(1-x^5)\dots},$$

$$(20) \quad \sum_{n=0}^{\infty} q(n) x^n = (1+x)(1+x^2)(1+x^3)\dots,$$

$$(21) \quad \sum_{n=0}^{\infty} q_{\text{odd}}^{\text{even}}(n) x^n = (1+x^2)(1+x^4)(1+x^6)\dots,$$

$$(21) \quad \sum_{n=0}^{\infty} q_{\text{odd}}^{\text{odd}}(n) x^n = (1+x)(1+x^3)(1+x^5)\dots.$$

このとき、

$$(22) \quad p^{\text{even}}(n) = \begin{cases} p(\frac{n}{2}) & (n: \text{even}) \\ 0 & (n: \text{odd}) \end{cases}, \quad p^{\text{odd}}(n) = f(n),$$

$$q^{\text{even}}(n) = \begin{cases} f(\frac{n}{2}) & (n: \text{even}) \\ 0 & (n: \text{odd}) \end{cases}$$

の関係がある。又、これから λ 量の漸近的評価は、 $C \stackrel{\text{def}}{=} \pi \sqrt{\frac{2}{3}}$

とするとき

$$(23) \quad \log p(n) \sim C\sqrt{n}, \quad \log f(n) \sim \log p^{\text{even}}(n) \sim C\sqrt{\frac{n}{2}},$$

$$\log q^{\text{even}}(n) \sim C\sqrt{\frac{n}{4}} \quad (n \rightarrow \infty)$$

で与えられる。

今

$$[KdV, MKdV] \quad \dim \{P(D_x) \mid P(D_x) \text{ は } D_0, D_1, D_2, \dots \text{ の weight } n \text{ の斉重多項式}\} = f(n),$$

$$[NLS] \quad \dim \{P(D_x) \mid P(D_x) \text{ は } D_1, D_2, D_3, \dots \text{ の weight } n \text{ の斉重多項式}\} = p(n)$$

である。

定理 (次元公式)

$$(24) \quad [KdV] \quad \dim \left\{ \frac{D^k f \cdot f}{f^2} \mid k=(k_0, k_1, k_2, \dots), |k|=n \right\} = q^{\text{even}}(n).$$

$$(25) \quad [MKdV] \quad \dim \left\{ \frac{D^k f \cdot g}{fg} \mid k=(k_0, k_1, k_2, \dots), |k|=n \right\} = q^{\text{odd}}(n).$$

$$(26) \quad [NLS] \quad \dim \left\{ \frac{D^k f \cdot g}{fg} \mid k=(k_1, k_2, k_3, \dots), |k|=n \right\} = q^{\text{odd}}(2n+1).$$

$$(27) \quad \dim \left\{ \frac{D^k f \cdot f}{f^2} \mid k=(k_1, k_2, k_3, \dots), |k|=n \right\} = q^{\text{odd}}(2n).$$

(表 8) に、0 から 59 までの n について、これから $p(n)$, $p^{\text{even}}(n)$, $f(n)$, $q^{\text{even}}(n)$ の数値をあげる [1]。

安定性定理 KdV 方程式に対しては、 $m \geq \frac{|k|-2}{4}$ 又は

$l_1 + 2m \geq |k| + 2$ のとき, MKdV 方程式に対しては $m \geq \frac{|k|-1}{4}$

又は $l_1 + 2m \geq |k| + 3$ のとき $l_1 + \dots + l_r = k_0 + 3k_1 + \dots + (2m+1)(k_m+1)$,

$l_1 > \dots > l_r > 0$ を満たす l と k に対して,
_{even (KdV)}
_{odd (MKdV)}

$$a_{l_1, \dots, l_r : k_0, \dots, k_{m-1}, k_m+1} = a_{l_1+2, \dots, l_r : k_0, \dots, k_{m-1}, k_m, 1} = a_{l_1+4, \dots, l_r : k_0, \dots, k_{m-1}, k_m, 0, 1}$$

$$= \dots = a_{l_1+2v, \dots, l_r : k_0, \dots, k_{m-1}, k_m, \underbrace{0, \dots, 0}_{v-1}, 1}$$

が成り立つ。

以下に, $\frac{D^k f \cdot f}{f^2}$, $\frac{D^k f \cdot g}{fg}$ 等を, 各々の場合に, basis $\{K_{l_1, \dots, l_r}\}$ の一次結合で表わした表を挙げる(表1, 3)。Bilinear Equations はこれらの表に含まれているが, 便利のため, 一部を書き出しておく(表5, 6, 7)。

(表1) MKIV KIV

	1	1
1	1	1

	K_1
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D_0	1
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	0	K_2
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D_0^2	0	1
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	K_3
--	-------

D_0^3	1
---------	---

D_1	1
-------	---

	$K_{3,1}$	K_4
--	-----------	-------

D_0^4	2	1
---------	---	---

$D_0 D_1$	-1	1
-----------	----	---

	K_5
--	-------

D_0^5	1
---------	---

$D_0^2 D_1$	1
-------------	---

D_2	1
-------	---

	$K_{5,1}$	K_6	$K_{4,2}$
--	-----------	-------	-----------

D_0^6	4	1	5
---------	---	---	---

$D_0^3 D_1$	1	1	-1
-------------	---	---	----

D_1^2	-2	1	2
---------	----	---	---

$D_0 D_2$	-1	1	0
-----------	----	---	---

MKIV KIV

	K_7
--	-------

D_0^7	1
---------	---

$D_0^4 D_1$	1
-------------	---

$D_0 D_1^2$	1
-------------	---

$D_0^2 D_2$	1
-------------	---

D_3	1
-------	---

	$K_{7,1}$	$K_{5,3}$	K_8	$K_{6,2}$
--	-----------	-----------	-------	-----------

D_0^8	6	14	1	14
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$D_0^5 D_1$	3	-1	1	2
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$D_0^2 D_1^2$	0	2	1	-1
---------------	---	---	---	----

