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ANALYSIS OF THE NAVIER-STOKES EQUATIONS BASED ON GREEN'S FUNCTION TECHNIQUE

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This work describes a novel approach for the numerical calculations of incompressible viscous flow problems. For the flow region one can think of a fluid contained between three fixed walls, while the upper wall moves in its own plane creating motion into fluid.

In contrast to the usual finite-difference scheme the Navier-Stokes equations are solved by an iterative-integral scheme using Green's function. The solutions for steady problem-stream function and vorticity are computed for Reynolds numbers 0,2, 4,8,16,32,64 and 100 in a square of side unity for the mesh size 1/10. Checking with finite-difference was in good agreement.

The analytical part of the time dependent Navier-Stokes equations is formulated in the same fashion as steady case and are based on the time dependent Green's function.The numerical part of it is in the process of computations and satisfactory results are expected.

1. INTRODUCTION

Streaming flows past an obstacle and the viscous flows occouring inside a closed domain have intensive

numerical literature based mostly on difference scheme, whereas integral approach in connection with Green's function have received comperatively little attention.

Many numerical schemes have been developed for solving steady and unsteady viscous incompressible Navier-Stokes flow equations for different regions.

The present problem, being of wide interest has been handled numerically as well as experimentally by several fluid dynamicists. Kawaguchi[1] seems to have been the first to consider these fluid motions numerically. Further contributions for such kind of steady problem are done by many workers.[2]

On the other hand previous studies concerning time dependent Navier-Stokes equations are due to Simuni[3], Pearson[4], and Greenspan[5]. But most of them are treated numerically by addopting difference scheme.

Recently R.D.Mills[6] presented the numerical results of the steady two-dimensional viscous motion within a circular cylinder generated by the rotation of part of the cylinder wall for low Reynolds numbers by applying Iterative-Integral technique which based on the biharmonic Green's function. He compaired and correlated his work to others. Harmonic Green's function have been used in viscous flow problems[7] but this approach still requires the use of finite-difference method for the solution of the vorticity equation.

Latest related work on the integral representations approach for the time dependent viscous flows is supported by J.C.Wu & Y.M.Rizk[8] who considered time dependent incompressible viscous

flows involving two special dimensions past non-rotating finite solids.

The present work is motivated by initiating a programme of research with the goal of establishing numerical approach with better accuracy and less time consumption than other schemes based totally on integral representations using Green's function.

The flow problem which is interest of fluid dynamicists,numerical analysts(Kawaguchi, Greenspan), aerodynamicists(Mills) and laboratory researchers (Ozawa, Pan & Acrivos) is considered We expect that present work will represent the first successful approach of the totally integral representations based on Green's function to the time dependent internal viscous flow problem.

2. FORMULATION OF THE PROBLEM

The physical cofiguration of the box shape mathematical model sketching in Fig.a consists of three fixed and one moving wall. The movement of the upper wall creats motion into the incompressible fluid contained in it. A cartesian co-ordinate (x,y) system is introduced with its centre at the lower left corner. The region of physical interest has dimensions A and L along x and y-axis respectively.

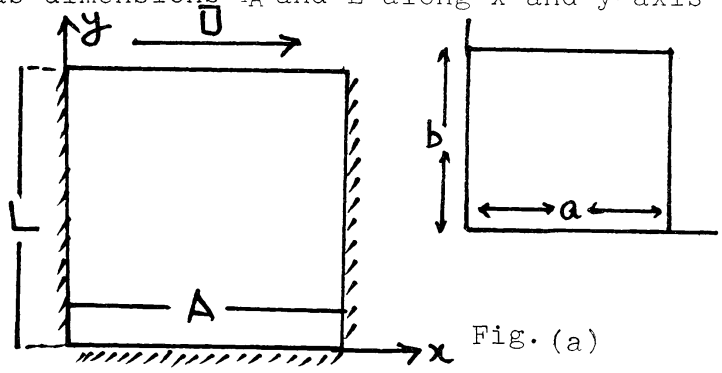


Fig. (a)

FUNDAMENTAL EQUATIONS

The familiar unsteady, viscous and incompressible Navier-Stokes (2.1), (2.2) and continuity equations (2.3) form a set of three simultanous partial differential equations in three unknown u, v, and p which could be solved for many problems.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}p_x + \nu\nabla^2 u \quad \text{-----(2.1)}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}p_y + \nu\nabla^2 v \quad \text{-----(2.2)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{----------(2.3)}$$

The relation between stream function ψ and velocity components u, v (2.4) plus the definition of vorticity ω (2.5) as the curl of the velocity vector are given by,

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x} \quad \text{---------- (2.4)}$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{----------(2.5)}$$

The relation between stream function and vorticity is obtained by using (2.4) and (2.5).

$$\nabla^2\psi = -\omega \quad \text{----------(2.6)}$$

Introduction of stream function automatically satisfies the equation of continuity (2.3). The dynamical equation of vorticity (2.7) is established by differenciating (2.1), (2.2) with respect to y and x respectively, subtracting and using (2.3), (2.4) and (2.5)

$$\frac{\partial \omega}{\partial t} - \frac{\partial(\psi,\omega)}{\partial(x,y)} = \nu\nabla^2\omega \quad \text{-------- (2.7)}$$

where $\frac{\partial(\psi,\omega)}{\partial(x,y)} = \frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}$

The pressure p has been eliminated by this process. For the present problem on fixed walls the velocity of the fluid vanishes while on the moving one it is prescribed arbitrarily parallel to itself.

$$u = 0 = v \qquad 0 \le x \le a;\ y = a \text{ ---------- } (2.8a)$$
$$u = 0 = v \qquad 0 \le y \le b;\ x = 0 \text{ ---------- } (2.8b)$$
$$u = 0 = v \qquad 0 \le y \le b;\ x = a \text{ ---------- } (2.8c)$$
$$u = \bar{U}\ v = 0 \qquad 0 \le x \le a;\ y = b \text{ ---------- } (2.8d)$$

DIMENSIONLESS FORM

For simplicity and convenience all the quantities are transformed in dimensionless form since these are useful in numerical computations. The dimensionless variables (capitals) are defined by taking reference length L and velocity $\bar{U}$ of the left and upper wall respectively.

$$X = x/L;\ Y = y/L;\ U = u/L,\ V = v/L$$
$$A = a/L;\ \Psi = \psi/\bar{U}L;\ \omega = \omega L/\bar{U}$$
$$T = t\bar{U}/L \text{ ---------- } (2.9)$$

The dimensionless forms of the vorticity (2.10) and the dynamical equation of vorticity (2.11) are given by,

$$\nabla^2\Psi = -\omega \text{ ---------- } (2.10)$$
$$\partial\omega/\partial T = -\partial(\Psi,\omega)/\partial(X,Y) = 1/R\,\nabla^2\omega \text{ ---- } (2.11a)$$
$$\nabla^2\omega = -R\,\partial(\Psi,\omega)/\partial(X,Y) \text{ (steady case) ---------- } (2.11b)$$

where $R = \bar{U}L/\nu$ is the Reynolds number

The boundary conditions demand that on the fixed walls the normal and tangential gradients of stream function vanish, while on the moving wall normal gradient is unity.

$$\Psi = 0 = \partial\Psi/\partial X \qquad 0 \le Y \le 1;\ X = 0 \text{ ---------- } (2.12a)$$
$$\Psi = 0 = \partial\Psi/\partial Y \qquad 0 \le X \le A;\ Y = 0 \text{ ---------- } (2.12b)$$

$$\psi = 0 = \partial\psi/\partial X \qquad 0 \le Y \le 1 \; ; \; X = A \qquad \text{-------(2.12c)}$$

$$\psi = 0 \; ; \; \partial\psi/\partial Y = 1 \qquad 0 \le X \le A \; ; \; Y = 1 \qquad \text{-----(2.12d)}$$

3. STEADY PROBLEM

We initiate with the steady problem. The region of numerical interest, flow equations and imposed boundary conditions are sketched in Fig. b. Differential flow equations in the form of integrals are prepared as the bases of our numerical approach.

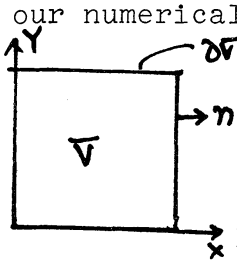
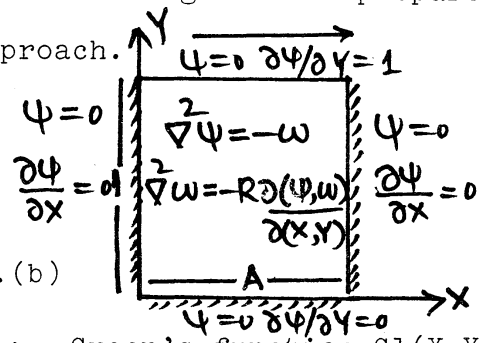


Fig.(b)



The two-point func. ie Green's function $G1(X,Y;X',Y')$ is defined uniquely by the delta function relation (3.1) and the prescribed boundary conditions (3.2) in the following form,

$$\nabla^2 G_1(\mathbf{X},\mathbf{X}') = \delta^2(\mathbf{X}-\mathbf{X}') \quad \mathbf{X},\mathbf{X}' \in \bar{V} \text{----------(3.1)}$$

$$G_1(\mathbf{X},\mathbf{X}') = 0 \quad \mathbf{X} \in \partial V, \mathbf{X}' \in \bar{V}+\partial V \text{----------(3.2)}$$

$\mathbf{X}=X,Y$ and $\mathbf{X}'=X',Y'$

INTEGRAL EXPRESSION FOR $\psi(X,Y)$ AND $\omega(X,Y)$

Starting with the Green's identity and substituting $f'=\psi(X',Y')$ and $g'=G1(X',Y';X,Y)$

$$\iint_V (f'\nabla'^2 g' - g'\nabla'^2 f')\, d^2\mathbf{X}' = \int_{\partial V} \left(f'\frac{\partial g'}{\partial n'} - g'\frac{\partial f'}{\partial n'}\right) ds' \text{----------(3.3)}$$