$D_0^3 D_2$	1	-1	1	-1
-------------	---	----	---	----

$D_1 D_2$	-2	-1	1	2
-----------	----	----	---	---

$D_0 D_3$	-1	0	1	0
-----------	----	---	---	---

	K_9	$K_{5,3,1}$
--	-------	-------------

D_0^9	1	42
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$D_0^6 D_1$	1	-6
-------------	---	----

$D_0^3 D_1^2$	1	0
---------------	---	---

D_1^3	1	6
---------	---	---

$D_0^4 D_2$	1	2
-------------	---	---

$D_0 D_1 D_2$	1	-1
---------------	---	----

$D_0^2 D_3$	1	0
-------------	---	---

D_4	1	0
-------	---	---

(表1)

MKIV KAV

	$K_{7,1}$	$K_{7,3}$	K_{10}	$K_{8,2}$	$K_{6,4}$
D_0^{10}	8	48	1	27	42
$D_0^7 D_1$	5	6	1	9	0
$D_0^4 D_1^2$	2	0	1	0	3
$D_0 D_1^3$	-1	3	1	0	-3
$D_0^5 D_2$	3	-2	1	2	-3
$D_0^2 D_1 D_2$	0	1	1	-1	0
D_2^2	-2	-2	1	2	2
$D_0^3 D_3$	1	-1	1	-1	0
$D_1 D_3$	-2	-1	1	2	0
$D_0 D_4$	-1	0	1	0	0

 K_{11} $K_{7,3,1}$

D_0^{11}	1	198
$D_0^8 D_1$	1	6
$D_0^5 D_1^2$	1	-6
$D_0^2 D_1^3$	1	0
$D_0^6 D_2$	1	-2
$D_0^3 D_1 D_2$	1	1
$D_1^2 D_2$	1	4
$D_0 D_2^2$	1	-2
$D_0^4 D_3$	1	2
$D_0 D_1 D_3$	1	-1
$D_0^2 D_4$	1	0
D_5	1	0

MKaV

KaV

	$K_{11,1}$	$K_{9,3}$	$K_{7,5}$	K_{12}	$K_{10,2}$	$K_{7,4}$	$K_{6,4,2}$
D_0^{12}	10	110	132	1	44	165	462
$D_0^9 D_1$	7	29	6	1	20	21	-42
$D_0^6 D_1^2$	4	2	6	1	5	3	3
$D_0^3 D_1^3$	1	2	-3	1	-1	3	3
D_1^4	-2	2	6	1	2	6	12
$D_0^7 D_2$	5	5	-8	1	9	-5	7
$D_0^4 D_1 D_2$	2	-1	1	1	0	1	-2
$D_0 D_1^2 D_2$	-1	2	1	1	0	-2	-2
$D_0^2 D_2^2$	0	0	2	1	-1	0	2
$D_0^5 D_3$	3	-2	-1	1	2	-3	0
$D_0^2 D_1 D_3$	0	1	-1	1	-1	0	0
$D_2 D_3$	-2	-2	-1	1	2	2	0
$D_0^3 D_4$	1	-1	0	1	-1	0	0
$D_1 D_4$	-2	-1	0	1	2	0	0
$D_0 D_5$	-1	0	0	1	0	0	0

(表1)

MKdV

MKdV

KdV

	K_{13}	$K_{7,3,1}$	$K_{7,5,1}$		$K_{13,1}$	$K_{11,3}$	$K_{9,5}$	K_{14}	$K_{12,2}$	$K_{10,4}$	$K_{8,6}$	$K_{8,4,2}$
D_0^{13}	1	572	858	D_0^{14}	12	208	572	1	65	429	429	3003
$D_0^{10}D_1$	1	92	-6	$D_0^{11}D_1$	9	76	77	1	35	49	33	33
$D_0^7D_1^2$	1	-10	12	$D_0^8D_1^2$	6	16	14	1	14	12	15	-30
$D_0^4D_1^3$	1	-4	-6	$D_0^5D_1^3$	3	1	5	1	2	6	-3	6
$D_0D_1^4$	1	2	-6	$D_0^2D_1^4$	0	4	-4	1	-1	0	6	-6
$D_0^8D_2$	1	12	-22	$D_0^9D_2$	7	28	-13	1	20	14	-21	-12
$D_0^5D_1D_2$	1	-3	-1	$D_0^6D_1D_2$	4	1	2	1	5	-1	3	3
$D_0^2D_1^2D_2$	1	0	2	$D_0^3D_1^2D_2$	1	1	-1	1	-1	2	0	0
$D_0^3D_2^2$	1	2	-2	$D_1^3D_2$	-2	1	5	1	2	-4	-3	6
$D_1D_2^2$	1	2	4	$D_0^4D_2^2$	2	-2	2	1	0	-1	4	-2
$D_0^6D_3$	1	-2	4	$D_0D_1D_2^2$	-1	1	2	1	0	-1	-2	-2
$D_0^3D_1D_3$	1	1	1	$D_0^7D_3$	5	5	-9	1	9	-5	-5	7
$D_1^2D_3$	1	4	-2	$D_0^4D_1D_3$	2	-1	0	1	0	1	-2	-2
$D_0D_2D_3$	1	-2	-1	$D_0D_1^2D_3$	-1	2	0	1	0	-2	1	-2
$D_0^4D_4$	1	2	0	$D_0^2D_2D_3$	0	0	1	1	-1	0	0	2
$D_0D_1D_4$	1	-1	0	D_3^2	-2	-2	-2	1	2	2	2	0
$D_0^2D_5$	1	0	0	$D_0^5D_4$	3	-2	-1	1	2	-3	0	0
D_6	1	0	0	$D_0^2D_1D_4$	0	1	-1	1	-1	0	0	0
				D_2D_4	-2	-2	-1	1	2	2	0	0
				$D_0^3D_5$	1	-1	0	1	-1	0	0	0
				D_1D_5	-2	-1	0	1	2	0	0	0
				D_0D_6	-1	0	0	1	0	0	0	0

(表1)

KdV

	K_{16}	$K_{14,2}$	$K_{12,4}$	$K_{10,6}$	$K_{10,4,2}$	$K_{8,6,2}$		K_{16}	$K_{14,2}$	$K_{12,4}$	$K_{10,6}$	$K_{10,4,2}$	$K_{8,6,2}$
D_0^{16}	1	90	910	2002	11440	15444	$D_0 D_1 D_2 D_3$	1	0	-1	-1	-2	0
$D_0^{13} D_1$	1	54	286	286	1144	0	$D_0^2 D_3^2$	1	-1	0	0	2	2
$D_0^{10} D_1^2$	1	27	58	55	-98	81	$D_0^7 D_4$	1	9	-5	-5	7	0
$D_0^7 D_1^3$	1	9	10	13	-8	-27	$D_0^4 D_1 D_4$	1	0	1	-2	-2	0
$D_0^4 D_1^4$	1	0	7	-2	10	0	$D_0 D_1^2 D_4$	1	0	-2	1	-2	0
$D_0 D_1^5$	1	0	-5	10	10	0	$D_0^2 D_2 D_4$	1	-1	0	0	2	0
$D_0^{11} D_2$	1	35	90	-33	110	-231	$D_3 D_4$	1	2	2	2	0	0
$D_0^8 D_1 D_2$	1	14	6	6	-16	0	$D_0^5 D_5$	1	2	-3	0	0	0
$D_0^5 D_1^2 D_2$	1	2	3	0	2	6	$D_0^2 D_1 D_5$	1	-1	0	0	0	0
$D_0^2 D_1^3 D_2$	1	-1	0	3	2	3	$D_2 D_5$	1	2	2	0	0	0
$D_0^6 D_2^2$	1	5	-5	7	5	-6	$D_0^3 D_6$	1	-1	0	0	0	0
$D_0^3 D_1 D_2^2$	1	-1	1	1	-1	0	$D_1 D_6$	1	2	0	0	0	0
$D_1^2 D_2^2$	1	2	-2	-5	2	6	$D_0 D_7$	1	0	0	0	0	0
$D_0 D_2^3$	1	0	0	-3	0	-6							
$D_0^4 D_3$	1	20	14	-28	-12	30							
$D_0^6 D_1 D_3$	1	5	-1	-1	3	0							
$D_0^3 D_1^2 D_3$	1	-1	2	-1	0	-3							
$D_1^3 D_3$	1	2	-4	-1	6	-6							
$D_0^4 D_2 D_3$	1	0	-1	2	-2	0							

(表 2)

KAV

	u		u_2	u^2		u_4	uu_2	u_1^2	u^3	
K_2	1		K_4	1	3	K_6	1	10	5	10
						$K_{4,2}$	0	1	-1	1

	u_6	uu_4	u_1u_3	u_2^2	u^2u_2	uu_1^2	u^4
K_8	1	14	28	21	70	70	35
$K_{6,2}$	0	1	-2	1	10	-5	5

	u_8	uu_6	u_1u_5	u_2u_4	u_3^2	u^2u_4	uu_1u_3	uu_2^2	$u_1^2u_2$	u^3u_2	$u^2u_1^2$	u^5
K_{10}	1	18	54	114	69	126	504	378	462	420	630	126
$K_{8,2}$	0	1	-2	2	-1	14	0	35	-28	70	0	21
$K_{6,4}$	0	0	0	1	-1	3	-12	6	7	20	-15	6

	u_{10}	uu_8	u_1u_7	u_2u_6	u_3u_5	u_4^2	u^2u_6	uu_1u_5	uu_2u_4	uu_3^2	$u_1^2u_4$	$u_1u_2u_3$	u_2^3
K_{12}	1	22	88	242	418	253	198	1188	2508	1518	1650	5676	1342
$K_{10,2}$	0	1	-2	2	-2	1	18	18	150	51	-72	-120	40
$K_{8,4}$	0	0	0	1	-2	1	3	-12	26	-20	12	-8	19
$K_{6,4,2}$	0	0	0	0	0	0	0	0	1	-1	-1	2	-1

	u^3u_2	$u^2u_1u_3$	$u^2u_2^2$	$uu_1^2u_2$	u_1^4	u^4u_2	$u^3u_1^2$	u^6
K_{12}	924	5544	4158	10164	1155	2310	4620	462
$K_{10,2}$	126	252	504	-42	-105	420	210	84
$K_{8,4}$	42	-84	147	-70	35	175	-70	35
$K_{6,4,2}$	1	-6	3	7	-5	5	-5	1