Right hand side of (3.3) vanishes due to the prescribed boundary condition (2.12) and (3.2), while the left hand side using (3.1) and (2.10) yields,

$$\psi(X,Y) = -\iint_V \omega(X',Y')\, G_1(\mathbf{X}',\mathbf{X})\, d^2\mathbf{X}' \text{----------(3.4)}$$

In the similar manner (3.1),(3.2) and (2.11a) provide the expression for vorticity.

$$\omega(\mathbf{x}) = -R \iint_{V} \partial(\psi',\omega')/\partial(x',y')\, G_1(\mathbf{x}',\mathbf{x})\, d^2\mathbf{x}' + \int_{\partial V} \omega(s') \frac{\partial G_1}{\partial n'}(s',\mathbf{x})\, ds'$$

$$\partial G_1(s',\mathbf{x})/\partial n' = \partial G_1(\mathbf{x}',\mathbf{x})/\partial n' \,|_{\mathbf{x}' \in \partial V} \qquad \text{----------(3.5)}$$

CONSTRUCTION OF GREEN'S FUNCTION (IMAGE METHOD)

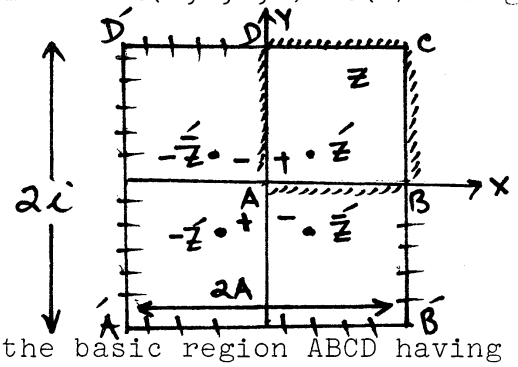
In the theory of function there exists function $f(Z,Z')$ analytic with respect to Z except at $Z=Z'$ and whose real part is the required Green's func. $G1(X,Y;X',Y')$.

$$G1(X,Y;X',Y') = Re(f(Z,Z')) \text{ ---------- } (3.6a)$$

where,

$Z=X+iY$ and $Z'=X'+iY'$

As $Z \longrightarrow Z'$ $\quad G1(X,Y;X',Y') = Re(1/2\pi\ Log(Z-Z')) \text{ ----------(3.6b)}$



Consider the basic region ABCD having a positive charge Z' and on which the boundary condition (3.2) is satified. Due to (3.2) Z' is reflected periodically with periods 2A and 2i along real and imaginary axis respectively. $A'B'C'D'$ is the periodic region. Here we introduced elliptic theta function which is doubly periodic and has paramount importance in numerical calculations due to its rapid convergence. Expression for theta product (3.7) and periodicity relationships (3.8) are given by,

$$\theta 1(Z,q) = 2Q^{1/4} \operatorname{Sin}\pi Z \prod_{1}^{L} (1-2q^{2L} \operatorname{Sin}\pi Z + q^{4}) \text{----------(3.7)}$$

$$\theta_1(z+1) = \theta_1(z)$$
$$\theta_1(z+\tau) = -M\,\theta_1(z) \qquad \text{----------(3.8)}$$
$$q = \exp(i\pi\tau) \qquad \tau = \omega_2/\omega_1$$
$$= \prod_{l=1}^{L}(1-q^{2l})$$
$$M = q^{-1}\exp(-2i\pi z)$$

ω1: Period along real axis

ω2: Period along Imaginary axis

In terms of theta function the analytic function f(Z,Z') is expressed by,

$$f(z,z') = \frac{1}{2\pi}\log\frac{\theta_1\left(\frac{z-z'}{2A}\right)\theta_1\left(\frac{z+z'}{2A}\right)}{\theta_1\left(\frac{z-\bar{z}'}{2A}\right)\theta_1\left(\frac{z+\bar{z}'}{2A}\right)} \qquad \text{==========(3.9)}$$

The expression for the derivative on the left (3.10a) and lower (3.10b) walls are obtained by taking real and imaginary parts of the derivative of f(Z,Z') with respect to Z'. On the other walls the sign will be opposite.

$$\frac{\partial G_1(\mathbf{x},\mathbf{s}')}{\partial n'} = -\frac{\partial G_1}{\partial y'} = \mathrm{Im}\,\frac{\partial}{\partial z'}\tilde{f}(z,z')\Big|_{z=\mathbf{x}\in\bar{V},\ z'=s'\in\gamma V} \qquad \text{----------(3.10a)}$$
$$\frac{\partial G_1(\mathbf{x},\mathbf{s}')}{\partial n'} = -\frac{\partial G_1}{\partial x'} = \mathrm{Re}\,\frac{\partial}{\partial z'}\tilde{f}(z,z') \qquad \text{----------(3.10b)}$$
$$\tilde{f}(z,z') = \frac{1}{2\pi}\log\frac{\theta_1\left(\frac{z-z'}{2A}\right)\theta_1\left(\frac{z+z'}{2A}\right)}{\theta_1\left(\frac{\bar{z}-z'}{2A}\right)\theta_1\left(\frac{\bar{z}+z'}{2A}\right)}$$

$\tilde{f}$(Z,Z') is an analytic function of Z' whose real part is also the required Green's function.

INTEGRAL EXPRESSION FOR VORTICITY ON THE BOUNDARY

In getting (3.4) the tangential conditions are satisfied,while the normal conditions will provide the informations of vorticity on the boundary.

$$\psi(\mathbf{x}) = -\iint_V \omega(\mathbf{x}'')\,G_1(\mathbf{x}'',\mathbf{x})\,d^2\mathbf{x}''$$

$$\omega(\mathbf{x}'') = -R \iint_V \frac{\partial(\psi',\omega')}{\partial(x',y')} G_1(\mathbf{x}',\mathbf{x}'')\, d^2\mathbf{x}' + \int_{\partial V} \tilde{\omega}(s') \frac{\partial G_1}{\partial n'}(s',\mathbf{x}'')\, ds'$$

$\Rightarrow$

$$\psi(\mathbf{x}) = -\iint_V G_1(\mathbf{x}'',\mathbf{x}) \left[-R \iint_V \frac{\partial(\psi',\omega')}{\partial(x',y')} G_1(\mathbf{x}',\mathbf{x}'')\, d^2\mathbf{x}' + \int_{\partial V} \tilde{\omega}(s') \frac{\partial G_1}{\partial n'}(s',\mathbf{x}'')\, ds' \right] d^2\mathbf{x}''$$

$$= R \iint_V \Omega(\mathbf{x}'') G_1(\mathbf{x}'',\mathbf{x})\, d^2\mathbf{x}'' - \int_{\partial V} \tilde{\omega}(s') \iint_V G_1(\mathbf{x}'',\mathbf{x}) \frac{\partial G_1}{\partial n'}(s',\mathbf{x}'')\, d^2\mathbf{x}''\, ds'$$

$$\Rightarrow \frac{\partial \psi(s)}{\partial n} = R \iint_V \Omega(\mathbf{x}'') \frac{\partial G_1}{\partial n}(s,\mathbf{x}'')\, d^2\mathbf{x}'' - \int_{\partial V} \tilde{\omega}(s') \iint_V \frac{\partial G_1(s',\mathbf{x}'')}{\partial n'} \frac{\partial G_1(\mathbf{x}'',s)}{\partial n}\, d^2\mathbf{x}''\, ds'$$

$$\Rightarrow \psi_n(s) = RA(s) - \int_{\partial V} \tilde{\omega}(s')\, \tilde{F}(s,s')\, ds'$$

where $\tilde{F}(s,s') = \iint_V \frac{\partial G_1(s',\mathbf{x}'')}{\partial n'} \frac{\partial G_1(\mathbf{x}'',s)}{\partial n}\, d^2\mathbf{x}''$

$$A(s) = \iint_V \Omega(\mathbf{x}'') \frac{\partial G_1}{\partial n}(s,\mathbf{x}'')\, d^2\mathbf{x}''$$

$$\Omega(\mathbf{x}'') = \iint_V \partial(\psi',\omega')/\partial(x',y')\, G_1(\mathbf{x}',\mathbf{x}'')\, d^2\mathbf{x}'$$

$$\Rightarrow \int_{\partial S} \tilde{\omega}(s')\, \tilde{F}(s,s') = RA(s) - \psi_n(s) = B(s) \qquad \text{----------(3.11)}$$

In the equation (3.11) $F(S,S')$ and $B(S)$ are known and hence $\tilde{\omega}(s')$ can be computed .

NUMERICAL PROCEDURE

The integral expressions (3.4), (3.5) and (3.11) for $\psi(X,Y)$, $\omega(X,Y)$ and $\tilde{\omega}(S)$ respectively are developed for the numerical calculations of stream function and vorticity for steady flow inside a square domain.

The following steps include the procedure of calculations.

Step1

The Green's function $G1(X,Y;X',Y')$ inside a square with side unity is computed with mesh size 1/10 for the whole region once and preserved in four dimensional array except for $X=Y=X'=Y'$ (the singularity behaviour) which required large demands on storage for all permutations of $(X,Y;X',Y')$. Otherwise, we should have to compute G1 for the whole region in every iteration, which would lead to large computing times due to sine and log functions.