(表2)

MKdV

	v				v_2	v^3				vv_2	v_1^2	v^4	
K_1	1				K_3	1	-2			$K_{3,1}$	1	-1	-1
	v_4	v^2v_2	vv_1^2	v^5					vv_4	vv_3	v_2^2	v^3v_2	v^6
K_5	1	-10	-10	6				$K_{5,1}$	1	-2	1	-10	4
	v_6	v^2v_4	vv_1v_3	vv_2^2	$v_1^2v_2$	v^4v_2	$v^3v_1^2$	v^7					
K_7	1	-14	-56	-42	-70	70	140	-20					
	vv_6	v_1v_5	v_2v_4	v_3^2	v^3v_4	$v^2v_1v_3$	$v^2v_2^2$	$vv_1^2v_2$	v_1^4	v^5v_2	$v^4v_1^2$	v^8	
$K_{7,1}$	1	-2	2	-1	-14	-28	-56	-14	21	70	70	-15	
$K_{5,3}$	0	0	1	-1	-2	12	-6	-14	1	14	-10	-3	
	v_8	v^2v_6	vv_1v_5	vv_2v_4	vv_3^2	$v_1^2v_4$	vv_2v_3	v_2^3					
K_9	1	-18	-108	-228	-138	-210	-756	-182					
$K_{5,3,1}$	0	0	0	1	-1	-1	2	-1					
	v^4v_4	$v^3v_1v_3$	$v^3v_2^2$	$v^2v_1^2v_2$	vv_1^4	v^6v_2	$v^5v_1^2$	v^9					
K_9	126	1008	756	3108	798	-420	-1260	70					
$K_{5,3,1}$	-1	8	-4	-14	11	6	-6	-1					

但し u_j は $(\frac{\partial}{\partial x_0})^j u = u_{\underbrace{x_0 \dots x_0}_j}$ の略記であり、 v_j も同様である。

Matsumo [5] の (3.2a) ~ (3.2d), Matsumo [6] の Appendix A, B, C 参照。

(表3)

NLS

	$f.g$	$f.f$		
	K_0	1		
1	1	1		
	K_1	0		
D_1	1	0		
	K_2	$K_{1,0}$		
D_1^2	1	1		
D_2	1	0		
	K_3	$K_{2,0}$		
D_1^3	1	0		
$D_1 D_2$	1	1		
D_3	1	0		
	K_4	$2K_{2,1,0}$	$K_{3,0}$	$K_{2,1}$
D_1^4	1	3	1	-3
$D_1^2 D_2$	1	-1	0	0
D_2^2	1	1	1	1
$D_1 D_3$	1	0	1	0
D_4	1	0	0	0

	$f \cdot g$		$f \cdot f$	
	K_5	$2K_{3,1,0}$	$K_{4,0}$	$K_{3,1}$
D_1^5	1	5	0	0
$D_1^3 D_2$	1	1	1	-2
$D_1 D_2^2$	1	-1	0	0
$D_1^2 D_3$	1	-1	0	0
$D_2 D_3$	1	1	1	1
$D_1 D_4$	1	0	1	0
D_5	1	0	0	0

	K_6	$2K_{4,1,0}$	$2K_{3,2,0}$	$K_{5,0}$	$K_{4,1}$	$K_{3,2}$
D_1^6	1	10	-5	1	-5	10
$D_1^4 D_2$	1	2	3	0	0	0
$D_1^2 D_2^2$	1	0	1	1	-1	-2
D_2^3	1	0	-3	0	0	0
$D_1^3 D_3$	1	1	-2	1	-2	1
$D_1 D_2 D_3$	1	-1	0	0	0	0
D_3^2	1	1	1	1	1	1
$D_1^2 D_4$	1	-1	0	0	0	0
$D_2 D_4$	1	1	0	1	1	0
$D_1 D_5$	1	0	0	1	0	0
D_6	1	0	0	0	0	0

(表3)