Step2

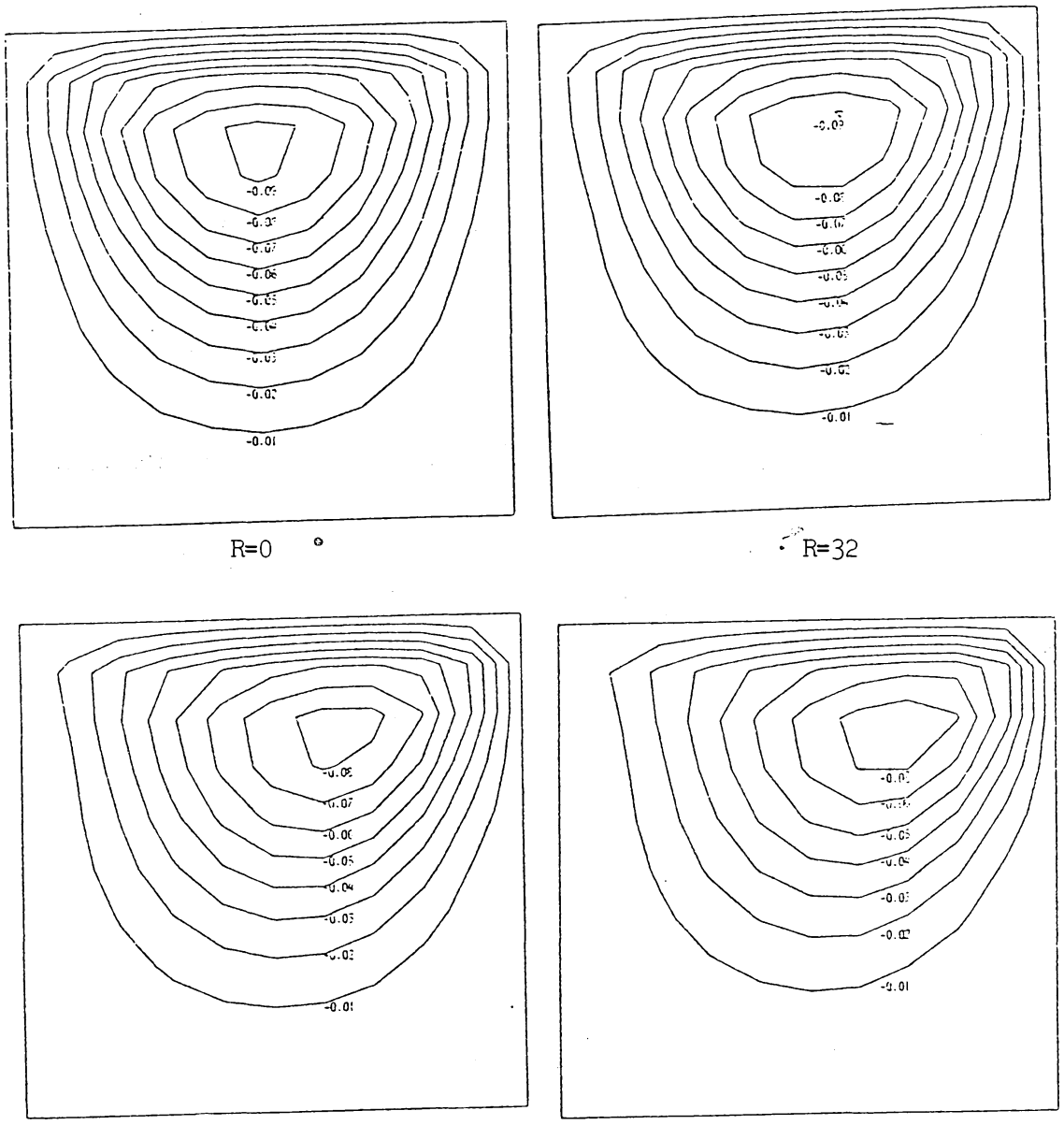
In the same fashion derivative of Green's function and $F(S,S')$ are calculated and preserved.

Step3

The integral equations (3.4) and (3.5) are then solved iteratively with due consideration to (3.11) with mesh size 1/10 for Reynolds numbers upto 100 at each mesh point of the square region of length unity. The region of singularity is treated seperately in (3.4).