NLS

s.g

s.g				s.f				K_8	$2K_{6,10}$	$2K_{5,20}$	$2K_{4,30}$	$2K_{4,2,1}$	
	K_7	$2K_{5,10}$	$2K_{4,30}$	$2K_{3,3,1}$	$K_{6,0}$	$K_{5,1}$	$K_{4,2}$						
D_1^7	1	14	0	-35	0	0	0	D_1^8	1	21	-7	35	-70
$D_1^5 D_2$	1	6	0	5	1	-4	5	$D_1^6 D_2$	1	9	5	-5	-10
$D_1^3 D_2^2$	1	0	4	-1	0	0	0	$D_1^4 D_2^2$	1	3	3	-1	4
$D_1 D_2^3$	1	0	0	3	1	0	-3	$D_1^2 D_2^3$	1	-1	3	3	0
$D_1^4 D_3$	1	2	0	1	0	0	0	D_2^4	1	1	-3	3	6
$D_1^2 D_2 D_3$	1	0	0	-1	1	-1	-1	$D_1^5 D_3$	1	6	-4	5	5
$D_2^2 D_3$	1	0	-2	-1	0	0	0	$D_1^3 D_2 D_3$	1	0	2	1	-1
$D_1 D_3^2$	1	-1	0	1	0	0	0	$D_1 D_2^2 D_3$	1	0	0	-1	1
$D_1^3 D_4$	1	1	-2	0	1	-2	1	$D_1^2 D_3^2$	1	0	-1	2	-1
$D_1 D_2 D_4$	1	-1	0	0	0	0	0	$D_2 D_3^2$	1	0	-1	-2	-1
$D_3 D_4$	1	1	1	0	1	1	1	$D_1^4 D_4$	1	2	0	-3	1
$D_1^2 D_5$	1	-1	0	0	0	0	0	$D_1^2 D_2 D_4$	1	0	0	-1	-1
$D_2 D_5$	1	1	0	0	1	1	0	$D_2^2 D_4$	1	0	-2	1	-1
$D_1 D_6$	1	0	0	0	1	0	0	$D_1 D_3 D_4$	1	-1	0	0	1
D_7	1	0	0	0	0	0	0	D_4^2	1	1	1	1	0
								$D_1^3 D_5$	1	1	-2	0	0
								$D_1 D_2 D_5$	1	-1	0	0	0
								$D_3 D_5$	1	1	1	0	0
								$D_1^2 D_6$	1	-1	0	0	0
								$D_2 D_6$	1	1	0	0	0
								$D_1 D_7$	1	0	0	0	0
								D_8	1	0	0	0	0

(表4)

NLS (f.g)

$(\frac{\partial}{\partial x_i})^i v = v_{\underbrace{x_1 \dots x_i}_i}$ と v_j と略記する。 u_j, w_j も同様である。

	v		w	v^2		v_2	vw	v^3		w_2	vw_2	$v^2 w$	w^2	$v^3 w$	v^4			
K_1	1		K_2	1	1		K_3	1	3	1		K_4	1	4	$\frac{3}{2}$	$\frac{3}{2}$	6	1
	v_4	vw_2	$v_1 w_1$	$v_2 w$	$v^3 v_2$	vv_1^2	vw^2	$v^3 w$	v^5		$2K_{2,10}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	
K_5	1	5	5	5	10	$\frac{15}{2}$	$\frac{15}{2}$	10	1									
$2K_{3,10}$	0	0	-1	1	0	$-\frac{3}{2}$	$\frac{3}{2}$	0	0									

	w_4	vv_4	$v_1 v_3$	v_2^2	ww_2	w_1^2	$v^2 w_2$	$vv_1 w_1$	$vv_2 w$	$v_1^2 w$	w^3	$v^3 v_2$	$v^2 v_1^2$	$v^2 w^2$	$v^4 w$	v^6
K_6	1	6	10	$\frac{15}{2}$	5	$\frac{5}{2}$	15	30	30	$\frac{25}{2}$	$\frac{5}{2}$	20	$\frac{45}{2}$	$\frac{45}{2}$	15	1
$2K_{4,10}$	0	0	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	-4	4	-1	1	0	-3	3	0	0
$2K_{3,20}$	0	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	-2	2	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{3}{2}$	$\frac{3}{2}$	0	0

	v_6	vw_4	$v_1 w_3$	$v_2 w_2$	$v_3 w_1$	$v_4 w$	$v^2 v_4$	$vv_1 v_3$	$v v_2^2$	$v_1^2 v_2$	$vw w_2$	vw_1^2
K_7	1	7	14	21	14	7	21	70	$\frac{105}{2}$	$\frac{105}{2}$	35	$\frac{35}{2}$
$2K_{5,10}$	0	0	-1	1	-1	1	0	-5	$\frac{5}{2}$	-5	5	$-\frac{5}{2}$
$2K_{4,20}$	0	0	0	1	-1	0	0	-2	3	2	2	-3
$2K_{3,21}$	0	0	0	0	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$

	$v_1 w w_1$	$v_2 w^2$	$v^3 w_2$	$v^2 v_1 w_1$	$v^2 v_2 w$	$vv_1^2 w$	vw^3	$v^4 v_2$	$v^3 v_1^2$	$v^3 w^2$	$v^5 w$	v^7
K_7	35	$\frac{35}{2}$	35	105	105	$\frac{175}{2}$	$\frac{35}{2}$	35	$\frac{105}{2}$	$\frac{105}{2}$	21	1
$2K_{5,10}$	0	5	0	-10	10	-5	5	0	-5	5	0	0
$2K_{4,20}$	-2	0	0	-8	8	0	0	0	-4	4	0	0
$2K_{3,21}$	1	$-\frac{1}{2}$	0	-1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0

(表4)

NLS (j.f)

	u			u_1	uv			u_2	u_1v	u^2	uv^2	
$K_{1,0}$	1			$K_{2,0}$	1	2		$K_{3,0}$	1	3	$\frac{3}{2}$	3
								$K_{2,1}$	0	1	$-\frac{1}{2}$	1

	u_3	u_2v	u_1v_1	uv_2	uu_1	u_1v^2	u^2v	uv^3
$K_{4,0}$	1	4	2	2	3	6	6	4
$K_{3,1}$	0	1	-1	0	0	3	0	2

	u_4	u_3v	u_2v_1	u_1v_2	uv_2	u_1^2	u_2v^2	u_1vv_1	uu_1v	uv_1^2	uvv_2	u^3	u_1v^3	u^2v^2	uv^4
$K_{5,0}$	1	5	5	5	5	$\frac{5}{2}$	10	10	15	5	10	$\frac{5}{2}$	10	15	5
$K_{4,1}$	0	1	-1	1	0	$-\frac{1}{2}$	4	-2	3	-1	2	$-\frac{1}{2}$	6	3	3
$K_{3,2}$	0	0	-1	0	1	$-\frac{1}{2}$	1	-2	0	-1	0	1	2	0	1