Step4

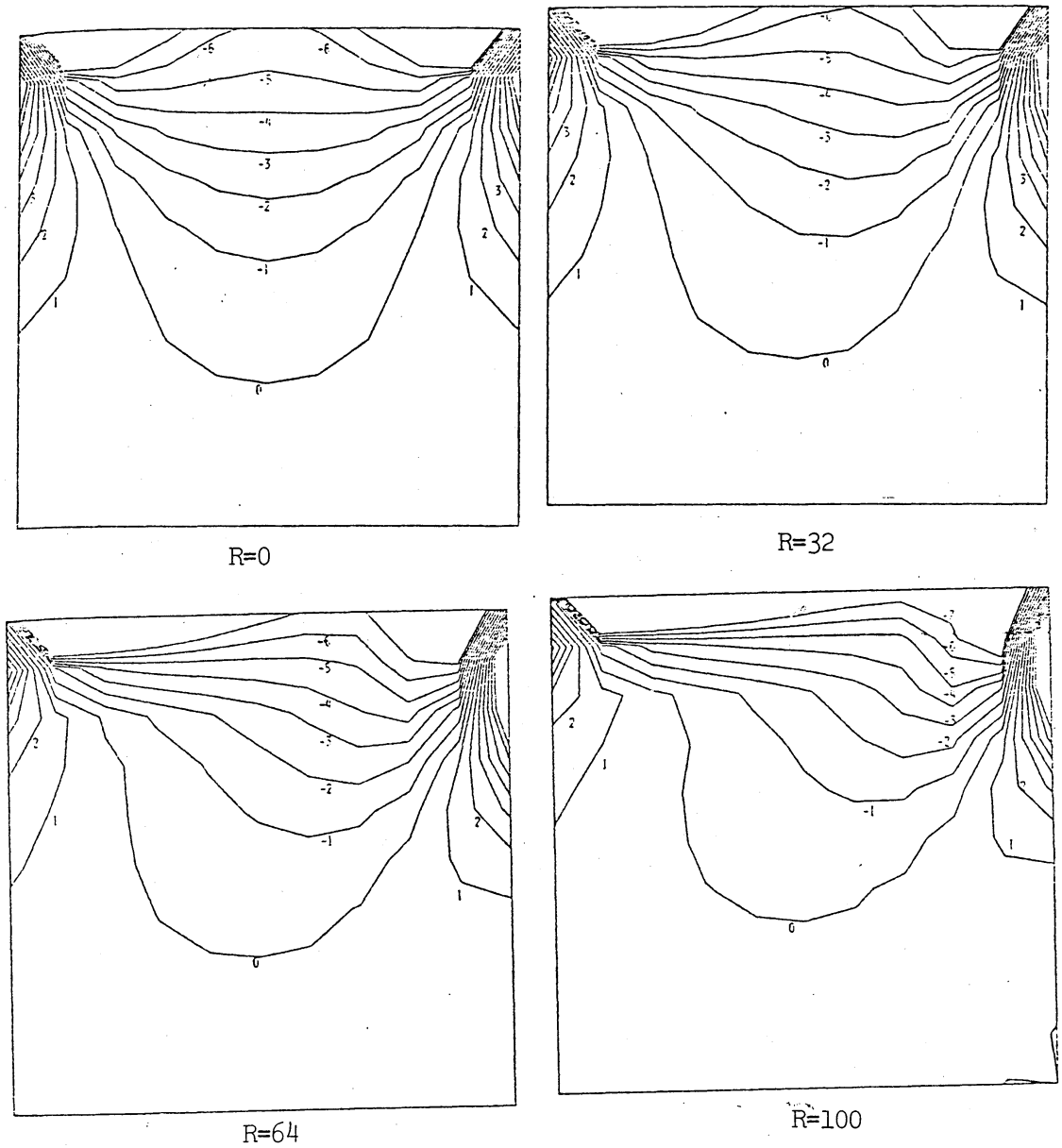
Numerical results— streamlines and equivorticity curves are plotted by automatic plotter and are shown for Reynolds numbers 0,32,64 and 100.



STREAMLINES Fig(1)

RESULTS AND DISCUSSIONS

Streamlines are shown for Reynolds numbers 0, 32, 64 and 100. For low Reynolds numbers symmetric flow pattern is obtained and the vortex centre is shifted towads right (since the movement was



Equivorticity curves Fig.(2)

from left to right) as the Reynolds numbers increased. The flow pattern (streamlines) and the equivorticity curves from this scheme have resemblence with those obtained by finite-difference method. As a check the same problem is solved by difference scheme at each mesh point and is found in good agreement.

4. UNSTEADY PROBLEM

For the next stage attention is directed to unsteady or time dependent problem inside a box and is shown in Fig.(C).

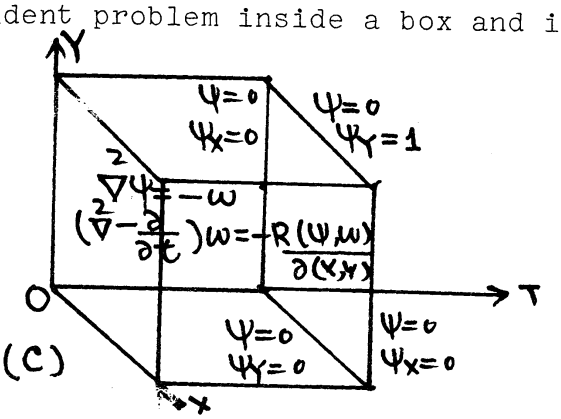


Fig (C)

Starting with the flow equations (4.12) and (4.13)

$$\nabla^2\psi(X,Y,t)=-\omega(X,Y,t) \text{ ---------- } (4.12)$$

$$(\nabla^2-\frac{\partial}{\partial t})\omega=-R\partial(\psi,\omega)/\partial(X,Y) \text{ ---------- } (4.13)$$

where,

$t\equiv R^{-1}T$ putting in (2.11a)

In formulating the present problem the time dependent Green's function is defined by (4.14),(4.15)—delta function relations (4.16)—boundary conditions and (4.17)—time dependent boundary conditions. Using Green's theorem (4.19), equations (4.12) and (4.13) are transformed into integral forms (4.18) and (4.20). Similar to steady case (4.21) is prepared to compute vorticity on the boundary.

$$L_1=\nabla^2 \qquad L_2=\nabla^2-\frac{\partial}{\partial t}$$

$$\hat{L}_2=\nabla^2+\frac{\partial}{\partial t} \text{ (adjoint to L2)}$$

$$L_1 G_1(\mathbf{x},\mathbf{x}')=\delta^2(\mathbf{x}-\mathbf{x}') \text{ ---------- } (4.14)$$

$$\hat{L}_2\hat{G}_2(\mathbf{x},t;\mathbf{x}',t')=\delta^2(\mathbf{x}-\mathbf{x}')\,\delta(t-t') \quad \mathbf{x},\mathbf{x}'\in S \text{ ---- } (4.15)$$

$$G_1(\mathbf{x},\mathbf{x}') = \hat{G}_2(\mathbf{x},t;\mathbf{x}',t') = 0 \qquad \text{----------} \quad (4.16)$$
$$\mathbf{x}\in\partial S\,,\ \mathbf{x}'\in S+\partial S$$

$$\hat{G}_2(\mathbf{x},t;\mathbf{x}',t') = 0 \qquad t < t' \qquad \text{----------} (4.17)$$

$$\psi(\mathbf{x},t) = -\iint_S \omega(\mathbf{x}',t)\,G_1(\mathbf{x}',\mathbf{x})\,d^2\mathbf{x}' - \int_{\partial S} \frac{\partial\psi'}{\partial n'}\,G_1(s',\mathbf{x})\,ds'$$
$$= -\iint_S \omega(\mathbf{x}',t)\,G_1(\mathbf{x}',\mathbf{x})\,d^2\mathbf{x}' \qquad \text{----------} (4.18)$$

$$\int_{t_0}^{T}\iint_S (u'L_2v' - v'\hat{L}_2u')\,d^2\mathbf{x}'\,dt' = \int_{t_0}^{T}\int_{\partial S}\left(u'\frac{\partial v'}{\partial n'} - v'\frac{\partial u'}{\partial n'}\right)ds'\,dt'$$

$$-\iint_S u'(T)\,v'(T)\,d^2\mathbf{x}' + \iint_S u'(t_0)\,v'(t_0)\,d^2\mathbf{x}' \qquad \text{------} (4.19)$$

$$u' = \hat{G}_2(\mathbf{x}',t';\mathbf{x},t)$$
$$v' = \omega(\mathbf{x}',t')$$

$$\omega(\mathbf{x},t) = -R\int_{t_0}^{t}\iint_S J(\psi',\omega')\,\hat{G}_2(\mathbf{x}',t';\mathbf{x},t)\,d^2\mathbf{x}'\,dt'$$
$$+\int_{t_0}^{t}\omega(\mathbf{x}',t')\,\frac{\partial}{\partial n'}\hat{G}_2(s',\mathbf{x}';\mathbf{x},t)\,ds'\,dt'$$
$$-\iint_S \omega(\mathbf{x}',t_0)\,\hat{G}_2(\mathbf{x}',t_0;\mathbf{x},t) \qquad \text{----------} (4.20)$$

$$\psi(\mathbf{x},t) = -\iint_S G_1(\mathbf{x}'',\mathbf{x})\Big[-R\int_{t_0}^{t}\iint_S J(\psi',\omega')\,\hat{G}_2(\mathbf{x}',t';\mathbf{x},t)\,d^2\mathbf{x}'\,dt'$$
$$+\int_{t_0}^{t}\int_{\partial S}\overline{\omega}(s',t')\,\frac{\partial}{\partial n'}\hat{G}_2(\mathbf{x}',t';\mathbf{x}'',t)\,ds'\,dt' - \iint_S \omega(\mathbf{x}',t_0)\,\hat{G}_2(\mathbf{x}',t_0;\mathbf{x}'',t)\Big]\,d^2\mathbf{x}''$$

$$\Rightarrow$$

$$\psi(\mathbf{x},t) = R\int_{t_0}^{t}\iiiint J(\psi',\omega')\,\hat{G}_2(\mathbf{x}',t';\mathbf{x}'',t)\,G_1(\mathbf{x}'',\mathbf{x})\,d^2\mathbf{x}'\,d^2\mathbf{x}''\,dt'$$

$$-\int_{t_0}^{t}\iint_{S}\int_{\partial S}\bar{\omega}(s',t')\,\frac{\partial\hat{G}_2}{\partial n'}(\mathbf{x}',t';\mathbf{x}'',t)\,G_1(\mathbf{x}'',\mathbf{x})\,ds'\,d^2\mathbf{x}''\,dt'$$

$$+\iiiint\omega(\mathbf{x}',t_0)\,\hat{G}_2(\mathbf{x}',t_0;\mathbf{x}'',t)\,G_1(\mathbf{x}'',\mathbf{x})\,d^2\mathbf{x}'\,d^2\mathbf{x}''$$

$\Rightarrow$

$$\frac{\partial\psi(s,t)}{\partial n}=R\int_{t_0}^{t}\iiiint J(\psi',\omega')\,\hat{G}_2(\mathbf{x}',t';\mathbf{x}'',t)\,\frac{\partial G_1}{\partial n}(\mathbf{x}'',s)\,d^2\mathbf{x}'\,d^2\mathbf{x}''\,dt'$$

$$-\int_{t_0}^{t}\iint_{S}\int_{\partial S}\bar{\omega}(s',t')\,\frac{\partial\hat{G}_2}{\partial n'}(s',t';\mathbf{x}'',t)\,\frac{\partial G_1}{\partial n}(\mathbf{x}'',s)\,ds'\,d^2\mathbf{x}''\,dt'$$

$$+\iiiint\omega(\mathbf{x}',t_0)\,\hat{G}_2(\mathbf{x}',t_0;\mathbf{x}'',t)\,\frac{\partial G_1}{\partial n}(\mathbf{x}'',s)\,d^2\mathbf{x}'\,d^2\mathbf{x}''$$