	u_5	u_4v	u_3v_1	u_2v_2	u_1v_3	uv_2	uu_3	u_1u_2	u_3v^2	u_2vv_1	u_1vv_2	$u_1v_1^2$	uu_2v
$K_{6,0}$	1	6	9	11	4	2	5	10	15	30	30	15	30
$K_{5,1}$	0	1	-1	1	-1	0	0	0	5	0	10	0	5
$K_{4,2}$	0	0	-1	1	0	0	1	-1	1	-6	2	-3	4

	u_1^2v	uu_1v_1	u^2v_2	u^2u_1	u_2v^3	$u_1v^2v_1$	uv^2v_2	uvv_1^2	uu_1v^2	u^3v	u_1v^4	u^2v^3	uv^5
$K_{6,0}$	15	20	10	$\frac{15}{2}$	20	30	30	30	45	15	15	30	6
$K_{5,1}$	0	-20	0	0	10	0	10	0	15	0	10	10	4
$K_{4,2}$	-3	15	2	$\frac{3}{2}$	4	-6	2	-6	3	3	5	2	2

(表5)

KIV 方程式

$$\begin{aligned}
& D_0(D_0^3 - D_1) f \cdot f = 0, \quad (D_0^3 - D_1)(D_0^3 + 2D_1) f \cdot f = 0, \quad D_0^2(D_0^3 - D_1)(D_0^3 - 4D_1) f \cdot f = 0, \\
& D_0(D_0^3 - D_1)(D_0^3 + 2D_1)(D_0^3 - 4D_1) f \cdot f = 0, \quad (D_0^3 - D_1)(D_0^3 - 4D_1)(D_0^6 + 27D_0^3D_1 + 16D_1^2) f \cdot f = 0 \\
& \text{weight 14 以上 } \angle \text{ は } D_0, D_1 \text{ の 4 の 方程式は存在しない。} \\
& D_0(D_0^3D_1^2 + D_1^3 - 2D_4) f \cdot f = 0, \quad (2D_0D_1^3 + 3D_2^2 - 3D_1D_3 - 2D_0D_4) f \cdot f = 0, \\
& D_1(2D_0^6D_1 + 3D_0D_1D_2 - 5D_4) f \cdot f = 0, \quad (D_0^4D_1D_2 - D_0D_1^2D_2 + D_0^5D_3 - D_1D_4) f \cdot f = 0, \\
& (D_0D_1D_2^2 - D_0^4D_1D_3 + D_2D_4 - D_1D_5) f \cdot f = 0, \quad (D_0^3D_1^2D_2 - D_2D_4 - 3D_0^5D_5 + 3D_0D_6) f \cdot f = 0, \\
& D_0^2(D_1D_m - D_0D_{m+1}) f \cdot f = 0, \quad (2D_0^3D_m + D_1D_m - 3D_0D_{m+1}) f \cdot f = 0, \\
& (2D_0^5D_m + 3D_2D_m - 5D_1D_{m+1}) f \cdot f = 0 \quad (m = 0, 1, 2, \dots : \text{安定性の系}).
\end{aligned}$$

(表6)

MKIV 方程式

$$\begin{aligned}
& D_0^2 f \cdot g = 0, \quad (D_0^3 - D_1) f \cdot g = 0, \quad D_0(D_0^3 + 2D_1) f \cdot g = 0, \quad D_0^2(D_0^3 - D_1) f \cdot g = 0, \\
& D_0^3(D_0^3 - 4D_1) f \cdot g = 0, \quad D_1(2D_0^3 + D_1) f \cdot g = 0, \quad D_0^4(D_0^3 - D_1) f \cdot g = 0, \\
& D_0D_1(D_0^3 - D_1) f \cdot g = 0, \quad D_0^2(D_0^3 + 2D_1)(D_0^3 - 4D_1) f \cdot g = 0, \quad D_0^3(D_0^3 - D_1)(D_0^3 + 8D_1) f \cdot g = 0, \\
& D_1(D_0^3 - D_1)^2 f \cdot g = 0, \quad D_0^4(D_0^3 - 4D_1)^2 f \cdot g = 0, \quad D_0D_1(2D_0^3 + D_1)(D_0^3 - 4D_1) f \cdot g = 0, \\
& D_0^2(D_0^3 - D_1)(D_0^6 - 32D_0^3D_1 - 32D_1^2) f \cdot g = 0, \quad D_0^2D_1(D_0^3 - D_1)(D_0^3 + 2D_1) f \cdot g = 0, \\
& D_0^3(D_0^3 + 2D_1)(D_0^3 - 4D_1)^2 f \cdot g = 0, \quad D_1(D_0^3 + 2D_1)(D_0^3 - 4D_1)(2D_0^3 + D_1) f \cdot g = 0, \\
& D_0^4(D_0^3 - D_1)(D_0^6 - 8D_0^3D_1 - 56D_1^2) f \cdot g = 0, \quad D_0D_1(D_0^3 - D_1)(D_0^6 + D_0^3D_1 + 16D_1^2) f \cdot g = 0, \\
& D_0^5(D_0^3 - 4D_1)^3 f \cdot g = 0, \quad D_0^2D_1(D_0^3 - 4D_1)^2(2D_0^3 + D_1) f \cdot g = 0, \quad \dots \dots \dots \\
& \text{weight 20 以上 } \tau \text{ は } D_0, D_1 \text{ の 4 の 方程式は存在しない。} \\
& (D_0^5 - D_2) f \cdot g = 0 \quad (\text{Lym の例}), \quad D_0(D_0^2D_1 + D_2) f \cdot g = 0, \quad (D_0^7 - D_3) f \cdot g = 0,
\end{aligned}$$