$\Rightarrow$

$$\psi_n(s,t)=R\alpha(s,t)-\int_{t_0}^{t}\int_{\partial S}\bar{\omega}(s',t')\,F(s',t';s,t)\,ds'\,dt'$$

$$+\beta(s,t)$$

where;

$$\iint_{S}\frac{\partial\hat{G}_2}{\partial n'}(s',t';\mathbf{x}'',t)\,\frac{\partial G_1}{\partial n}(\mathbf{x}'',s)\,d^2\mathbf{x}'\,d^2\mathbf{x}''=F(s',t';s,t)$$

$\Rightarrow$

$$\int_{t_0}^{t}\int_{\partial S}\bar{\omega}(s',t')\,F(s',t';s,t)\,ds'\,dt'=R\alpha(s,t)-\beta(s,t)-\psi_n(s,t)$$

$$=T(s,t) \qquad (4.21)$$

The known functions $F(s',t';S,t)$ and $T(S,t)$ will provide the informations of $\bar{\omega}(s',t')$

In constructing time dependent Green's function G2 by Reflection method (Fig.d) the following steps are taken into consideration

$$\left(\nabla^2+\frac{\partial}{\partial t}\right)\hat{F}_2(\mathbf{x},t;\mathbf{x}',t')=\delta^2(\mathbf{x}-\mathbf{x}')\,\delta(t-t') \qquad (4.22)$$

$$\hat{F}_2(\mathbf{x},t;\mathbf{x}',t')=0 \qquad t>t' \qquad (4.23)$$

Fundamental solution satisfying conditions (4.22) and (4.23) is found to be,

$$\hat{F}_2(\mathbf{x},t;\mathbf{x}',t') = -\frac{H(t'-t)}{4\pi(t'-t)} \exp\left[\frac{(X-X')^2+(Y-Y')^2}{-4(t'-t)}\right] \qquad \text{----------(4.24)}$$

where $H(t'-t)$ is the step function.

$$-H(t'-t) = \begin{matrix} 0 & t > t' \\ 1 & t < t' \end{matrix}$$

also,

$$\hat{F}_2(\mathbf{x},t) = \frac{-H(t)}{4\pi t} \exp\left[\frac{-\mathbf{x}^2}{4t}\right]$$

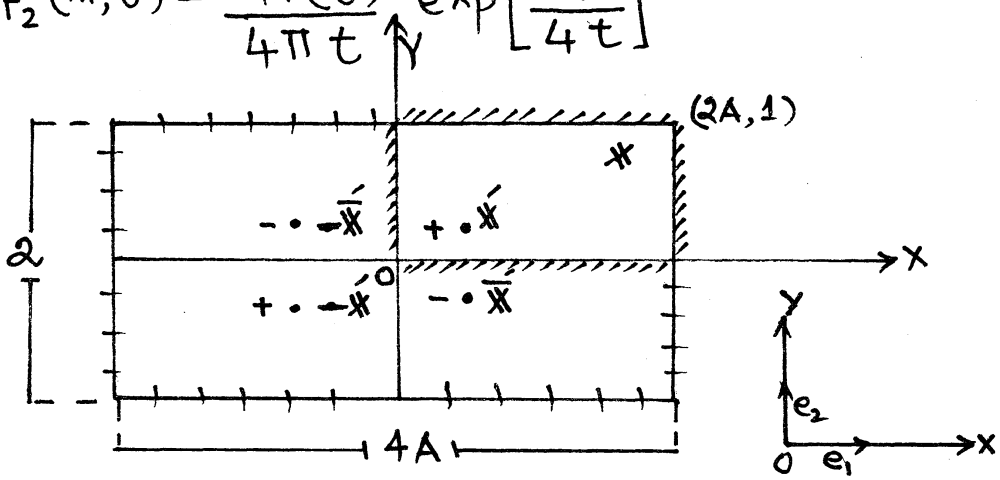


Fig.(d)

$$\hat{G}_2(\mathbf{x},t;\mathbf{x}',t') = \bar{G}_2(\mathbf{x}-\mathbf{x}',t'-t) + \bar{G}_2(\mathbf{x}+\mathbf{x}',t'-t) - \bar{G}_2(\mathbf{x}-\bar{\mathbf{x}}',t'-t) - \bar{G}_2(\mathbf{x}+\bar{\mathbf{x}}',t'-t) \qquad \text{----------(4.25)}$$

$$\bar{G}_2(\mathbf{x},t) = \sum_{l,m} \hat{F}_2(\mathbf{x}+4al\,e_1+2m\,e_2,t) \qquad \text{----------(4.26)}$$

where,

$\mathbf{x}=Xe_1+Ye_2$ and $\mathbf{x}'=X'e_1+Y'e_2$ $\qquad$ e_1,e_2 are unit vectors

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