$$\begin{aligned}
& D_0 (D_0 D_1^2 + 2 D_0^2 D_2 + 2 D_3) f \cdot f = 0, \quad D_0^2 (D_0 D_1^2 - D_3) f \cdot f = 0, \quad D_0 (D_0^3 D_1^2 + 2 D_4) f \cdot f = 0, \\
& D_0^2 (D_1^3 - D_4) f \cdot f = 0, \quad D_0 D_2 (D_0^5 - D_2) f \cdot f = 0, \quad D_0 (D_0^3 D_1 D_2 - D_1^2 D_2 - 3 D_0^2 D_4) f \cdot f = 0, \\
& (D_0^2 D_1^2 D_2 + D_1^2 D_3 - 2 D_0^4 D_4) f \cdot f = 0, \quad D_0^2 D_1 (D_1^3 - 4 D_4) f \cdot f = 0, \\
& (D_0^2 D_m - D_{m+1}) f \cdot f = 0, \quad \left(\sum_{v=1}^m (-1)^v D_v D_{m-v} \right) f \cdot f = 0 \quad (\text{但し } m \text{ odd } \lambda \neq 1 \text{ is trivial}), \\
& (D_0^2 \sum_{v=0}^m D_v D_{m-v} + 2 D_0 D_{m+1}) f \cdot f = 0, \\
& (D_0^3 D_m - D_1 D_m + 3 D_0 D_{m+1}) f \cdot f = 0, \quad (D_0^4 D_m + 2 D_0 D_1 D_m - 3 D_0^2 D_{m+1}) f \cdot f = 0, \\
& (D_0^5 D_m - D_0^2 D_1 D_m - 3 D_0^3 D_{m+1}) f \cdot f = 0, \quad (D_0^5 D_m - D_2 D_m + 5 D_0 D_{m+2}) f \cdot f = 0, \\
& (D_0^7 D_m - D_3 D_m - 7 D_0^2 D_1 D_{m+1} + 7 D_0 D_{m+3}) f \cdot f = 0 \quad (m=0, 1, 2, \dots : \text{安定性定理系}).
\end{aligned}$$

(表7)

NLS 方程式

$$(D_1^2 - D_2) f \cdot g = 0, \quad D_1 (D_1^2 - D_2) f \cdot g = 0, \quad (D_1^2 - D_2)(D_1^2 + 2D_2) f \cdot g = 0,$$

$$D_1 (D_1^2 - D_2)(D_1^2 - 2D_2) f \cdot g = 0, \quad (D_1^2 - D_2)(D_1^2 - 2D_2)^2 f \cdot g = 0,$$

weight 7 以上では D_1, D_2 のみの方程式は存在しない。

$$(D_1^3 - D_3) f \cdot g = 0, \quad D_1 (D_1^3 + 3D_1 D_2 - 4D_3) f \cdot g = 0, \quad (D_1^4 - 3D_2^2 + 2D_1 D_3) f \cdot g = 0,$$

$$D_1 (D_2^2 - D_1 D_3) f \cdot g = 0, \quad D_2 (D_1^3 - D_3) f \cdot g = 0, \quad (D_1^5 + 2D_1 D_2^2 - 3D_2 D_3) f \cdot g = 0,$$

$$D_1 (D_1^4 + 5D_1 D_3 - 6D_4) f \cdot g = 0, \quad (D_1^2 D_2^2 - D_3^2 + D_2 D_4 - D_6) f \cdot g = 0,$$

$$(D_1 D_2^3 + 3D_1^2 D_2 D_3 - 4D_7) f \cdot g = 0, \quad D_1 (D_1^3 D_3 - D_3^2 + 3D_1 D_5 - 3D_6) f \cdot g = 0,$$

$$(D_1^3 D_2 D_3 + D_2^2 D_4 - 2D_8) f \cdot g = 0, \quad (D_1 D_2^2 D_3 + D_2^2 D_4 - D_1^3 D_5 - D_1^2 D_6) f \cdot g = 0,$$

$$(D_1^8 - 5D_1^6 D_2 + 20 D_1^4 D_4 - 16 D_1^3 D_5) f \cdot g = 0,$$

$$(D_1 D_m - D_{m+1}) f \cdot g = 0, \quad (D_1^2 D_m + D_2 D_m - 2D_1 D_{m+1}) f \cdot g = 0 \quad (m=1, 2, 3, \dots).$$

$$D_1^2 (D_1^4 + 3 D_2^2 - 4 D_1 D_3) f \cdot f = 0, \quad (D_1^2 D_2^2 + 2 D_3^2 - D_2 D_4 - 2 D_1 D_5) f \cdot f = 0,$$

$$\begin{aligned}
& (D_1^4 D_2^2 + 2 D_1^3 D_3 + 5 D_2 D_4 - 8 D_1 D_5) f \cdot f = 0, \quad (D_1^2 D_2 D_3 + D_3 D_4 - 2 D_1 D_6) f \cdot f = 0, \\
& (D_1^2 D_2 D_3 + D_1^3 D_4 + 3 D_2 D_5 - 5 D_1 D_6) f \cdot f = 0, \quad (D_1 D_2^3 + 3 D_3 D_4 - 3 D_2 D_5 - D_1 D_6) f \cdot f = 0, \\
& (D_1^3 D_m - D_3 D_m + 3 D_2 D_{m+1} - 3 D_1 D_{m+2}) f \cdot f = 0 \quad (m=1, 2, 3, \dots).
\end{aligned}$$

Jimbo-Miwa[8] の (31), (33) 式 参照.

[付録 1] Painlevé II 型方程式: $y_{xx} = 2y^3 + xy + \alpha$, $y = (\log \frac{f}{g})_x$,

$$\begin{cases} D_x^2 f \cdot g = 0 \\ (D_x^3 - \alpha D_x - \alpha) f \cdot g = 0 \end{cases} \quad ([4]), \quad (D_x^6 - 4\alpha D_x^4 - 4\alpha x D_x - 4\alpha^2) f \cdot g = 0.$$

[付録 2] 沢田-小寺 KdV 方程式: $(D_x^6 - D_x D_t) f \cdot f = 0$, $D_x = D_0$, $D_t = D_2$

$$D_0(D_0^5 - D_2) f \cdot f = 0 \quad ([2]), \quad D_0(3D_0^7 + 7D_0^2 D_2 - 10D_3) f \cdot f = 0,$$

$$(D_0^5 - D_2)(D_0^5 + 4D_2) f \cdot f = 0, \quad (6D_0^5 D_2 - D_2^2 - 5D_0^3 D_3) f \cdot f = 0,$$

$$D_0^3(D_0^7 - 21D_0^2 D_2 + 20D_3) f \cdot f = 0, \quad (D_0^{12} + 7D_0^7 D_2 + 7D_0^2 D_2^2 - D_0^5 D_3 - 14D_2 D_3) f \cdot f = 0.$$

[付録 3] SIT 方程式: $\begin{cases} D_x^2 f \cdot g = 0 \dots\dots\dots \textcircled{1} \\ (-D_t - D_x^2 D_t + D_x) f \cdot g = 0 \dots\dots \textcircled{2} \end{cases} \quad (\text{Hirota, Oishi [4]})$

$D_x = D_0$, $D_t = \sum_{m=0}^{\infty} (-)^m D_m$ と取れば, $\textcircled{1}$ のもとで, 1個の SIT 方程式②

$$\text{と 1 系列の MKdV 方程式: } (D_0^2 D_m - D_{m+1}) f \cdot g = 0 \quad (m=0, 1, 2, \dots) \quad \text{と 同値である.}$$

$$\text{更に, 1 個の SIT 方程式: } (D_x^2 D_t^2 - 2D_x D_t) f \cdot g = 0 \dots\dots \textcircled{3}$$

$$\text{と 1 系列の MKdV 方程式: } (D_0^2 \sum_{v=0}^m D_v D_{m-v} + 2D_0 D_{m+1}) f \cdot g = 0 \quad (m=0, 1, 2, \dots)$$

も同値である。

(表8)

n	p(n)	g(n)				n	p(n)	g(n)				n	p(n)	g(n)			
		even p(n)	odd p(n)	even f(n)	odd f(n)			even p(n)	odd p(n)	even f(n)	odd f(n)			even p(n)	odd p(n)	even f(n)	odd f(n)
0	1	1	1	1	1	20	627	42	64	10	7	40	37338	627	1113	64	46
1	1	0	1	0	1	21	792	0	76	0	8	41	44583	0	1260	0	49
2	2	1	1	1	0	22	1002	56	89	12	8	42	53174	792	1426	76	52
3	3	0	2	0	1	23	1255	0	104	0	9	43	63261	0	1610	0	57
4	5	2	2	1	1	24	1575	77	122	15	11	44	75175	1002	1816	89	63
5	7	0	3	0	1	25	1958	0	142	0	12	45	89134	0	2048	0	68
6	11	3	4	2	1	26	2436	101	165	18	12	46	105558	1255	2304	104	72
7	15	0	5	0	1	27	3010	0	192	0	14	47	124754	0	2590	0	78
8	22	5	6	2	2	28	3718	135	222	22	16	48	147273	1575	2910	122	87
9	30	0	8	0	2	29	4565	0	256	0	17	49	173525	0	3264	0	93
10	42	7	10	3	2	30	5604	176	296	27	18	50	204226	1958	3658	142	98
11	56	0	12	0	2	31	6842	0	340	0	20	51	239943	0	4097	0	107
12	77	11	15	4	3	32	8349	231	390	32	23	52	281589	2436	4582	165	117
13	101	0	18	0	3	33	10143	0	448	0	25	53	329931	0	5120	0	125
14	135	15	22	5	3	34	12310	297	512	38	26	54	386155	3010	5718	192	133
15	176	0	27	0	4	35	14883	0	585	0	29	55	451276	0	6378	0	144
16	231	22	32	6	5	36	17977	385	668	46	33	56	526823	3718	7108	222	157
17	297	0	38	0	5	37	21637	0	760	0	35	57	614154	0	7917	0	168
18	385	30	46	8	5	38	26015	490	864	54	37	58	715220	4565	8808	256	178
19	490	0	54	0	6	39	31185	0	982	0	41	59	831820	0	9792	0	192

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